WSVD basis for RBF and Krylov subspaces

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Eigenbasis - properties

Let K be a continuous, positive definite kernel on a bounded $\Omega \subset \mathbb{R}^s$.
Then the operator $\mathcal{T} : \mathcal{N}_K(\Omega) \to \mathcal{N}_K(\Omega)$,

$$\mathcal{T}[f](x) = (f, K(\cdot, x))_{L_2(\Omega)} \ \forall x \in \Omega$$

is bounded, compact and self-adjoint. It has an enumerable set of eigenvalues and eigenvectors $\{\boldsymbol{\lambda_j}, \boldsymbol{\varphi_j}\}_{j>0}$ which are a basis for $\mathcal{N}_K(\Omega)$.

Moreover:

P1 $\{\varphi_j\}_{j>0}$ is $\mathcal{N}_K(\Omega)$-orthonormal

P2 $\{\varphi_j\}_{j>0}$ is $L_2(\Omega)$-orthogonal

P3 $\|\varphi_j\|^2_{L_2(\Omega)} = \lambda_j \quad \forall j > 0$

P4 $\lambda_j \xrightarrow[j\to\infty]{} 0$, $\sum_{j>0} \lambda_j = K(0,0)\ |\Omega|$

Problem and definition

→ How to obtain a **stable basis** for RBF approximation?

→ How to embed information about the kernel K and the set Ω into it?

→ We can approximate the ***eigenbasis*** of the **native space** $\mathcal{N}_K(\Omega)$ of RBF kernel K.

Definition: Given $X = \{x_1, \dots, x_n\} \subset \Omega$, a weighted-SVD (WSVD) basis $\mathcal{U} = \{u_j\}_{1\leqslant j\leqslant n}$ is a basis for $\mathcal{N}_K(X) = \mathrm{span}\{K(\cdot, x_i), 1 \leqslant i \leqslant n\}$ obtained by the matrix of change of basis

$$C_{\mathcal{U}} = \sqrt{W} \cdot Q \cdot \Sigma^{-1}$$

where $A_W := \sqrt{W} \cdot A \cdot \sqrt{W} = Q \cdot \Sigma^2 \cdot Q^T$ is a SVD of the scaled kernel matrix $A = (K(x_i, x_j))_{1\leqslant i,j\leqslant n}$, and $\mathcal{W} = \{W_{ii}\}_{1\leqslant i\leqslant n}$ are the weigths of a cubature rule $(X, \mathcal{W})$.

WSVD basis - properties

Let K be a continuous, positive definite kernel on a bounded $\Omega \subset \mathbb{R}^s$, and $(X, \mathcal{W})$ a cub. rule.
Then the operator $\mathcal{T}_n : \mathcal{N}_K(\Omega) \to \mathcal{N}_K(X)$,

$$\mathcal{T}_n[f](x) = (f, K(\cdot, x))_{\ell_2^w(X)} \ \forall x \in \Omega$$

has finite rank and is self-adjoint. It has n eigenvalues and eigenvectors $\{\boldsymbol{\sigma_j^2}, \boldsymbol{u_j}\}_{1\leqslant j\leqslant n}$ which are a basis for $\mathcal{N}_K(X)$.

Moreover:

p1 $\{u_j\}_{1\leqslant j\leqslant n}$ is $\mathcal{N}_K(\Omega)$-orthonormal

p2 $\{u_j\}_{1\leqslant j\leqslant n}$ is $\ell_2^w(X)$-orthogonal

p3 $\|u_j\|^2_{\ell_2^w(X)} = \sigma_j^2, \quad 1 \leqslant j \leqslant n$

p4 $\sum_{j=1}^n \sigma_j^2 = K(0,0)\ |\Omega|$

WSVD approximation

Interpolant: for all $f \in \mathcal{N}_K(\Omega)$, since $\mathcal{U}$ is $\mathcal{N}_K(\Omega)$-o.n.,

