RADON TRANSFORM

The mathematical model for the CT scan is the Radon transform. For a given bivariate function $f$ on $[1, k]^2$, which represents the attenuation coefficient of a body slice on a grid of $k^2$ pixels, it is defined for each $(t, \theta)$ as:

$$R[f](t, \theta) = \int_{-\infty}^{\infty} f(t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta)) ds$$

It has an explicit inversion formula: the Filtered back projection formula, that is the fundamental basis for image reconstruction technique based on inversion strategies. ART method, otherwise, is an approach based on linear algebra. While the Fourier transform approach solves continuous problem and then moves to discrete one, ART considers the discrete problem from the beginning. The image is formed by $k \times k$ pixels, and defining pixel basis functions $b_1, b_2, \ldots, b_{k^2}$ we have the problem:

$$R[f](t_j, \theta_j) = \sum_{i=1}^{k^2} c_i R[R[b_i](t_j, \theta_j)] \forall j = 1, \ldots, n.$$ 

THE MODEL

We seek for a generalized Hermite-Birkhoff approximation $s$ of $f$ as

$$s(x) = \sum_{j=1}^{n} c_j R[R[K_{w,w}(x,y)](t_j, \theta_j), \forall x \in [1, k]^2$$

where $K_{w,w}$ is a double weighted version of an RBF kernel $K$, namely

$$K_{w,w}(x,y) = K(x,y)w(x)w(y) = \varphi(||x-y||)w(||x||)w(||y||)$$

Then we ask that $R^2[s]$ coincides with $R[f]$ on the points $(t_j, \theta_j)$ for all $j = 1, \ldots, n$, which is equivalent to solve the linear system

$$A_{K,K} \cdot c = b$$

where

$$(A_{K,K})_{i,j} = R[R[K_{w,w}]], \quad b_i = R[f]_i.$$ 

Our approach overcomes the problem of the standard kernel method: if no weights are used, each radial kernel gives singular matrices.

NUMERICAL EXAMPLES

AIR Tools package [6]

GAUSSIAN KERNEL

Crescent - Shape
(full data - no noise )

Bull’s Eye
(white noise with $\sigma = 0.05$ )

Shepp - Logan
(missing 40% of the Radon lines)

REFERENCES

- S. De Marchi, A. Iske and A. Sironi Kernel-based image reconstruction from scattered Radon data by positive definite function. preprint
- M. Hesse and B. Schaback Bases for Kernel-Based Spaces Journal of Computational and Applied Mathematics Volume 236 Issue 4, September, 2011 Pages 575-588

WORK IN PROGRESS

- The kernel and the weights shape strongly depend on some parameters which are not yet fully optimized.
- The method is for now limited to the Gaussian kernel: we plan to extend it to other RBF kernels.
- The computational cost of the algorithm can be reduced using compactly supported kernels and weights functions.

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- We can make use of theory of change of basis for RBF, in particular the Newton basis [3] allows to recursively select the Radon lines which minimize the error.
- It is possible to reconstruct images from missing or scattered data.
- The dimension of the linear system is independent on $k$. 

BENEFITS

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