A New Stable Basis for RBF Approximation

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Abstract

It's well know that Radial Basis Function interpolants suffers of bad conditioning if the simple basis of translates is used. A recent work of M.Pazouki and R.Schaback [2] gives a quite general way to build stable, orthonormal bases for the native space $\mathcal{N}_\Phi(\Omega)$ based on a factorization of the kernel matrix A_Φ.
Starting from that setting we describe a particular $\mathcal{N}_\Phi(\Omega)$-orthonormal, $\ell_2^w(X)$-orthogonal basis that arises from a weighted singular value decomposition of A_Φ. This basis is related to a discretization of a compact operator $T_\Phi : \mathcal{N}_\Phi(\Omega) \to \mathcal{N}_\Phi(\Omega)$,

$$T_\Phi[f](x) = \int_\Omega \Phi(x,y) f(y) dy \quad \forall x \in \Omega$$

and provides a connection with the continuous basis that arises from an eigen-decomposition of T_Φ.
We give convergence estimates and stability bound for the interpolation and the discrete least-squares approximation based on this basis, which involves the eigenvalues of such an operator.

Tools

Change of Basis [1, 2]:
Any basis $\mathcal{U}$ for $\mathcal{N}_\Phi(X)$ arises from a factorization

$$A_\Phi = V_\mathcal{U} \cdot {C_\mathcal{U}}^{-1}$$

where $V_\mathcal{U} = (u_j(x_i))_{1 \leqslant i,j \leqslant N}$, $U(x) = T(x) \cdot C_\mathcal{U}$

Orthonormal Basis [1, 2]: Each Φ-orthonormal basis $\mathcal{U}$ arises from a decomposition $A_\Phi = B^T \cdot B$ with $V_\mathcal{U} = B^T$ and $C_\mathcal{U} = B^{-1}$. Each $\ell_2(X)$-orthonormal basis $\mathcal{U}$ arises from a decomposition $A_\Phi = Q \cdot B$ with $Q^T \cdot Q = I$, $V_\mathcal{U} = Q$ and $C_\mathcal{U} = B$.

Cubature Rule: $(X,\ \mathcal{W})_N$ such that

$$\textstyle\sum_{j=1}^N f(x_j) w_j \approx \int_\Omega f(y) dy \quad \forall f \in \mathcal{C}(\Omega)$$

Symmetric Nyström Method: $\forall\ i,j = 1,\dots,N$

$$\lambda_j ({w_i}^{\frac{1}{2}} \varphi_j(x_i)) = \textstyle\sum_{h=1}^N ({w_i}^{\frac{1}{2}} \Phi(x_i, x_h)\ {w_h}^{\frac{1}{2}} ({w_h}^{\frac{1}{2}} \varphi_j(x_h))$$

Key Features

- the basis is Φ-orthonormal
- the basis is $\ell_2^w(X)$-orthogonal
- the basis approximates the Eigenbasis $\{\varphi_j\}_{j>0}$ (strong connection with the kernel and Ω)
- the interpolant $P_X[f]$ and the weighted-DLS approximant $\Lambda_M[f]$ are stable
- the weighted-DLS approximant $\Lambda_M[f]$ can be obtained as a simple truncation of the interpolant, at an index M s.t. the corresponding singular value $\sigma_M < tol$ (low-rank approximation)

Eigenbasis

Definition: Let Φ be a continuous, positive definite kernel on a bounded $\Omega \subset \mathbb{R}^n$. Then the operator

$$T_\Phi : \mathcal{N}_\Phi(\Omega) \to \mathcal{N}_\Phi(\Omega), T_\Phi[f](x) = \int_\Omega \Phi(x,y) f(y) dy \ \forall x \in \Omega$$

is bounded, compact and self-adjoint. It has an enumerable set of eigenvalues and eigenvectors $\{\lambda_j, \varphi_j\}_{j>0}$ which forms a basis for $\mathcal{N}_\Phi(\Omega)$.

Properties:

P1 $\{\varphi_j\}_{j>0}$ is orthonormal in $\mathcal{N}_\Phi(\Omega)$

P2 $\{\varphi_j\}_{j>0}$ is orthogonal in $L_2(\Omega)$, $\|\varphi_j\|^2_{L_2(\Omega)} = \lambda_j \quad \forall j > 0$

P3 $\lambda_j \xrightarrow{j\to\infty} 0$, $\sum_{j>0} \lambda_j = \Phi(0,0)\ |\Omega|$

Weighted-SVD Basis

Definition: A weighted-SVD basis $\mathcal{U}$ is a basis for $\mathcal{N}_\Phi(X)$ characterized by the following matrices:

$$V_\mathcal{U} = \sqrt{W^{-1}} \cdot Q \cdot \Sigma, \ C_\mathcal{U} = \sqrt{W} \cdot Q \cdot \Sigma^{-1}$$

where

$$\sqrt{W} \cdot A_\Phi \cdot \sqrt{W} = Q \cdot \Sigma^2 \cdot Q^T$$

is an SVD (and an unitary diagonalization) of the scaled kernel matrix A_W, and $\{w_j\}_{j=1}^N$ are the weigths of a cubature rule $(X,\ \mathcal{W})_N$.
Each element of the basis takes the form

$$u_j(x) = \frac{1}{\sigma_j^2} \sum_{i=1}^N w_i u_j(x_i) \Phi(x, x_i) \approx \frac{1}{\sigma_j^2} T_\Phi[u_j](x) \quad \forall\ 1 \leqslant j \leqslant N,\ \forall x \in \Omega$$

