Polynomial approximation on Lissajous curves on the d-cube [1]

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UNIVERSITÀ DEGLI STUDI DI PADOVA

[1] Joint work with Len Bos (Verona), Wolfgang Erb (Luebeck), Francesco Marchetti (Padova) and Marco Vianello (Padova),

Outline

Introduction and known results

1. Are parametric curves studied by Bowditch (1815) and Lissajous (1857) of the form

$$\gamma(t) = (A_x \cos(\omega_x t + \alpha_x), A_y \sin(\omega_y t + \alpha_y))\,.$$

A_x, A_y are amplitudes, ω_x, ω_y are pulsations and α_x, α_y are phases.

Lissajous curves

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2. Chebyshev polynomials (T_k or U_k) are Lissajous curves (cf. J. C. Merino 2003). In fact a parametrization of $y = T_n(x)$, $|x| \leq 1$ is

$$\begin{cases} x = \cos t \\ y = -\sin\left(nt - \frac{\pi}{2}\right) \quad 0 \leq t \leq \pi \end{cases}$$

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3. Padua points (of the first family) [JAT 2006] lie on $[-1,1]^2$ on the π-periodic Lissajous curve $T_{n+1}(x) = T_n(y)$ called generating curve given in parametric form as

$$\gamma_n(t) = (\cos nt, \cos(n+1)t), \ \ 0 \le t \le \pi \, , \ n \ge 1 \, .$$

The generating curve of the Padua points ($n = 4$)

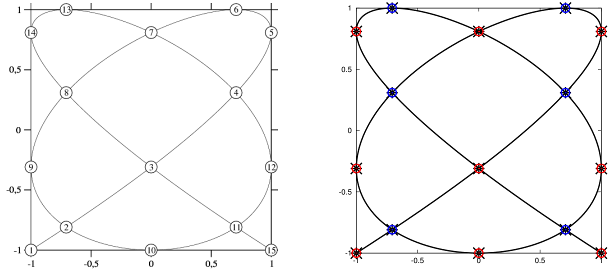

Figure : $\mathrm{Pad}_n = C^{\mathrm{O}}_{n+1} \times C^{\mathrm{E}}_{n+2} \cup C^{\mathrm{E}}_{n+1} \times C^{\mathrm{O}}_{n+2} \subset C_{n+1} \times C_{n+2}$

$C_{n+1} = \left\{ z_j^n = \cos\left(\frac{(j-1)\pi}{n}\right),\ j = 1, \dots, n+1 \right\}$: Chebsyhev-Lobatto points on $[-1, 1]$

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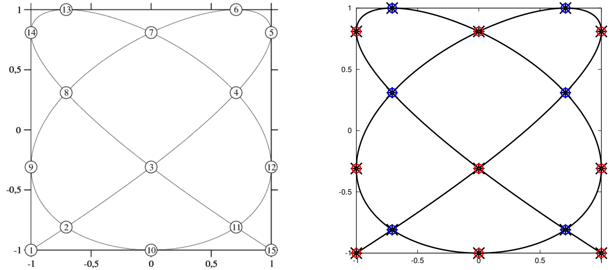

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Note: $|Pad_n| = \binom{n+2}{2} = dim(\mathbb{P}_n(\mathbb{R}^2))$

Lemma (cf. JAT 2006)

For all $p \in \mathbb{P}_{2n}(\mathbb{R}^2)$ *we have*

$$\frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy = \frac{1}{\pi}\int_0^{\pi} p(\gamma_n(t))dt.$$

The generating curve and cubature

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Proof. Check the property for all $p(x,y) = T_j(x)T_k(y),\ \ j+k \leq 2n$. $\square$

Lissajous points in 2D: non-degenerate case

[Erb et al. NumerMath16 (to appear)] in the framework of Magnetic Particle Imaging applications, considered

$$\gamma_{n,p}(t) = (\sin nt, \sin((n+p)t)) \quad 0 \le t < 2\pi\,,$$

$n, p \in \mathbb{N}$ s.t. n and $n+p$ are relative primes.

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Università degli Studi di Padova

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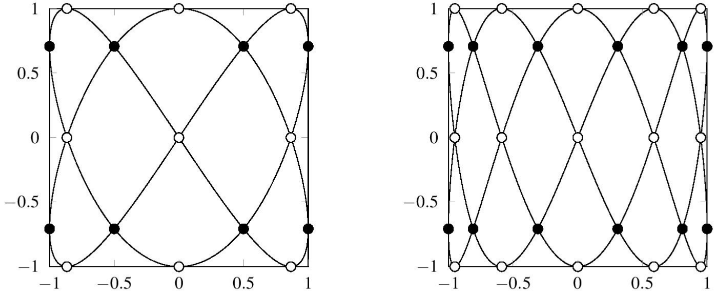
$n, p \in \mathbb{N}$ s.t. n and $n+p$ are relative primes.

- $\gamma_{n,p}$ is non-degenerate iff p is odd.
- Consider $t_k = 2\pi k/(4n(n+p)),\ k = 1, ..., 4n(n+p)$.

$$Lisa_{n,p} := \left\{\gamma_{n,p}(t_k),\ \ k = 1, \ldots, 4n(n+p)\right\},\ |Lisa_{n,p}| = 2n(n+p) + 2n + p\,.$$

Notice: $|Lisa_{n,1}| = 2n^2 + 4n + 1$ while $|Pad_{2n}| = 2n^2 + 3n + 1$ is obtained with $p = 1/2$.

