Stability of Kernel–Based Interpolation: examples

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In this file, we collect all examples we ran in order to illustrate the results proved in [1]. Hence, we invite interested readers to refer to the paper [1] for better understanding the content of this note.

Figure 1 shows the values Λ_X of the Lebesgue constants for the Sobolev/Matern kernel $(r/c)^\nu K_\nu(r/c)$ for $\nu = 1.5$ at scale $c = 20$. In this and other examples for kernels with finite smoothness, one can see that our bounds on the Lebesgue constants are valid, but the experimental Lebesgue constants seem to be uniformly bounded. In all cases, the maximum of the Lebesgue function is attained in the interior of the domain.

Things are different for infinitely smooth kernels. Figure 2 shows the behavior for the Gaussian. The maximum of the Lebesgue function is attained near the corners for large scales, while the behavior in the interior is as stable as for kernels with limited smoothness. The Lebesgue constants do not seem to be uniformly bounded.

A second series of examples was run on 225 regular points in $[-1, 1]^2$ for different kernels at different scales using a parameter c as $\Phi_c(x) = \Phi(x/c)$.

Figures 3 to 5 show how the scaling of the Gaussian kernel influences the shape of the associated Lagrange basis functions. The limit for large scales is called the *flat limit* [3] which is a Lagrange basis function of the de Boor/Ron polynomial interpolation [4]. It cannot be expected that such Lagrange basis functions are uniformly bounded.

In contrast to this, Figure 6 shows the corresponding Lagrange basis function for the Sobolev/Matern kernel at scale 320. The scales were such that the conditions of the kernel matrices were unfeasible for the double scale. Figure 7 shows the Lebesgue function in the situation of Figure 5, while Figure 8 shows the Sobolev/Matern case in the situation of Figure 6.

Figures 9 and 10 show how the same Sobolev kernel behaves on scattered data given in Figure 11. The encircled point is where the Lagrange function is taken for Figure 9. Note that the situation does not change dramatically when scattered data are used.

We also checked if the large errors in the corners of the domain in Figure 7 disappeared for

