

Local bases for kernel spaces on manifolds

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joint work with:

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Computing with kernels on manifolds

- A d dimensional compact Riemannian manifold $\mathbb{M}$.
- A continuous, positive definite kernel $k : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$.
 - for any Ξ ,

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is positive definite.

- A set of centers $\Xi \subset \mathbb{M}$.

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Examples:

- RBFs on $\mathbb{M} = \mathbb{R}^d$, e.g. $k(x, \alpha) = e^{-|x-\alpha|^2}$,
- SBFs on $\mathbb{M} = \mathbb{S}^d$, e.g. $k(x, \alpha) = e^{-x \cdot \alpha}$.

Positive definiteness:

- $(k(\cdot, \xi))_{\xi \in \Xi}$ forms a basis for

$$S(\Xi) := \text{span}_{\xi \in \Xi} k(\cdot, \xi)$$

- **Interpolation** at centers: There is a unique $l_{\Xi} f \in S(\Xi)$:

$$l_{\Xi} f(\xi) = f(\xi), \quad \forall \xi \in \Xi$$

- There is a Hilbert space $\mathcal{H}_k$ with reproducing kernel k for which l_{Ξ} is the orthogonal projector onto $S(\Xi)$:

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Challenges:

- Collocation matrix $(k(\zeta, \xi))_{\zeta, \xi \in \Xi}$ becomes full as $\#\Xi \rightarrow \infty$
- Basis $(k(\cdot, \xi))_{\xi \in \Xi}$ becomes ill-conditioned.

1. Find a 'robust' basis $(B_\xi)_{\xi \in \Xi}$ for $S(\Xi)$

- **Localness:** $|B_\xi(x)|$ decays rapidly as $\text{dist}(x, \xi)$ grows
- **L_p Condition:** $\left\| \sum_{\xi \in \Xi} a_\xi B_\xi \right\|_p \sim \|a\|_{\ell_p(\Xi)}.$

2. Establish stable approximation schemes

- Bound $\|I_\Xi\|_{\infty \rightarrow \infty}$ independently of $\#\Xi$.
- Bound the L_2 projector on L_p , $1 \leq p \leq \infty$.

3. Do this in a computationally feasible way

- Find kernels for which 1&2 hold with nice, closed form representations.
- B_ξ computable in $\mathcal{O}(1)$ or $\mathcal{O}(\log N)$ time.

Organization

① Sobolev kernels

- Reproducing kernels for Sobolev spaces
- Treats all compact Riemannian manifolds

② Kernels of polyharmonic and related type

- Conditionally positive definite kernels
- Theory works for compact 2 point homogeneous spaces
- Includes restricted surface splines on $\mathbb{S}^d$ and surface splines on $SO(3)$

③ Computable bases

- Covariant derivatives: $\nabla^k f(x)$ is an order k tensor
- Inner product $\langle \nabla^k f, \nabla^k g \rangle_x$ is independent of coordinates.
- For $\Omega \subset \mathbb{M}$, Sobolev space $W_2^m(\Omega)$:

$$\langle f, g \rangle_{W_2^m(\Omega)} := \sum_{k=0}^m \int_{\Omega} \langle \nabla^k f, \nabla^k g \rangle_x \, dx.$$

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- Strong equivalence via exp. maps: For $\Omega \subset B(q, \frac{r_{\mathbb{M}}}{2})$,
 $\text{Exp}_q : B(0, \frac{r_{\mathbb{M}}}{2}) \subset \mathbb{R}^d \rightarrow B(q, \frac{r_{\mathbb{M}}}{2})$

$$\|f\|_{W_2^m(\Omega)} \sim \|f \circ \text{Exp}_q\|_{W_2^m(\text{Exp}_q^{-1}(\Omega))}$$

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Sobolev kernel ($m > \frac{d}{2}$):

κ_m : the **reproducing kernel** for $\mathcal{H}_{\kappa_m} = W_2^m(\mathbb{M})$
 & the fundamental solution to $\mathcal{L}_m = \sum_{k=0}^m (\nabla^k)^* \nabla^k$

Zeros lemma (H – Narcowich – Ward, '11)

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- Let $X \subset \Omega$ have mesh norm $h := \max_{x \in \Omega} \text{dist}(x, X)$ satisfying

$$h \leq h_0 R, \quad h_0 := h_0(m, \mathbb{M}, \varphi).$$

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If $u \in W_2^m(\Omega)$ satisfies $u|_X = 0$, for $k \leq m$

$$\|u\|_{W_2^k(\Omega)} \leq C_{m,\varphi,\mathbb{M}} h^{m-k} \|u\|_{W_2^m(\Omega)}$$

and (for $m > d/2$)

$$\|u\|_{L_\infty(\Omega)} \leq C_{m,\varphi,\mathbb{M}} h^{m-d/2} \|u\|_{W_2^m(\Omega)}.$$

Basis:

Lagrange functions $(\chi_\xi)_{\xi \in \Xi}$.

