Abstract

It’s well known that Radial Basis Function interpolants suffers of bad conditioning, if the simple basis of translates is used. A recent work of M.Pazouki and R.Schaback [2] gives a quite general way to build stable, orthonormal bases for the native space $N_0^2(\Omega)$ based on a factorization of the kernel matrix $A_\Phi$. Starting from that setting we describe a particular $N_0^2(\Omega)$-orthonormal, $\ell^2(\mathcal{X})$-orthogonal basis that arises from a weighted singular value decomposition of $A_\Phi$. This basis is related to a discretization of a compact operator $T_\Phi : N_0^2(\Omega) \to N_0^2(\Omega)$,

$$T_\Phi[f](x) = \int_\Omega \Phi(x,y)f(y)dy \quad \forall x \in \Omega$$

and provides a connection with the continuous basis that arises from an eigen-decomposition of $T_\Phi$.

We give convergence estimates and stability bound for the interpolation and the discrete least-squares approximation based on this basis, which involves the eigenvalues of such an operator.

Tools

Change of Basis [1, 2]: Any basis $B$ for $N_0^2(\Omega)$ arises from a factorization

$$A_\Phi = V_B \cdot C_B^{-1}$$

where $V_B = (u_i(x))_{i \in \mathcal{X}, r \in \mathcal{N}}$, $U(x) = T(x) \cdot C_B$

Orthonormal Basis [1, 2]: Each $\Phi$-orthonormal basis $B$ arises from a decomposition $A_\Phi = B^T \cdot B$ with $V_B = B^T$ and $C_B = B^{-1}$. Each $\ell^2(\mathcal{X})$-orthogonal basis $B$ arises from a decomposition $A_\Phi = Q \cdot B$ with $Q^T \cdot Q = I$, $V_B = Q$ and $C_B = B$.

Cubature Rule: $(X, \mathcal{W})_N$ such that

$$\sum_{j=1}^N f(x_j)w_j = \int f(y)dy \quad \forall f \in C(\Omega)$$

Symmetric Nyström Method: $\forall i, j = 1, \ldots, N$

$$\lambda_j = \sum_{i=1}^N (u_i(x_j)) w_i \cdot (w_i(x_j))$$

Key Features

- the basis is $\Phi$-orthonormal
- the basis is $\ell^2(\mathcal{X})$-orthogonal
- the basis approximates the Eigenbasis $\{\varphi_j\}_{j>0}$ (strong connection with the kernel and $\Omega$)
- the interpolant $P_N[f]$ and the weighted-DLS approximant $\Lambda_M[f]$ are stable
- the weighted-DLS approximant $\Lambda_M[f]$ can be obtained as a simple truncation of the interpolant, at an index $M$ s.t. the corresponding singular value $\sigma_M < tol$ (low-rank approximation)

Weighted-SVD Basis

Definition: A weighted-SVD basis $B$ is a basis for $N_0^2(\Omega)$ characterized by the following matrices:

$$V_B = \sqrt{W^{-1}} \cdot Q \cdot \Sigma, \quad C_B = \sqrt{W} \cdot Q \cdot \Sigma^{-1}$$

where

$$\sqrt{W} \cdot A_\Phi \cdot \sqrt{W} = Q \cdot \Sigma^2 \cdot Q^T$$

is an SVD (and an unitary diagonalization) of the scaled kernel matrix $A_\Phi$, and $\{w_j\}_{j=1}^N$ are the weight of a cubature rule $(X, \mathcal{W})_N$.

Each element of the basis takes the form

$$u_j(x) = \frac{1}{\sigma_j^2} \sum_{i=1}^N w_i u_j(x_i) \Phi(x,x_i) \approx \frac{1}{\sigma_j^2} T_\Phi [u_j](x) \quad \forall 1 \leq j \leq N, \forall x \in \Omega$$

Properties:

- $p1 \{u_j\}_{j=1}^N$ is orthonormal in $N_0^2(\Omega)$
- $p2 \{u_j\}_{j=1}^N$ is $\ell^2(\mathcal{X})$-orthogonal, $\|u_j\|_{\ell^2(\mathcal{X})}^2 = \sigma_j^2$ for $j = 1, \ldots, N$
- $p3 \sum_{j=1}^N \sigma_j^2 = \|\Phi(0)\|^2$ |O|

Interpolant and Weighted-DLS Approximant: for all $f \in N_0^2(\Omega)$,

$$P_N[f](x) = \sum_{j=1}^N (f(u_j)u_j)(x), \quad \Lambda_M[f](x) = \sum_{j=1}^M (f(u_j)u_j)(x) \quad \forall x \in \Omega$$

$L_2(\Omega)$-Convergence: $\Omega \subset \mathbb{R}^n$ compact, for all $f \in N_0^2(\Omega)$,

$$\|f - P_N[f]\|_{L_2(\Omega)} \leq \sqrt{\left(\|f\|_{L_2(\Omega)}^2 - \sum_{j=1}^N \lambda_j + C \cdot \sum_{j=1}^N \|\varphi_j\|_{L_2(\Omega)} \|\varphi_j\|_{L_2(\Omega)}\right)}^{\frac{1}{2}} \|f\|_{\Omega}$$

(consistent with $\Lambda_M[f]$ if $N$ is replaced with $M < N$)

Stability: for all $f \in N_0^2(\Omega)$, $\forall x \in \Omega$,

$$|P_N[f](x)|, |\Lambda_M[f](x)| \leq \sqrt{\Omega(t)} \|f\|_{\Omega}$$

Numerical Example

The lens

$\varepsilon = 9$ $\varepsilon = 1$ and $\sigma_M < 10^{-17}$

Reconstruction of the Franke function on a lens with the MQ kernel, for different shape parameters $\varepsilon$. Comparison between the RMSE obtained using the standard basis (dotted line) and our basis centered on trigonometric-Gauss nodes [4, 5]. More example can be found in [3].

References