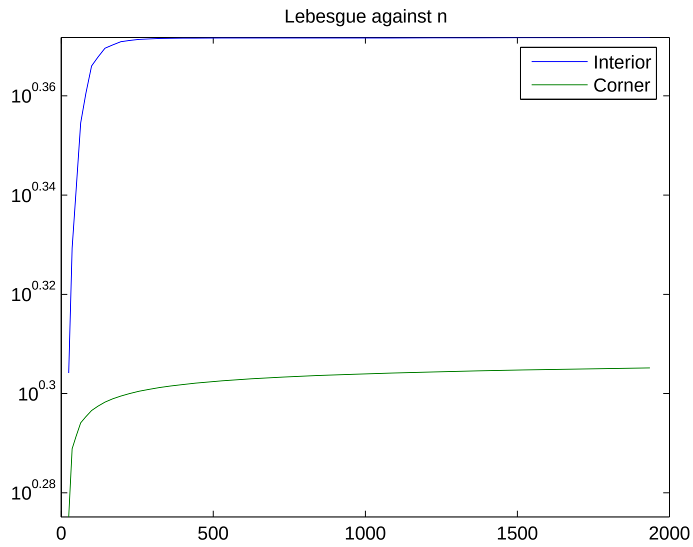

Figure 1: Lebesgue constants for the Sobolev/Matern kernel

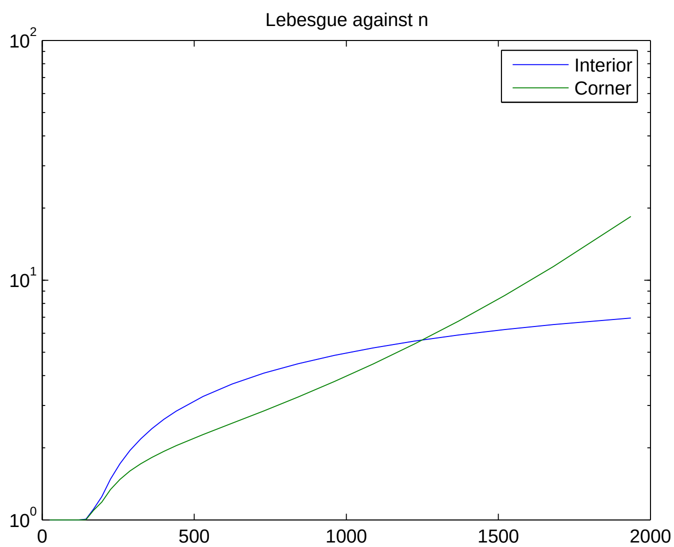

Figure 2: Lebesgue constants for the Gauss kernel

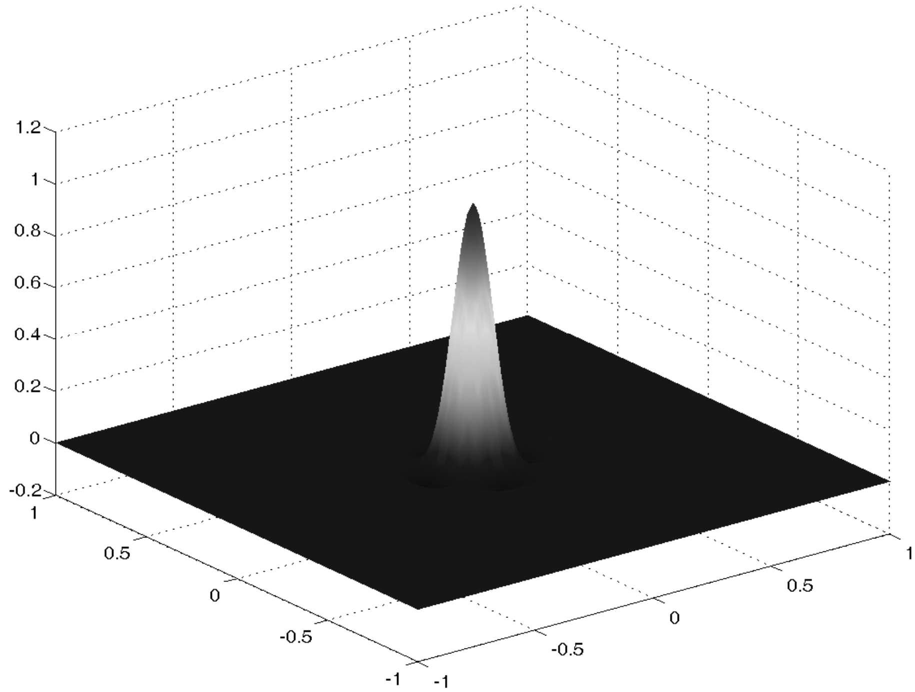

Figure 3: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.1

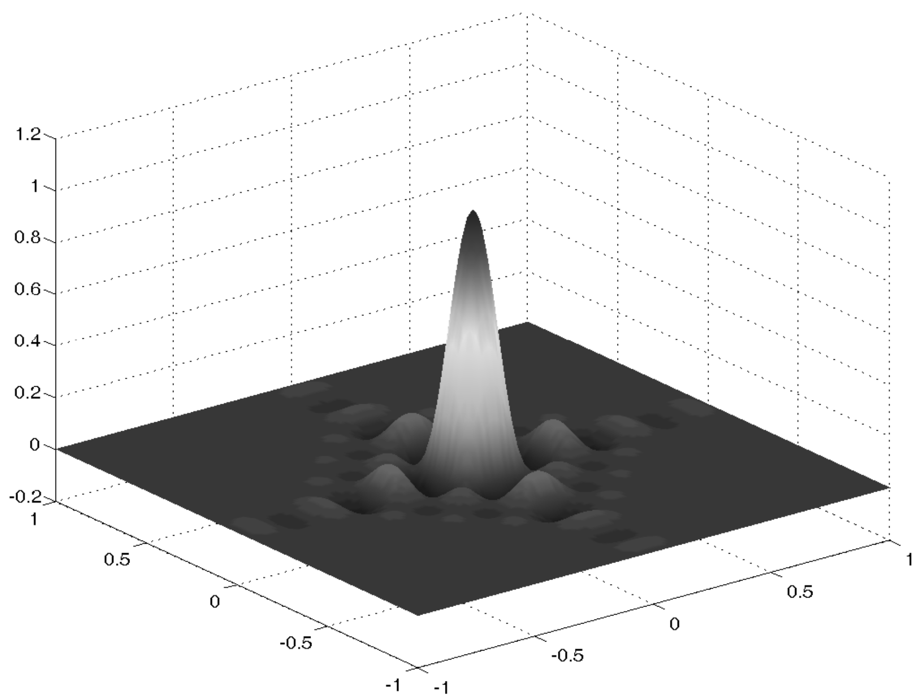

Figure 4: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.2

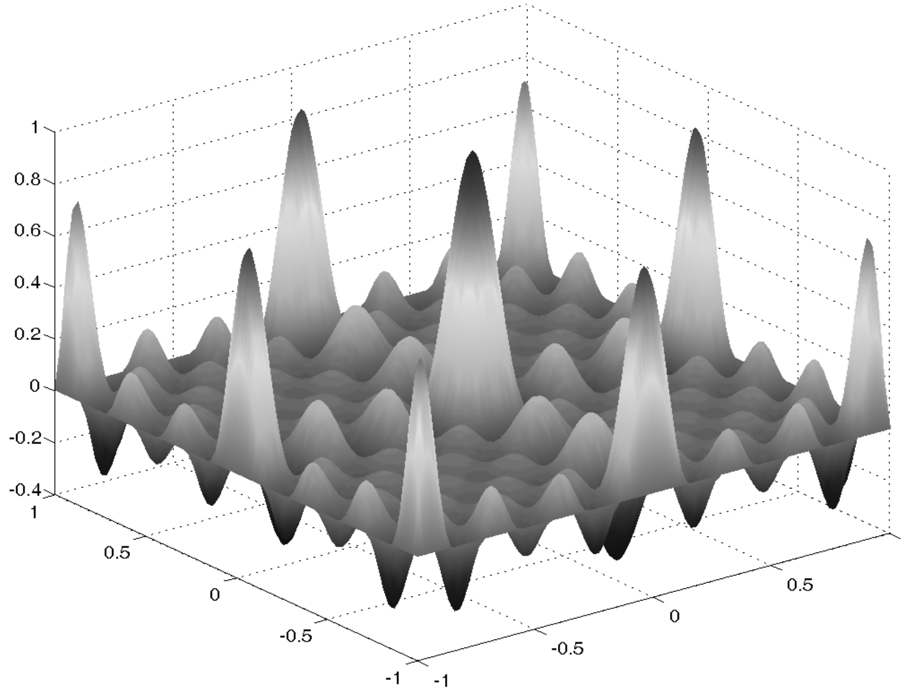

Figure 5: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.4

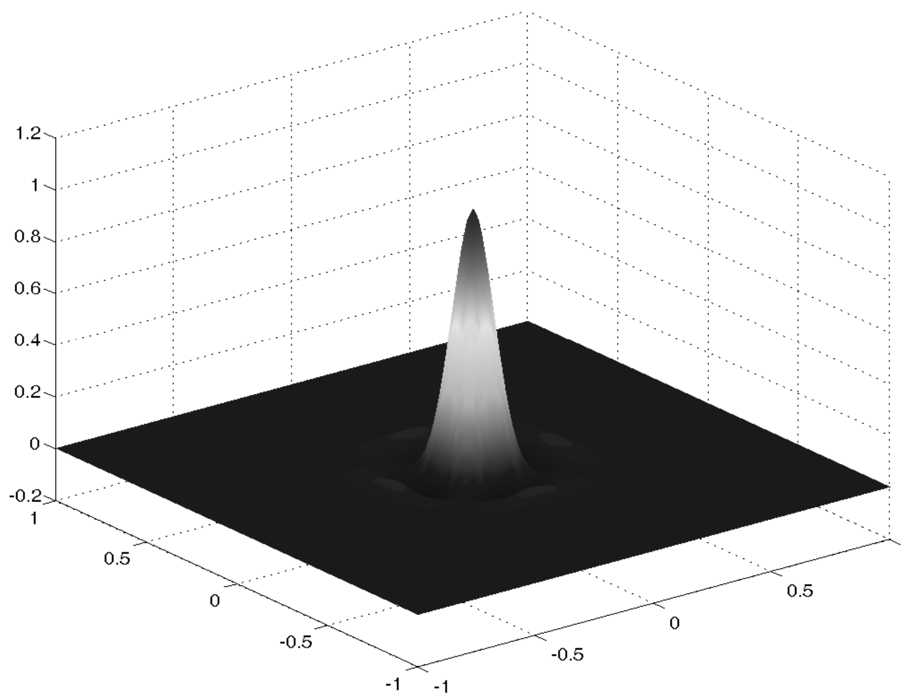

Figure 6: Lagrange basis function on 225 data points, Sobolev/Matern kernel with scale 320