- for $\xi \in \Xi$, $\chi_\xi \in \mathcal{S}(\Xi)$,
- $\chi_\xi(\xi) = 1$
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Energy estimate

For $T < r_{\mathbb{M}}/2$

$$\begin{aligned} & \|\chi_\xi\|_{W_2^m(\mathbb{M} \setminus B(\xi, T))} \\ & \lesssim e^{-\nu(\frac{T}{h})} \|\chi_\xi\|_{W_2^m(\mathbb{M})} \end{aligned}$$

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$q = \frac{1}{2} \min_{\xi \neq \zeta} (\text{dist}(\xi, \zeta))$ is the separation distance

- ① Lagrange basis is local (H-N-W, '10):

$$|\chi_\xi(x)| \leq C \exp \left[-\nu \left(\frac{\min[\text{dist}(\xi, x), r_{\mathbb{M}}]}{h} \right) \right].$$

- ② Lebesgue constant is bounded (H-N-W, '10):

$$\mathcal{L}_\Xi := \max_{x \in \mathbb{M}} \sum_{\xi \in \Xi} |\chi_\xi(x)| \leq C.$$

- ③ The Lagrange basis is stable (H-N-Sun-W, '11):

$$c_1 q^{d/p} \|a\|_{\ell_p} \leq \left\| \sum_{\xi \in \Xi} a_\xi \chi_\xi \right\|_p \leq c_2 q^{d/p} \|a\|_{\ell_p}$$

- ④ L_2 projector $T_\Xi : L_p(\mathbb{M}) \rightarrow S(\Xi)$, $\ker T_\Xi \perp S(\Xi)$ is bounded on each L_p , $1 \leq p \leq \infty$ (H-N-Sun-W, '11):

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Constants do not depend on $\#\Xi$, only on $m, \mathbb{M}$ and $\rho := \frac{h}{q}$!!!

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Interpolation I_Ξ is near-best approximation:

$$\|I_\Xi f - f\|_\infty \leq (1 + C) \text{dist}_\infty(f, S(\Xi)).$$

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T_Ξ is **universal** near-best approximation:

$$\|T_\Xi f - f\|_p \leq (1 + C) \text{dist}_p(f, S(\Xi)).$$

Problems:

- Sobolev kernels $\kappa_{m,\mathbb{M}}$ lack **direct** representations.
- Exception: $\mathbb{M} = \mathbb{R}^d$ – these are Matérn kernels
- It is hard to establish good approximation results for $\kappa_{m,\mathbb{M}}$:
how rapidly does $\text{dist}_p(f, S(\Xi))$ decay?
- Exception $\mathbb{M} = \mathbb{S}^d$: $\kappa_{m,\mathbb{S}^d}$ treated by Mhaskar, Narcowich, Prestin, Ward '10.
- How big is ν ?

Kernels $k : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$ of polyharmonic and related type

- Defined indirectly via an operator $\mathcal{L}_m = \sum_{j=0}^m a_j \Delta^j = Q(\Delta)$ and a reproducing formula:

$$f(x) = \int_{\mathbb{M}} [\mathcal{L}_m f](\alpha) k(x, \alpha) d\alpha$$

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- $S(\Xi, \Pi) := \Pi + \{\sum a_\xi k(\cdot, \xi) \mid \sum_{\xi \in \Xi} a_\xi \psi(\xi) = 0\}$
- Reproducing kernel semi-Hilbert space

$$\mathcal{H}_k := \{f = \sum_{j=0}^{\infty} \hat{f}_j \phi_j \mid \sum_{j \notin \mathcal{J}} \frac{|\hat{f}_j|^2}{\sqrt{Q(\lambda_j)}} < \infty\}$$

- Includes restricted surface splines on $\mathbb{S}^d$: $k(x, \alpha) = \phi(x \cdot \alpha)$

$$\phi(t) = \begin{cases} (1-t)^{m-d/2} & \text{for } d \text{ odd} \\ (1-t)^{m-d/2} \log(1-t) & \text{for } d \text{ even} \end{cases}$$

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- For $f \in B_{p,q}^s(\mathbb{S}^d), W_p^s(\mathbb{S}^d), C^s(\mathbb{S}^d), s \leq 2m$

$$\text{dist}_p(f, S(\Xi, \Pi)) = \mathcal{O}(h^s).$$

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- Surface splines on $SO(3)$: $k(x, \alpha) = \phi(\omega(\alpha^{-1}x))$

$$\phi(t) = (\sin(t/2))^{m-3/2}$$

For $f \in B_{p,q}^s(SO(3)), W_p^s(SO(3)), C^s(SO(3)), s \leq 2m$

$$\text{dist}_p(f, S(\Xi, \Pi)) = \mathcal{O}(h^s).$$

(H-Schmid '11)