Properties:

p1 $\{u_j\}_{j=1}^N$ is orthonormal in $\mathcal{N}_\Phi(\Omega)$

p2 $\{u_j\}_{j=1}^N$ is $\ell_2^w(X)$-orthogonal, $\|u_j\|^2_{\ell_2^w(X)} = \sigma_j^2 \quad \forall j = 1,\dots,N$

p3 $\sum_{j=1}^N \sigma_j^2 = \phi(0)\ |\Omega|$

Approximation

Interpolant and Weighted-DLS Approximant: for all $f \in \mathcal{N}_\Phi(\Omega)$,

$$P_X[f](x) = \sum_{j=1}^N (f, u_j)_\Phi u_j(x), \quad \Lambda_M[f](x) = \sum_{j=1}^M (f, u_j)_\Phi u_j(x) \quad \forall x \in \Omega$$

$\mathbf{L_2(\Omega)}$-Convergence: $\Omega \subset \mathbb{R}^n$ compact, for all $f \in \mathcal{N}_\Phi(\Omega)$,

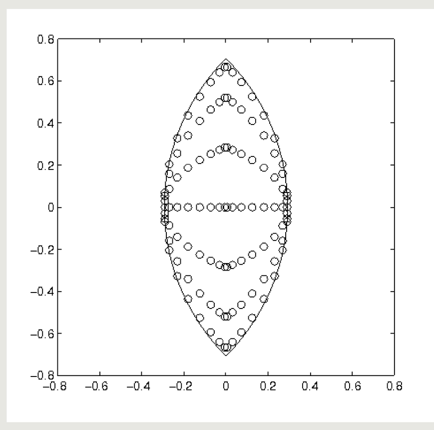
$$\|f - P_X[f]\|_{L_2(\Omega)} \leqslant \left(|\Omega| \cdot \phi(0) - \sum_{j=1}^N \lambda_j + C \cdot \sum_{j=1}^N \|u_j - \varphi_j\|_{L_2(\Omega)} \right)^{\frac{1}{2}} \|f\|_\Phi$$

(the same for $\Lambda_M[f]$ if N is replaced with $M < N$)

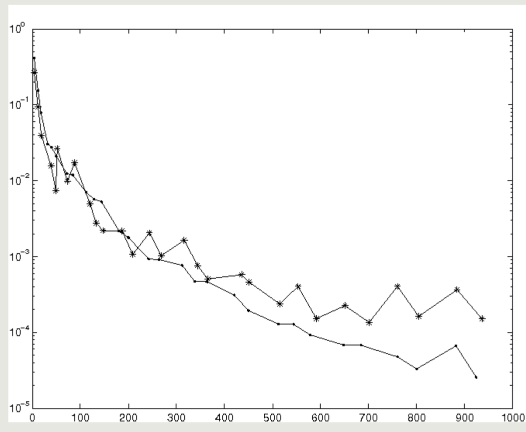
Stability: for all $f \in \mathcal{N}_\Phi(\Omega)$, $\forall x \in \Omega$,

$$|P_X[f](x)|, |\Lambda_M[f](x)| \leqslant \sqrt{\phi(0)}\ \|f\|_\Phi$$

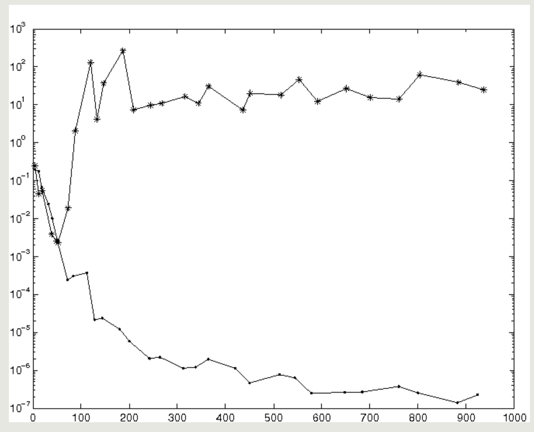
Numerical Example



The *lens*



$\varepsilon = 9$



$\varepsilon = 1$ and $\sigma_M < 10^{-17}$

Reconstruction of the Franke function on a lens with the IMQ kernel, for different shape parameters ε. Comparison between the RMSE obtained using the standard basis (dotted lines) and our basis centered on *trigonometric-Gauss* nodes [4, 5]. More example can be found in [3].

References

[1] S.Müller and R.Schaback, *A Newton basis for kernel spaces*, J. Approx. Th. 161 (2009), 645-655

[2] M.Pazouki and R.Schaback, *Bases for Kernel-Based Spaces*, Computational and Applied Mathematics 236 (2011), pp. 575-588

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[4] M.Vianello and G.Da Fies, *Algebraic cubature on planar lenses and bubbles*, Dolomites Res. Notes Approx. DRNA 5 (2012), 7–12

[5] M.Vianello and L.Bos, *Subperiodic trigonometric interpolation and quadrature*, Appl. Math. Comput., published online 16 May (2012)