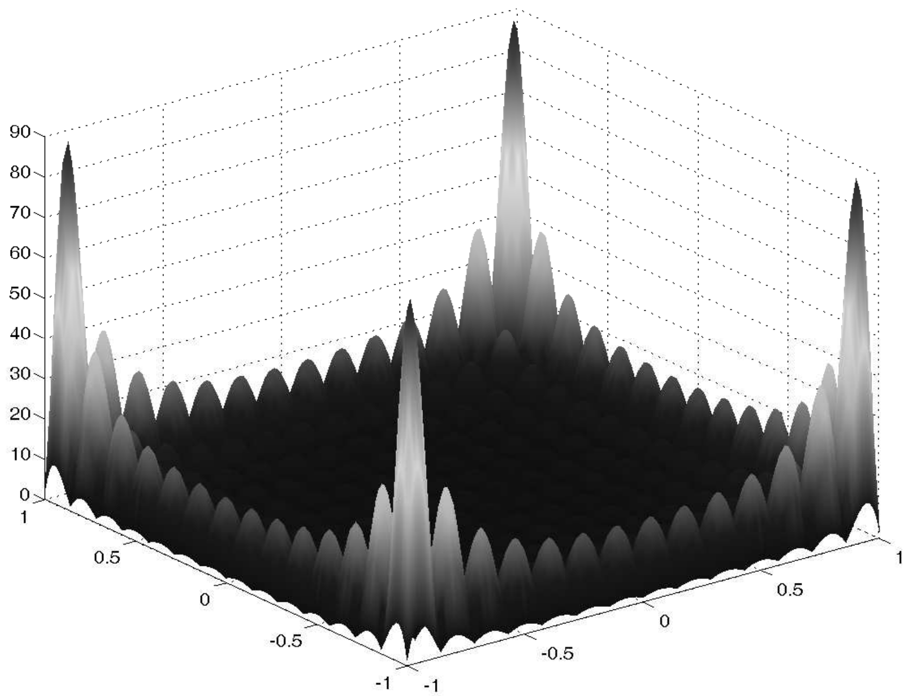

Figure 7: Lebesgue function on 225 regular data points, Gaussian kernel with scale 0.4

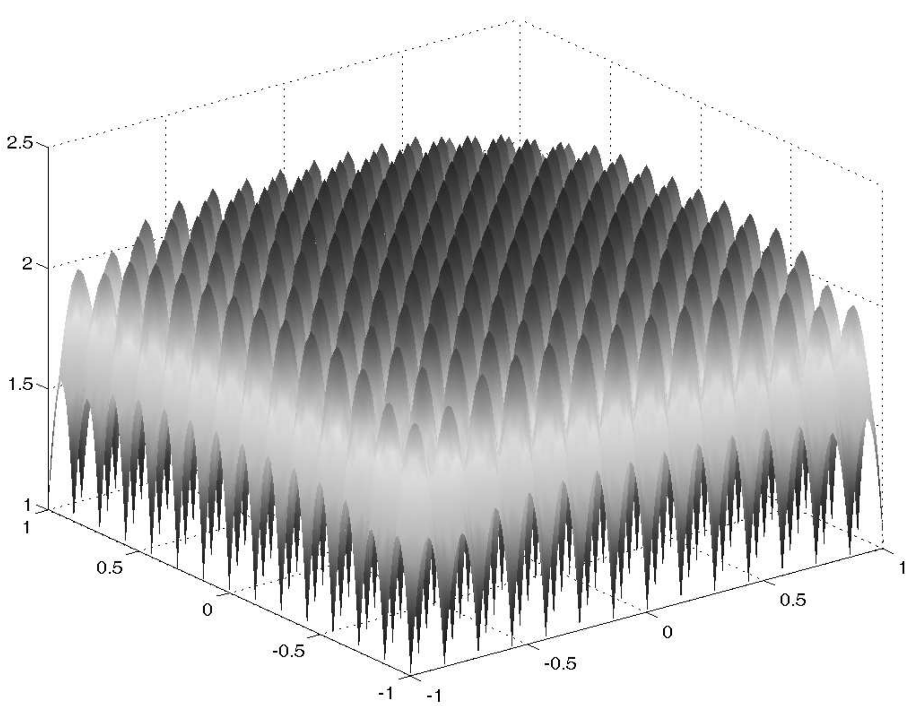

Figure 8: Lebesgue function on 225 regular data points, Sobolev/Matern kernel with scale 320

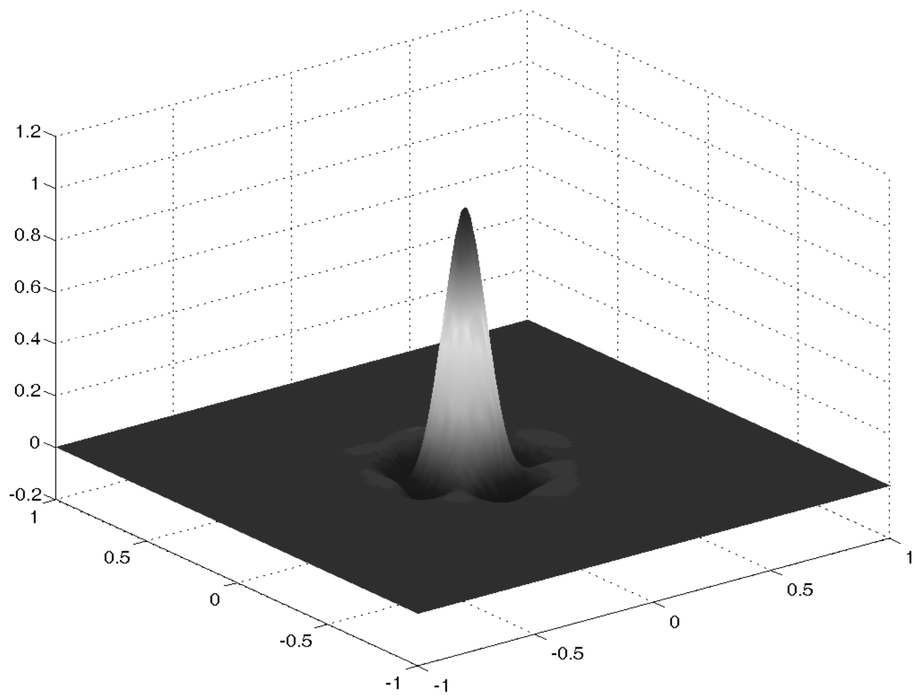

Figure 9: Lagrange basis function on 225 scattered data points, Sobolev/Matern kernel with scale 320