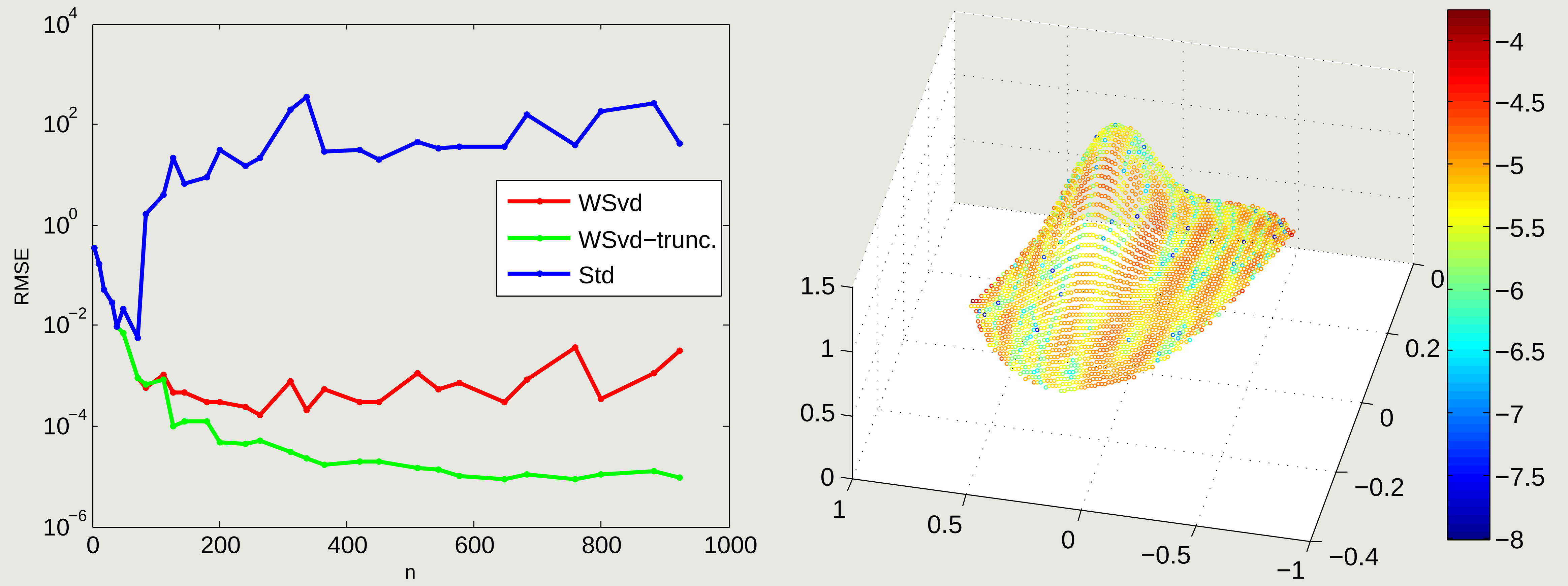
$$s(f) = \sum_{j=1}^{n} (f, u_j)_{\mathcal{N}_K(\Omega)} u_j.$$

Weighted-DLS Approximant: we can define the WDLS approximant $s^m(f)$ of order m as the minimizer of $\|f - g\|_{\ell_2^w(X)}$ between functions $g \in \mathrm{span}\{u_1, \dots, u_m\}$.
It turns out that, for all $f \in \mathcal{N}_K(\Omega)$,

$$\mathbf{s^m(f)} = \sum_{\mathbf{j=1}}^{\mathbf{m}} (\mathbf{f}, \mathbf{u_j})_{\mathcal{N}_{\mathbf{K}}(\boldsymbol{\Omega})} \mathbf{u_j},$$

so it is a truncation of the interpolant, using just the first m basis' element.
Since $\|u_j\|_{\ell_2^w(X)} = \sigma_j^2$, we are using a **low-rank** approximation of the kernel matrix.

Numerical example



Reconstruction of the Franke function on a *lens* with the gaussian kernel with shape parameter $\varepsilon = 2$ and with $n = 1, \dots, 900$ *trigonometric gaussian points*. Comparison between the RMSE obtained using the standard basis (blue line), the WSVD basis (red line) and the WVD basis with truncation at m s.t. $\sigma^2_{m+1} < 1e-17$ (left); approximant obtained by the truncated WSVD basis with $n = 900$ and m as above (right, false coloured by absolute error).

