Rational stable RBF-PU interpolation via VSKs

S. De Marchi, A. Martínez, E. Perracchione

Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Italia

Summary

- For univariate functions with steep gradients, the rational polynomial interpolation is particularly suitable.
- Its extension to high dimensions is still a challenging problem. Moreover, because of the dependence on meshes, it is not easy to implement for complex domains (refer e.g. to **[Hu et al.]**).
- We perform a local computation via the Partition of Unity (PU) method of rational Radial Basis Function (RBF) interpolants.
- The resulting scheme, implemented by means of the DACG (Deflation-Accelerated Conjugate Gradient) method **[Bergamaschi et al.]**, enables us to deal with huge data sets.
- Furthermore, thanks to the use of Variably Scaled Kernels (VSKs) **[Bozzini et al.; De Marchi et al.]**, it turns out to be stable.

Framework

- Let $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1,\dots,N\} \subseteq \Omega$ be a set of distinct data, $\mathcal{F}_N = \{f_i = f(\boldsymbol{x}_i), i = 1,\dots,N\}$ a set of data values and $\Phi : \Omega \times \Omega \to \mathbb{R}$ a strictly positive definite and symmetric kernel.
- Letting $\mathcal{X}_m = \{\boldsymbol{x}_i, i = k,\dots,k+m-1\} \subseteq \mathcal{X}_N$ and $\mathcal{X}_n = \{\boldsymbol{x}_i, i = j,\dots,j+n-1\} \subseteq \mathcal{X}_N$, we define a rational RBF expansion as

$$\mathscr{R}(\boldsymbol{x}) = \frac{\mathcal{R}^{(1)}(\boldsymbol{x})}{\mathcal{R}^{(2)}(\boldsymbol{x})} = \frac{\sum_{i_1 \in \mathcal{X}_m} \alpha_{i_1} \Phi(\boldsymbol{x}, \boldsymbol{x}_{i_1})}{\sum_{i_2 \in \mathcal{X}_n} \beta_{i_2} \Phi(\boldsymbol{x}, \boldsymbol{x}_{i_2})},$$

  provided $\mathcal{R}^{(2)}(\boldsymbol{x}) \neq 0$, $\boldsymbol{x} \in \Omega$.
- This problem leads to a homogeneous system $B\boldsymbol{\xi} = \boldsymbol{0}$, where $\boldsymbol{\xi} = (\boldsymbol{\alpha}, \boldsymbol{\beta})^T$, $\boldsymbol{0} \in \mathbb{R}^N$ and the entries of B are determined by the interpolation conditions.

Example. $N = 26$, $\Omega = [0,1]$, $f(x_1) = 1$. Let us fix $\mathcal{X}_m = \{\boldsymbol{x}_i, i = 1,\dots m\} \equiv \mathcal{X}_n$, with $m = n = 13$. The matrix B is singular. A possible way, illustrated in **Fig.1 (left)**, consists in computing

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^N, ||\boldsymbol{\xi}||_2 = 1} ||B\boldsymbol{\xi}||_2. \qquad (1)$$

Note that the method described by equation (1) can be used to obtain a non-trivial least square approximate solution when the matrix is non-singular, see **Fig.1 (right)**.