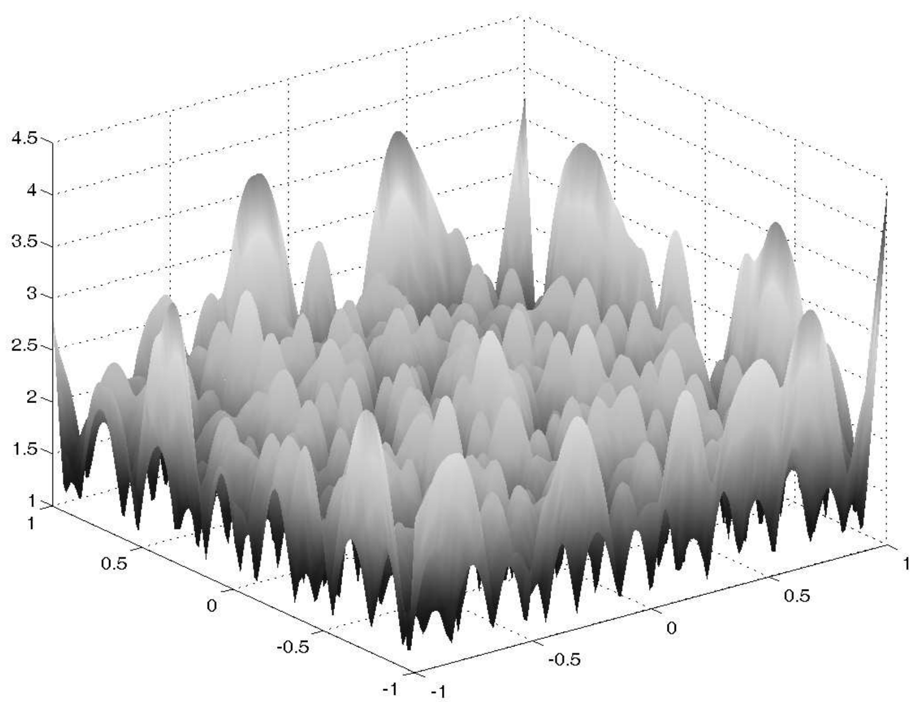

Figure 10: Lebesgue function on 225 scattered data points, Sobolev/Matern kernel with scale 320

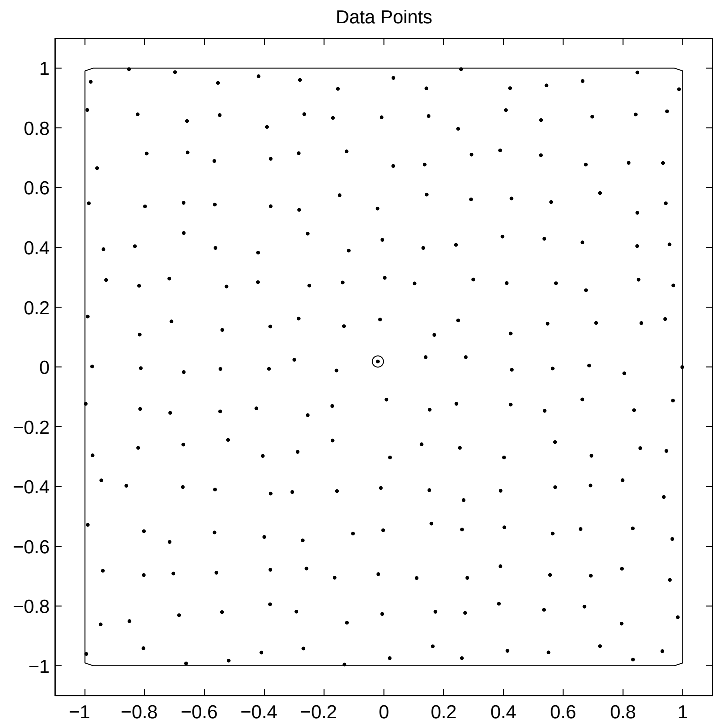

Figure 11: Data points for Figures 9 and 10

domains without corners. Figures 12 and 13 show how the same Gaussian behaves on scattered data on the circle given by Figure 14. It turns out that the boundary behavior is even more dramatic here, since there are no data points on the boundary.

Using Gaussians, with other dilations, did not improve the situation. New results of a forthcoming Ph.D. thesis by Christian Rieger of the University of Göttingen, suggest that an $\mathcal{O}(h^2)$ oversampling in a strip close to the boundary should have a positive effect. To check this indirectly, we used the greedy method of [2] to determine good interpolation points by iteratively adding maxima of the power function. Figures 15 and 16 show the dramatic improvement, while the points are now distributed as in Figure 17. One could also choose new data points via the maximum of the Lebesgue function, but this strategy turned out to be inferior.

We also ran some other examples on a cardioidal domain with an incoming cusp, but the results were not much different.

The improvement by oversampling on the boundary seems to be connected to analytic kernels, since the corresponding examples for non–smooth kernels showed a much weaker effect. We add figures for the C^2 Wendland function, but we remark that Matern/Sobolev kernels behave similarly. Note that all functions are chopped at the boundary of the cardioid.

