Zeros of functions with Matlab: Newton's method

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Introduction

- Finding the zeros of a function means finding:

$$\bar{x}: \quad f(\bar{x}) = 0.$$

- We already know two approaches to solve such problem:

  1. bisection method,
  2. fixed-point iteration.

- In this lecture we will see the Newton's method.
- It is the faster one provided that we take an initial condition *close* to the zero.

Bisection

Property

Given a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ *such that* $f(a)f(b) < 0$ *then* $\exists\ \bar{x} \in (a, b)$ *so that* $f(\bar{x}) = 0$.

- Thus, at the first step of the bisection method, we evaluate the mid point m_1 of $[a, b] = [a_1, b_1]$. If $f(m_1)f(a) < 0$ then

$$[a_2, b_2] = [a_1, m_1],$$

else

$$[a_2, b_2] = [m_1, b_1].$$

- We proceed in this way until a stopping criterion is satisfied, such as $f(m_k) < \tau$, for a fixed tolerance τ, at a certain step k.

Fixed point

We want to find $\bar{x}$ so that $f(\bar{x}) = 0$ or equivalently the fixed point of $g(x) = x$, for $f(x) = x - g(x) = 0$.

Theorem

Let us consider $x^{(k+1)} = g(x^{(k)})$, *for* $k \geq 0$, *with* $x^{(0)}$ *given. If*

1. $g : [a, b] \rightarrow [a, b]$;
2. $g \in C^1([a, b])$;
3. $\exists K < 1$ *such that* $|g^{'}(x)| < K \; \forall x \in [a, b]$;

then g *has a unique fixed point* $\bar{x}$ *in* $[a, b]$ *and* $\{x^{(k)}\}_{k>1}$ *converges to* $x^{(0)} \in [a, b]$.

Newton

Theorem (Convergence of Newton's method)

Let $f(x) \in \mathcal{C}^2([a,b])$*. Suppose* $f(\bar{x}) = 0$ *and* $f(\bar{x}) \neq 0$ *for some* $\bar{x} \in (a,b)$*. Then, there exists* $\delta \in \mathbb{R}$*,* $\delta > 0$ *such that for* $x_0 \in [\bar{x} - \delta, \bar{x} + \delta]$ *the iterations via Newton's method*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 1, 2, \dots,$$

converges quadratically to $\bar{x}$*.*

- The main drawback is that with Newton's method we need to start with an initial condition *sufficiently close* to the zeros of functions.

Exercise 1

Exercise

Write the Matlab function `newton.m` *for computing the zeros of functions with the Newton's method.*

Hint

Use an exit criterion in case the method does not converge.

The inputs that we need are `f,fp,x0,tol,maxiter`.

Fix the stopping criterion as `abs(f(x0))<tol`, *where* `x0` *is the k-th approximation.*

Exercise 2

Exercise

Consider the function $f(x) = e^x - 4x^2$*. Write a Matlab script that*

- *plot the function in* $[-2, 5]$ *and observe that in this interval the function has* 3 *zeros:* $\xi_1 \in (-1, 0)$, $\xi_2 \in (0, 1)$ *and* $\xi_3 \in (4, 4.5)$;
- *find* ξ_1 *with the bisection method,* ξ_2 *with Newton's method and* ξ_3 *with the fixed point iteration* $x_{i+1} = \log(4x_i^2)$*. Does this last method converge for every initial point* x_0*?*

Exercise 3

Exercise

Take the function

$$f(x) = x^2 - c,$$

$c \geq 0$, *and the following two iteration functions:*

1. $g_1(x) = x - \dfrac{x^2 - c}{2x}.$
2. $g_2(x) = x - \dfrac{x^2 - c}{2x} - \dfrac{\left(x - \dfrac{x^2 - c}{2x}\right)^2 - c}{2x}.$

Study the convergence of these two iterative methods. In particular take $c = 2$ *or* $c = 3$.