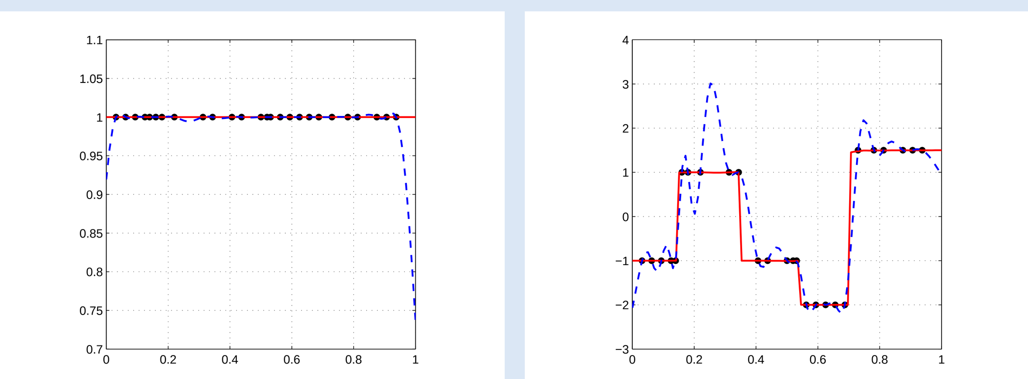

Figure 1. The black dots represent the set of scattered data, the red solid line and the blue dotted one are the curves reconstructed via the rational and classical RBF approximation.

Method

To deal with well-posed problems, we impose extra constraints and we take $\mathcal{X}_m \equiv \mathcal{X}_n \equiv \mathcal{X}_N$. If we assume to know the function values of $R^{(1)}$ and $R^{(2)}$, namely $\boldsymbol{p}$ and $\boldsymbol{q}$, to compute the rational interpolant, we need to solve $A\boldsymbol{\alpha} = \boldsymbol{p}$ and $A\boldsymbol{\beta} = \boldsymbol{q}$, where A is the standard kernel matrix.

According to **[Jakobsson et al.]**, this is equivalent to find the eigenvector $\boldsymbol{q}$ associated to the smallest eigenvalue in absolute value of the following generalized eigenvalue problem

$$\Lambda \boldsymbol{q} = \lambda \Theta \boldsymbol{q}, \quad \text{with} \quad \Lambda = 1/||\boldsymbol{f}||_2^2 D^T A^{-1} D + A^{-1} \quad \text{and} \quad \Theta = 1/||\boldsymbol{f}||_2^2 D^T D + I_N,$$

where I_N is the $N \times N$ identity matrix. Then, the only vector $\boldsymbol{p}$ such that $\mathcal{R}^{(1)}(\boldsymbol{x}_k) = f_k \mathcal{R}^{(2)}(\boldsymbol{x}_k)$, $k = 1,\dots,N$, is given by $\boldsymbol{p} = D\boldsymbol{q}$, where $D = \mathrm{diag}(f_1,\dots,f_N)$.

We define the rational PU interpolant which makes use of VSKs (RVSK-PU) as **[Wendland]**

$$\mathcal{I}_{\psi}(\boldsymbol{x}) = \sum_{j=1}^{d} \mathscr{R}_{\psi_j}(\boldsymbol{x}) W_j(\boldsymbol{x}), \quad \mathscr{R}_{\psi_j}(\boldsymbol{x}) = \frac{\sum_{i=1}^{N_j} \alpha_i^j \mathcal{K}((\boldsymbol{x}, \psi_j(\boldsymbol{x})), (\boldsymbol{x}_i^j, \psi_j(\boldsymbol{x}_i^j)))}{\sum_{k=1}^{N_j} \beta_k^j \mathcal{K}((\boldsymbol{x}, \psi_j(\boldsymbol{x})), (\boldsymbol{x}_k^j, \psi_j(\boldsymbol{x}_k^j)))},$$

where $\mathcal{K}$ is a VSK, $\psi_j : \Omega_j \mapsto \mathbb{R}$ is a scale function and W_j are the Shepard's weights.

Since for VSKs the analysis of the error on $\mathbb{R}^M$ coincides with the analysis of a fixed-scale problem on a submanifold in $\mathbb{R}^{M+1}$, in the following proposition we consider a rational PU interpolant $\mathcal{I}$ computed with a kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$.