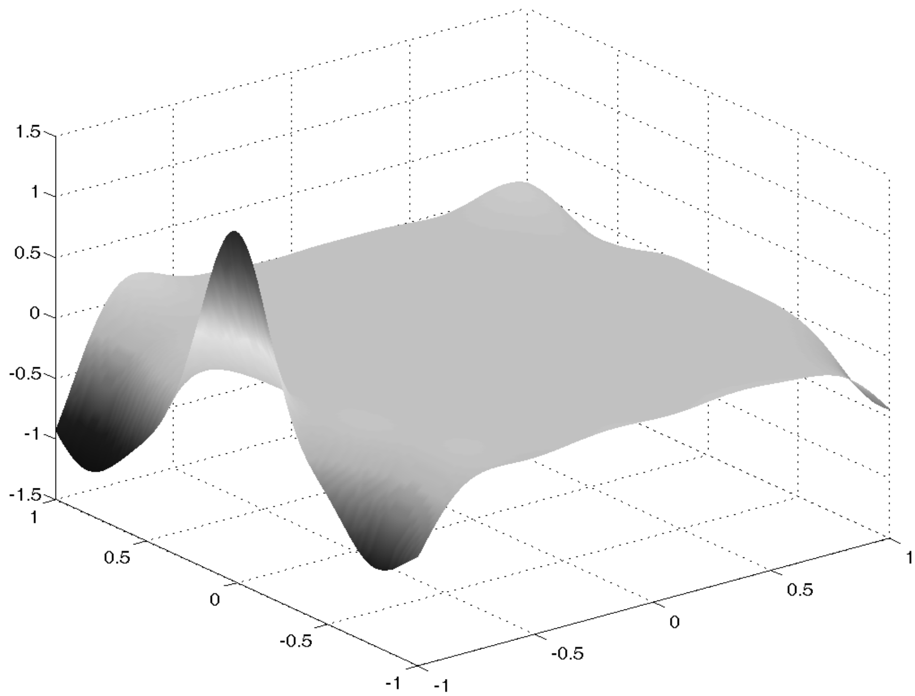

Figure 12: Lagrange basis function for 168 scattered data points on the circle, Gaussian kernel with scale 0.4

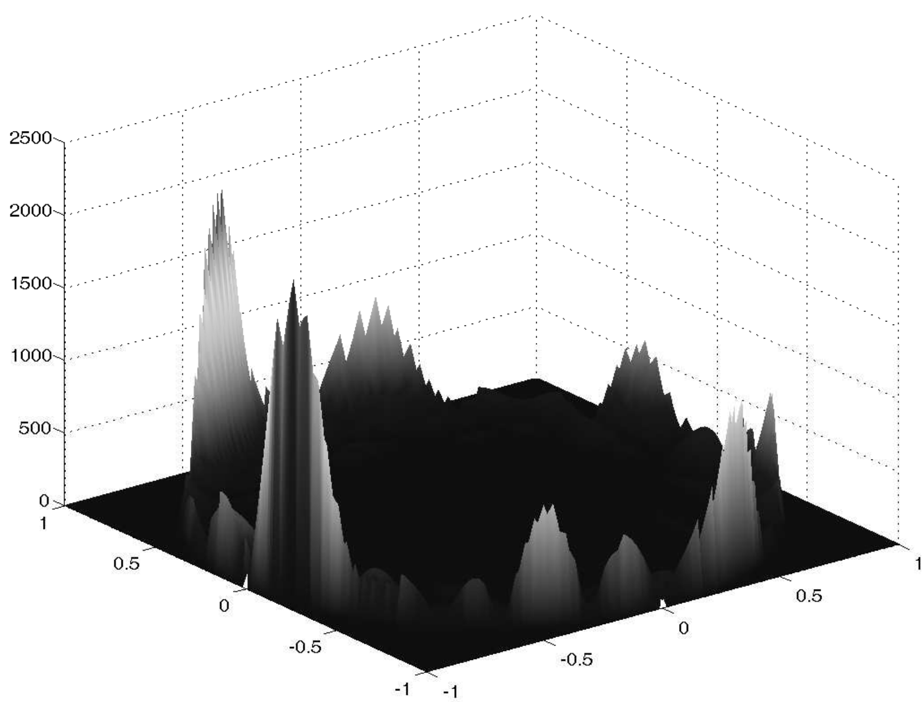

Figure 13: Lebesgue function for 168 scattered data points on the circle, Gaussian kernel with scale 0.4