Tools

- general theory of **Change of basis** for RBF native spaces (introduced in [2]).
- Symmetric Nyström Method to discretize the integral operator (see e.g [1]).
- Cubature rule on Ω (see e.g. [3, 4] for the one used here).

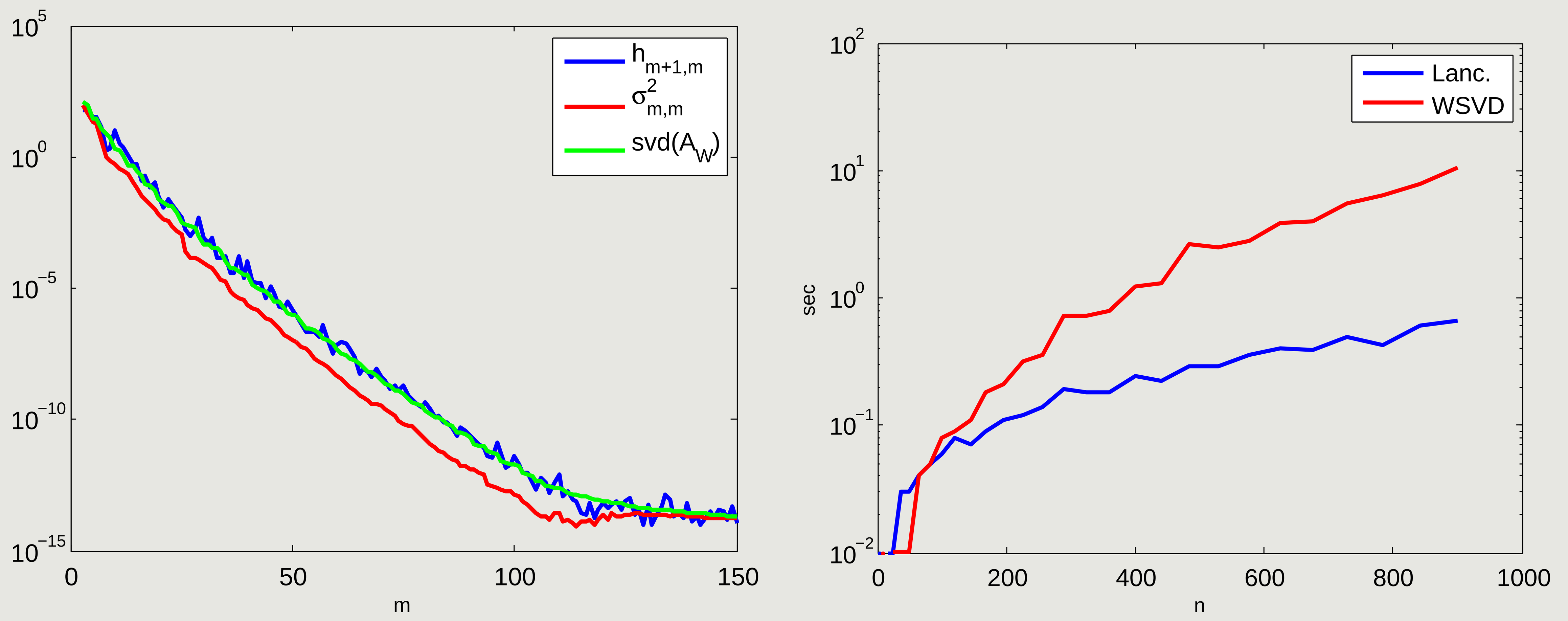
New idea

We need to compute a full SVD of A_W also if we want to use just a little part of it.
We plan to use Krylov-subspace methods, in particular the **Lanczos algorithm**:

$$A_W P_m = P_m H_m + h_{m+1} p_{m+1} e_m^T$$

where the columns of P_m are an o.n. basis of the Krylov subspace $\mathcal{K}_m(A_W, f)$, and the SVD of H_m (tridiagonal) gives a good approximation of the first m terms of the one of A_W.

Lanczos algorithm - some tests



Test in the same setting as above. Decay of the singular values of A_W and of H_m (left) and time needed to compute them (right).
From all the test made the approximation of the singular values of A_W by the one of H_m seems good and well controlled by $h_{m+1,m}$. The computational time is also promising.

References

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