Proposition. Suppose $\Phi \in C_k^{\nu}(\mathbb{R}^M)$ is strictly positive definite. Let $\{\Omega_j\}_{j=1}^d$ be a regular covering for $(\Omega, \mathcal{X}_N)$ and let $\{W_j\}_{j=1}^d$ be 0-stable for $\{\Omega_j\}_{j=1}^d$. Then, there exists an index $t \in \{1,\dots,d\}$ and a constant C (independent of the fill distance $h_{\mathcal{X}_N}$) such that for all $\boldsymbol{x} \in \Omega$:

$$|f(\boldsymbol{x}) - \mathcal{I}(\boldsymbol{x})| \leq 1/|\mathcal{R}_t^{(2)}(\boldsymbol{x})| C h_{\mathcal{X}_N}^{(k+\nu)/2} (|f(x)| ||q_t||_{\mathcal{N}_{\Phi}(\Omega_t)} + ||p_t||_{\mathcal{N}_{\Phi}(\Omega_t)}).$$

Proof sketch. Use **[Wendland, Th. 10.47, p. 170]** to the local setting and observe that

$$f_{|\Omega_j}(\boldsymbol{x}) - \mathscr{R}_j(\boldsymbol{x}) = [(\mathcal{R}_j^{(2)}(\boldsymbol{x}) f(\boldsymbol{x}) - q_j(\boldsymbol{x}) f(\boldsymbol{x})) + (q_j(\boldsymbol{x}) f(\boldsymbol{x}) - \mathcal{R}_j^{(1)}(\boldsymbol{x}))]/\mathcal{R}_j^{(2)}(\boldsymbol{x}).$$

Results

The results of using RVSK-PU and standard PU (SRBF-PU) with the Gaussian C^{∞} kernel as local approximant are shown in **Tab.1** and **Fig.2**.

N	h	Method	ε	RMSE	MCN	λ	CPU
4225	2.91E − 02	SRBF-PU	3	5.48E − 04	2.41E + 21	14.3	1.33
		RVSK-PU	–	6.04E − 06	7.61E + 21	10.4	1.39
16641	1.03E − 02	SRBF-PU	4	3.46E − 05	2.75E + 23	2.65	2.08
		RVSK-PU	–	6.97E − 07	4.67E + 21	2.07	2.85
66049	4.93E − 03	SRBF-PU	5	1.22E − 06	5.60E + 22	4.53	9.31
		RVSK-PU	–	2.76E − 07	3.09E + 22	1.25	11.0

Table 1. Fill distances (h), Root Mean Square Errors (RMSEs), Maximum Condition Numbers (MCNs), convergence rates (λ) and CPU times computed on Halton points. Results are obtained using $f_1(x_1, x_2) = \tan[9(x_2 - x_1) + 1]/(\tan 9 + 1)$.

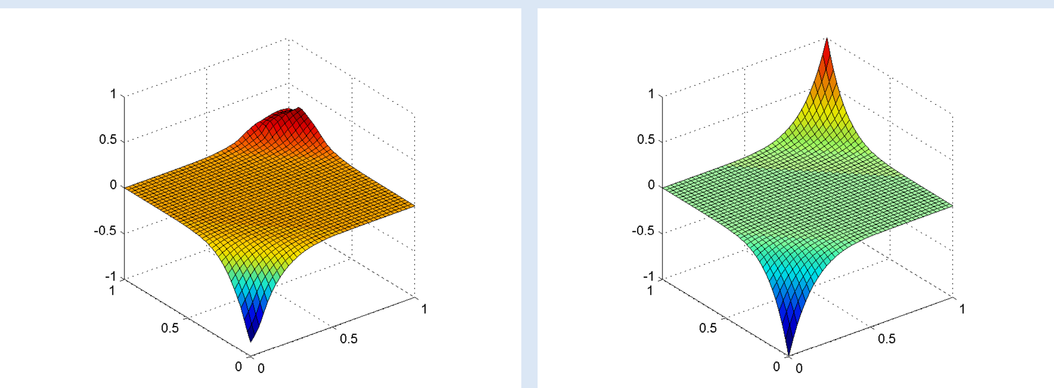

Figure 2. The approximate surface $f_2(x_1, x_2) = (x_1 + x_2 - 1)^7$ obtained with the SRBF-PU (left) and RVSK-PU (right) methods with $N = 1089$ Non-Conformal points. The experiments have been carried out with the MATLAB software.

References

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