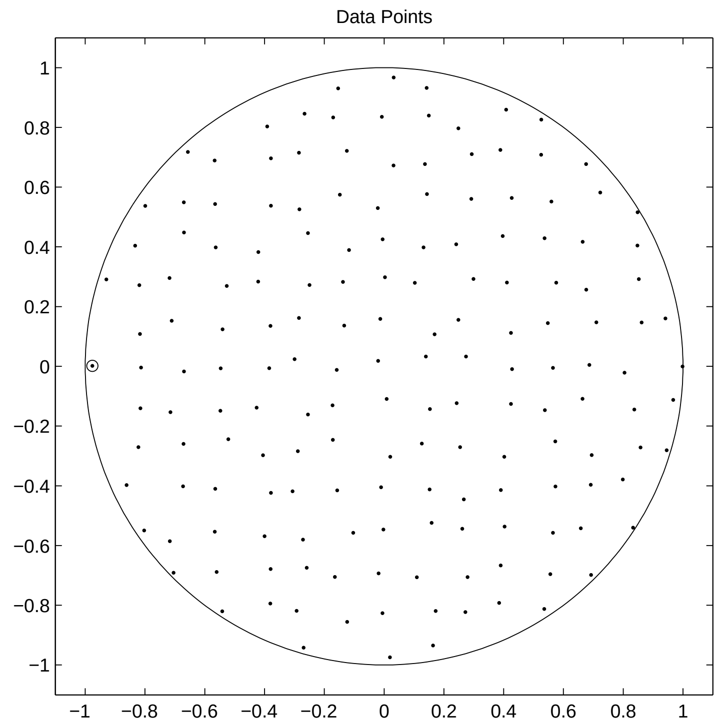

Figure 14: Data points for Figures 12 and 13

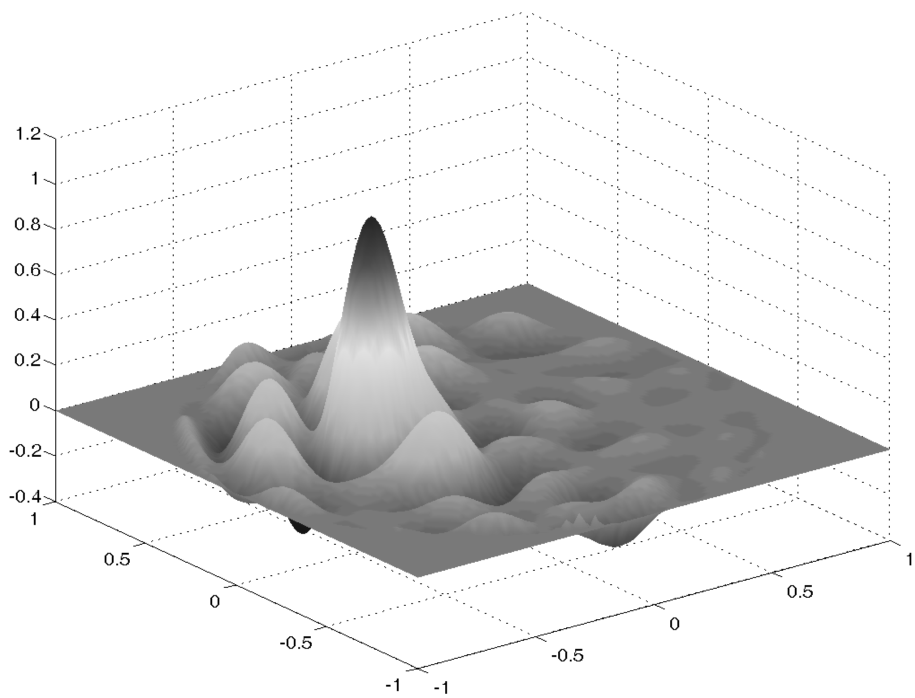

Figure 15: Lagrange basis function for 168 optimized data points on the circle, Gaussian kernel with scale 0.4

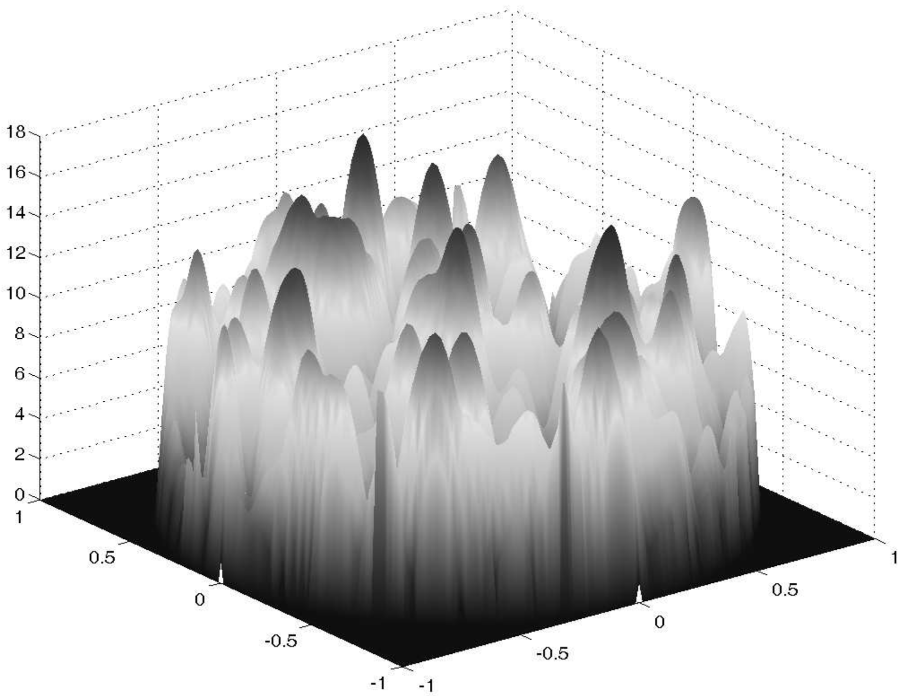

Figure 16: Lebesgue function for 168 optimized data points on the circle, Gaussian kernel with scale 0.4

