Kernel methods for Radon transform

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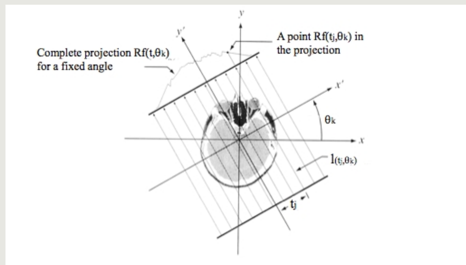
Radon Transform



The mathematical model for the CT scan is the *Radon transform*. For a given bivariate function f on $[1,\ k]^2$, which represents the attenuation coefficient of a body slice on a grid of k^2 pixel, it is defined for each (t,θ) as:

$$\mathcal{R}[f](t,\theta) = \int_{\ell_{t,\theta}} f(s)ds = \int_{-\infty}^{\infty} f(t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))ds$$

It has an explicit inversion formula: the Filtered back projection formula, that is the fundamental basis for image reconstruction tecnique based on inversion strategies. ART method, otherwise, is an approach based on linear algebra. While the **Fourier transform** approach solves continuous problem and then moves to discrete one, **ART** considers the discrete problem from the beginning. The image is formed by $k*k$ pixels, and defining pixel basis functions $b_1, b_2, \ldots, b_{k^2}$ we have the problem:

$$\mathcal{R}[f](t_j,\theta_j) = \sum_{i=1}^{k^2} x_i \mathcal{R}[b_i](t_j,\theta_j)\ \forall j = 1,\ldots,n.$$

The model

We seek for a generalized Hermite-Birkhoff approximation s of f as

$$s(x) = \sum_{j=1}^{n} c_j \mathcal{R}^y[\mathcal{K}_{w,w}(x,y)](t_j,\theta_j),\ \forall x \in [1,\ k]^2$$

where $\mathcal{K}_{w,w}$ is a **double weighted** version of an RBF kernel $\mathcal{K}$, namely

$$\boldsymbol{\mathcal{K}_{w,w}(x,y) = \mathcal{K}(x,y)w(x)w(y) = \varphi(\|x-y\|)w(\|x\|)w(\|y\|)}$$

Then we ask that $\mathcal{R}^x[s]$ coincides with $\mathcal{R}[f]$ on the points (t_j,θ_j) for all $j = 1,\ldots,n$, which is equivalent to solve the linear system

$$A_{\mathcal{K},\mathcal{R}} \cdot c = b, \quad \text{where}$$

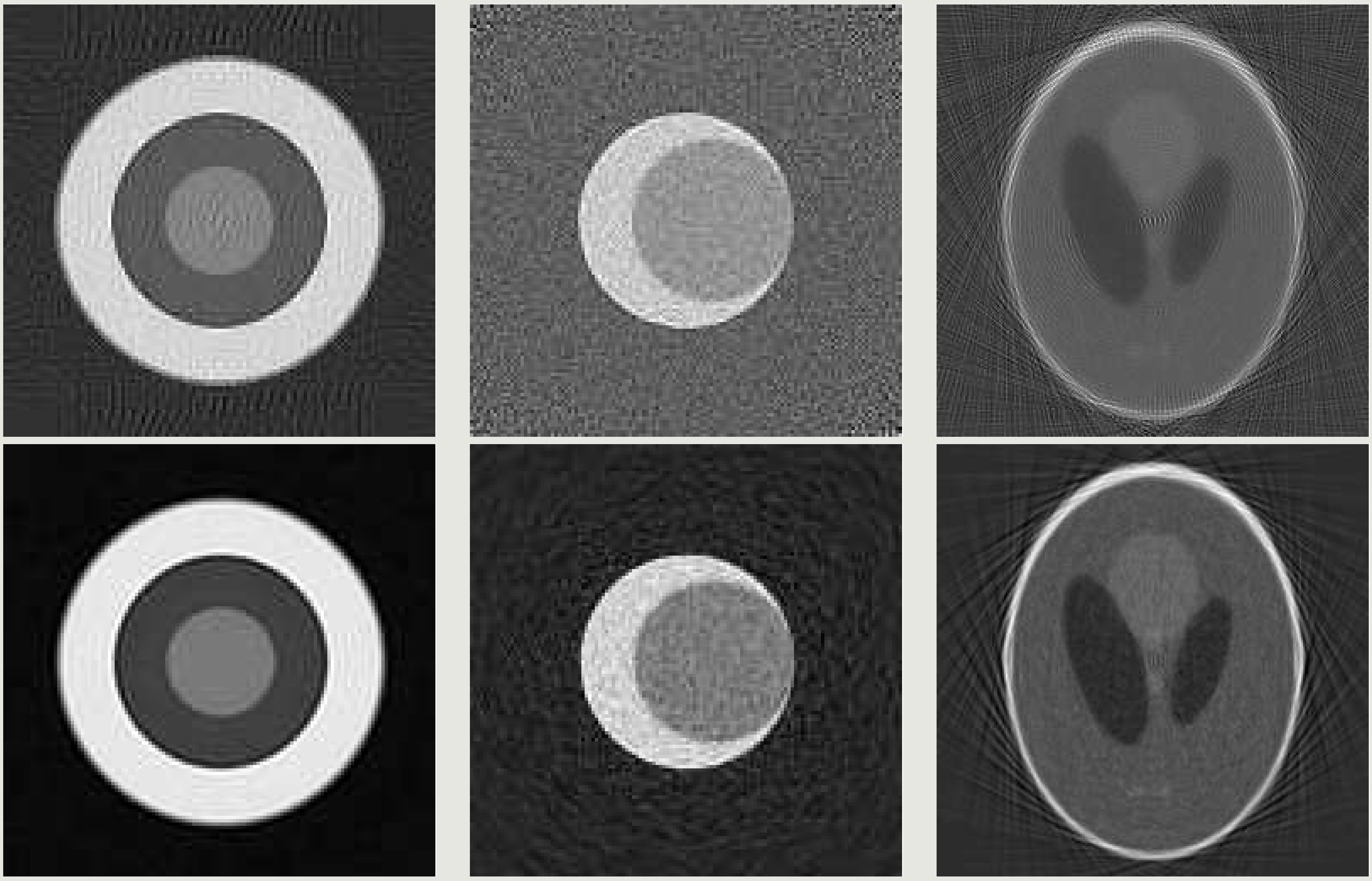
$$(A_{\mathcal{K},\mathcal{R}})_{i,j} = \mathcal{R}_i^x[R_j^y[\mathcal{K}_{w,w}]], \quad b_i = \mathcal{R}[f]_i.$$

Our approach overcomes the problem of the standard kernel method: if no weights are used, each radial kernel gives singular matrices.

Numerical examples

AIR Tools package [6]

Gaussian Kernel



Crescent - Shape (full data - no noise)	Bull's Eye (white noise with $\sigma = 0.05$)	Shepp -Logan (missing 40% of the Radon lines)

Benefits

- The reconstruction matrix is **symmetric** and **positive definite**.
- We can make use of theory of change of basis for RBF, in particular the Newton basis [3] allows to **recursively select** the Radon lines which minimize the error.
- It is possible to reconstruct images from **missing** or scattered **data**.
- The dimension of the linear system is independent on k.

Work in progress

- The kernel and the weights shape strongly depend on some parameters which are not yet fully optimized.
- The method is for now limited to the Gaussian kernel: we plan to extend it to other RBF kernels.
- The computational cost of the algorithm can be reduced using compactly supported kernels and weights functions.

References

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