- On two point homogeneous spaces,

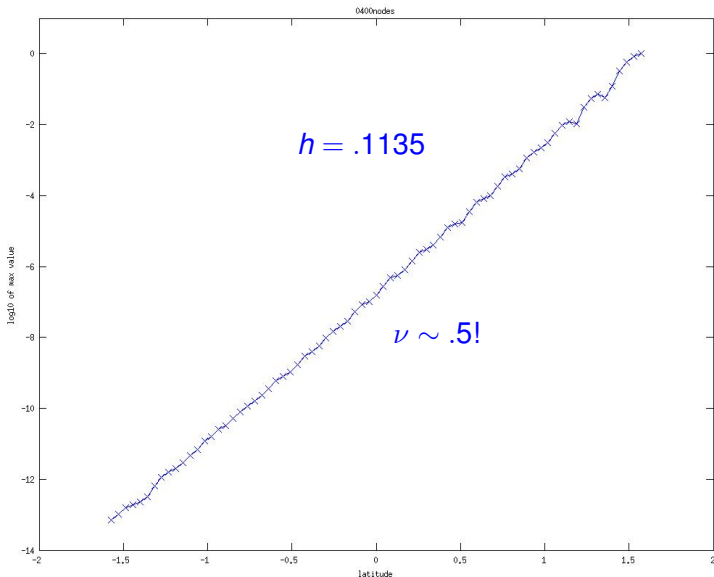
$$\mathcal{L}_m = \sum_{j=0}^m a_j \Delta^j = \sum_{j=0}^m \tilde{a}_j (\nabla^j)^* \nabla^j$$

- If $\mathcal{L}_m \Pi = \{0\}$, then Lagrange functions have exponential decay as with Sobolev kernels.
- If $\mathcal{L}_m \Pi \neq \{0\}$, then Lagrange functions have slower decay:

$$|\chi_\xi(x)| \leq C \max \left(\exp \left[-\nu \left(\frac{\text{dist}(\xi, x)}{h} \right) \right], h^{2m} \right).$$

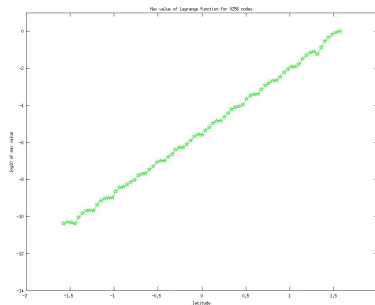
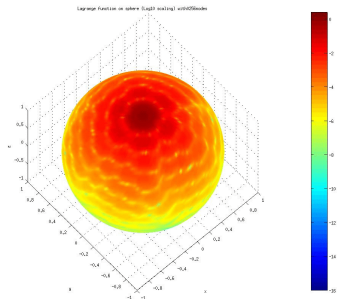
- In either case, the other results (bounded Lebesgue constant, L_p stability, boundedness of L_2 projector on L_p) all hold

$\Xi = 400$ 'minimum energy' nodes on $\mathbb{S}^2$, $\phi(t) = (1-t)\log(1-t)$
 the maximum value of χ over each latitude...

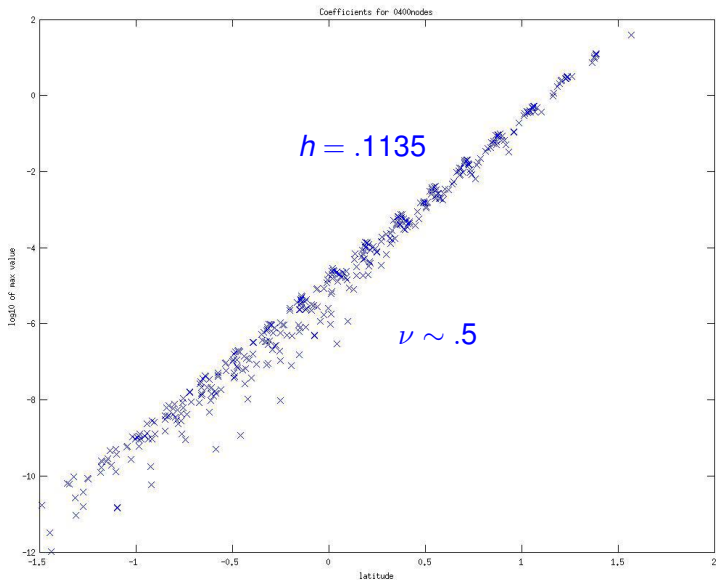


Cycling through minimum energy nodes from Rob Womersley:

<http://web.maths.unsw.edu.au/~rsw/Sphere/Energy/>



What about coefficients? Using 400 nodes again:



- $\chi_\xi = \sum_{\zeta \in \Xi} A_{\xi, \zeta} k(\cdot, \zeta) + p$
- $A_{\xi, \zeta} = \langle \chi_\xi, \chi_\zeta \rangle_{\mathcal{H}_k}$
- When $\mathcal{L}_m \Pi = \{0\}$,

$$|A_{\xi, \zeta}| \leq Cq^{d-2m} \exp(-\nu \frac{\text{dist}(\xi, \zeta)}{h})$$

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- Most coefficients can be ignored!
- Keep the coefficients within $Ch|\log h|$ with $C > (2m - d)/\nu$.
- For N quasi-uniform centers, this is only $\mathcal{O}(|\log N|^2)$ centers!