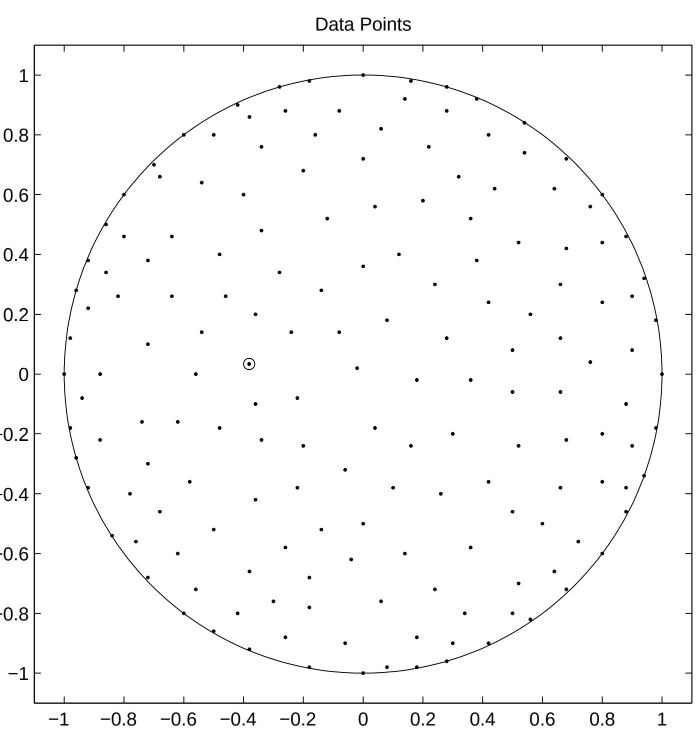

Figure 17: Data points for Figures 15 and 16

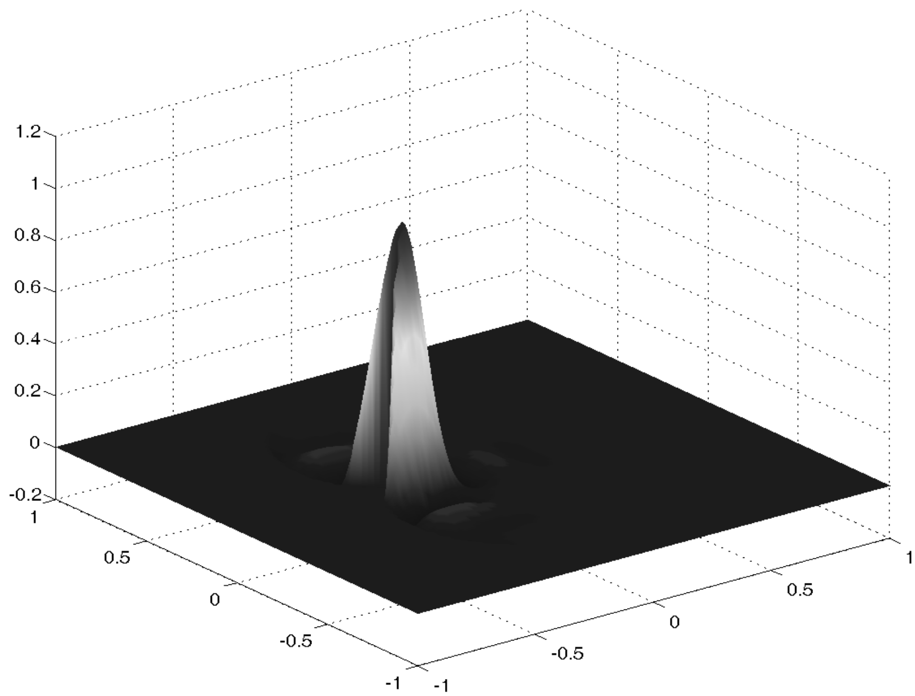

Figure 18: Lagrange basis function for 104 scattered data points on the cardioid, Wendland C^2 kernel with scale 30

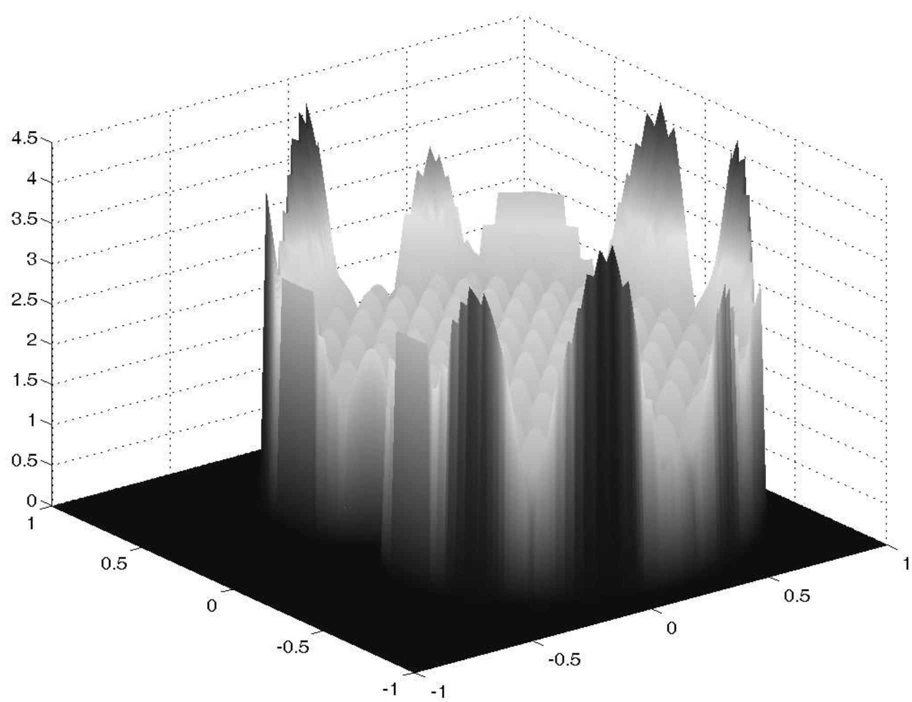

Figure 19: Lebesgue function for 104 scattered data points on the cardioid, Wendland C^2 kernel with scale 30

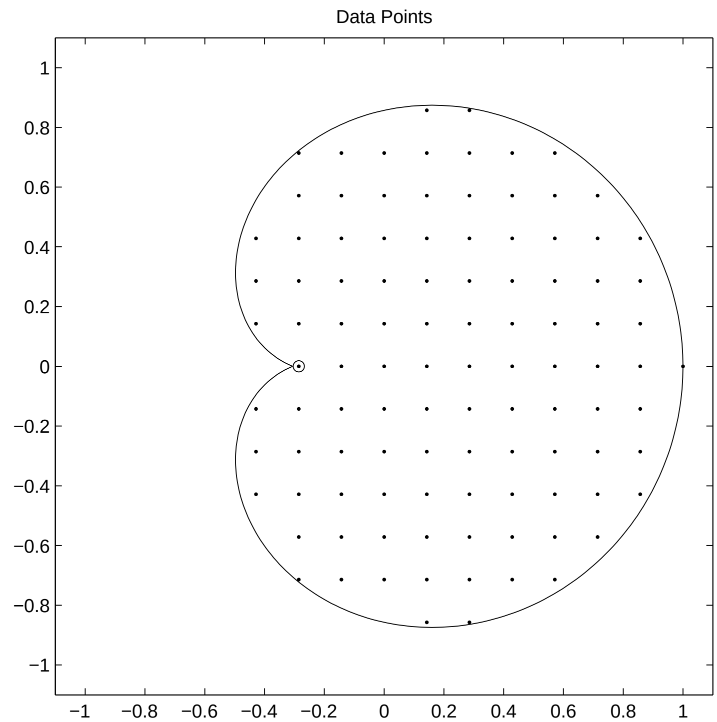

Figure 20: Data points for Figures 18 and 19

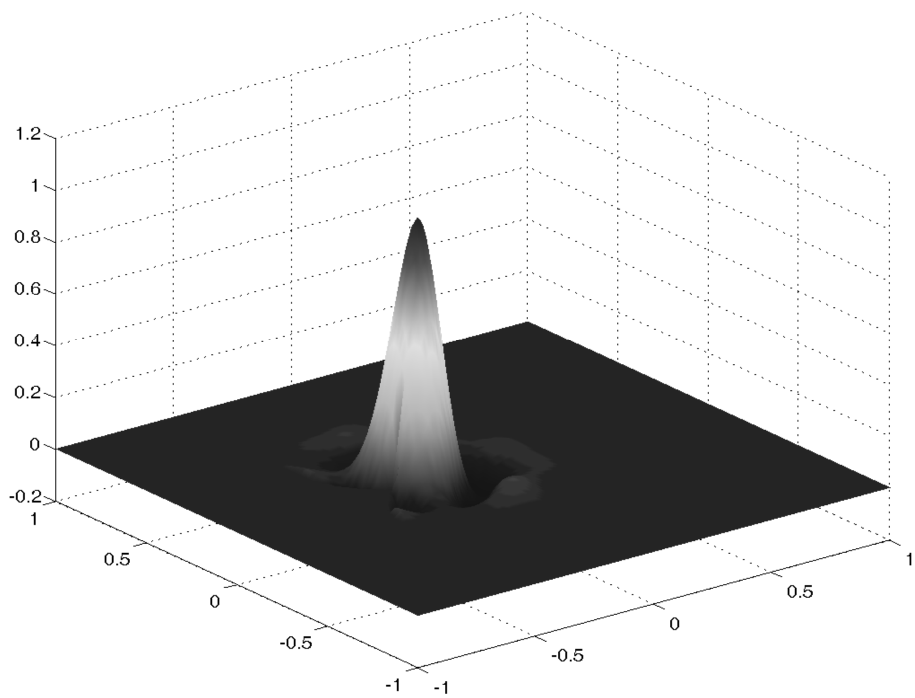

Figure 21: Lagrange basis function for 104 optimized data points on the cardioid, Wendland C^2 kernel with scale 30

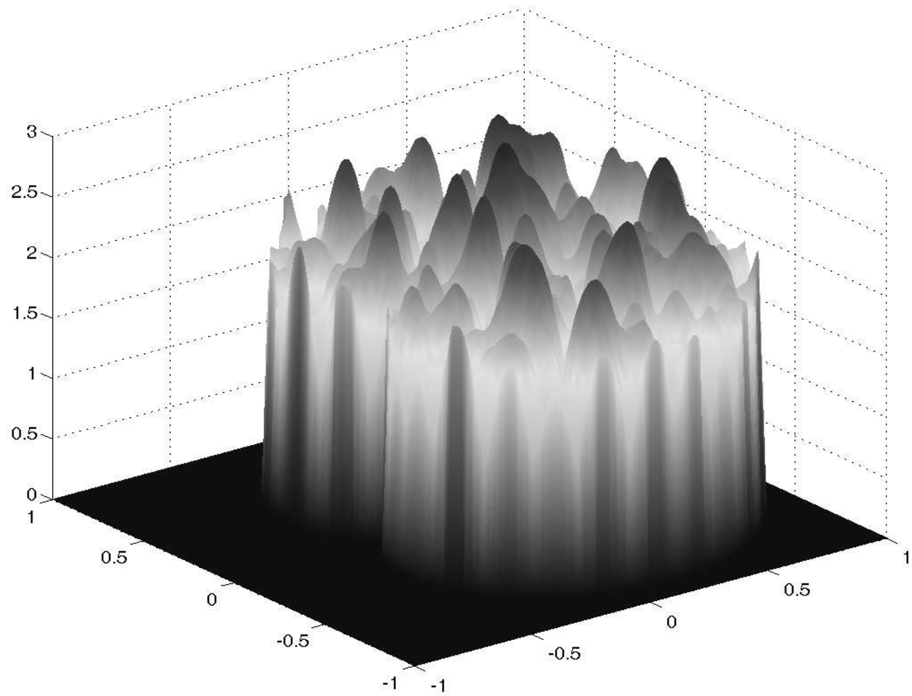

Figure 22: Lebesgue function for 104 optimized data points on the cardioid, Wendland C^2 kernel with scale 30

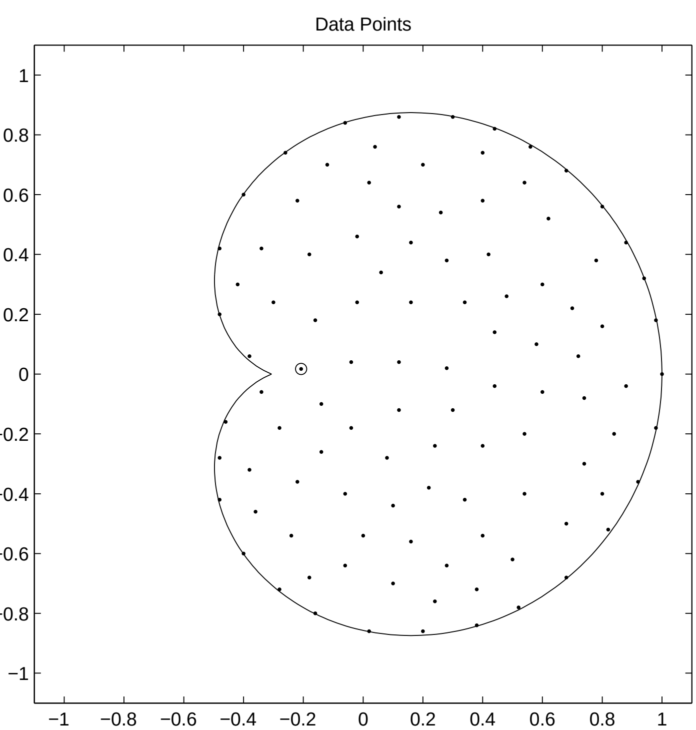

Figure 23: Data points for Figures 21 and 22

References

[1] De Marchi, S. and Schaback, R. *Stability of kernel-based interpolation*, Adv. Comput. Math. 2008, under revision. A draft is available at `http://profs.sci.univr.it/∼demarchi/papers/SoKBI-revised1.pdf`.

[2] De Marchi, S., Schaback, R. and Wendland, H. *Near-Optimal Data-independent Point Locations for Radial Basis Function Interpolation*, Adv. Comput. Math., Vol. 23(3) (2005), 317-330.

[3] Driscoll, T.A. and Fornberg, B. *Interpolation in the limit of increasingly flat radial basis functions*, Comput. Math. Appl., Vol. 43 (2002), 413–422.

[4] R. Schaback, *Multivariate interpolation by polynomials and radial basis functions*, Constr. Approx. **21** (2005), 293–317.