Lissajous points: non-degenerate case



(a) Lissajous figure $\gamma_{2,1}$, $|\mathrm{Lisa}_{2,1}| = 17$.
(b) Lissajous figure $\gamma_{2,3}$, $|\mathrm{Lisa}_{2,3}| = 27$.

Figure : From the paper by Erb et al. NM2016 (cf. arXiv 1411.7589)

Lissajous points: degenerate case

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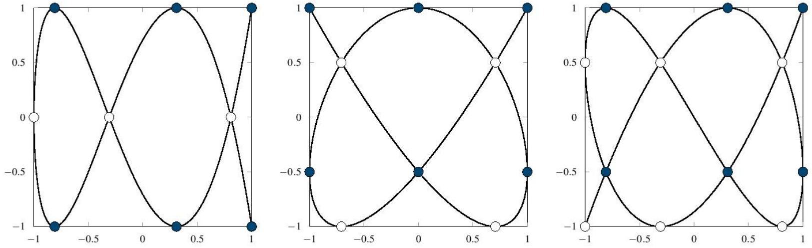
$$\gamma_{n,p}(t) = (\cos nt, \cos((n+p)t)) \quad 0 \le t < 2\pi \,,$$

- Consider $t_k = \pi k/(n(n+p)),\ k = 0, 1, ..., n(n+p)$.

$$LD_{n,p} := \left\{\gamma_{n,p}(t_k),\ \ k = 0, 1, \ldots, n(n+p)\right\},\ |LD_{n,p}| = \frac{(n+p+1)(n+1)}{2}\,.$$

Notice: for $p = 1$, $|LD_{n,1}| = |Pad_n| = \mathtt{dim}(\mathbb{P}_n(\mathbb{R}^2))$ and correspond to the Padua points themselves.

Lissajous points: degenerate case



(a) $\gamma_{2,3}$, $|\mathrm{LD}_{2,3}| = 9$. (b) $\gamma_{3,1}$, $|\mathrm{LD}_{3,1}| = 10$. (c) $\gamma_{3,2}$, $|\mathrm{LD}_{3,2}| = 12$.

Figure : From the paper by Erb AMC16, (cf. arXiv 1503.00895)

$\longrightarrow$ Work in progress with W. Erb and F. Marchetti.

Ideas to avoid Gibbs phenomenon at discontinuites [Gottlieb&Shu SIAMRev97,Tadmor&Tanner IMAJN05]

An application

Image reconstruction with adaptive filters

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Ideas to avoid Gibbs phenomenon at discontinuites [Gottlieb&Shu SIAMRev97,Tadmor&Tanner IMAJN05]

1. Sampling with Lissajous for finding the interpolating polynomial
2. Initial non-adaptive filter

$$\sigma(x;\alpha) = \begin{cases} e^{x^\alpha/(x^2-1)} & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

   (α can vary with the point x in the adaptive case): this allow to avoid the Gibbs phenomenon
3. Detect the discontinuities by Canny edge-detector algorithm
4. Apply adptively the filter

Image reconstruction with adaptive filters: examples

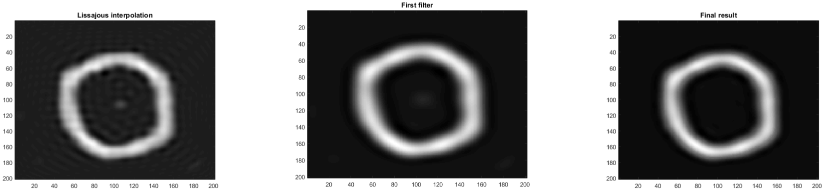
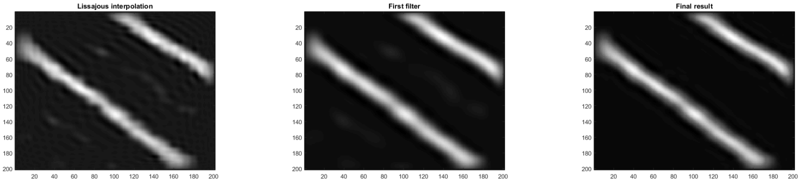


Figure : Original image: 115×115. Lissajous non degenerate curve with $(n, p) = (32, 33)$; Chebfun2 (modified) for the coefficients; $\alpha = 4$ for the initial filter and α chosen "Ad hoc" for the remainig adapted filtering

Image reconstruction with adaptive filters: examples



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3d case: notation

- $\Omega = [-1,1]^3$: the standard 3-cube
- The product Chebyshev measure

$$d\mu_3(\mathbf{x}) = w(\mathbf{x})d\mathbf{x}\,, \quad w(\boldsymbol{x}) = \frac{1}{\pi^3}\frac{1}{\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}}\,. \tag{1}$$

- $\mathbb{P}_k^3$: space of trivariate polynomials of degree k in $\mathbb{R}^3$ $(\mathtt{dim}(\mathbb{P}_k^3) = (k+1)(k+2)(k+3)/6)$.

Fundamental theorem

This results shows which are the admissible 3d Lissajous curves

Theorem (cf. Bos, DeM, Vianello 2015, IMA J. NA to appear)

Let $n \in \mathbb{N}^+$ and (a_n, b_n, c_n) be the integer triple

$$(a_n, b_n, c_n) = \begin{cases} \left(\frac{3}{4}n^2 + \frac{1}{2}n, \frac{3}{4}n^2 + n, \frac{3}{4}n^2 + \frac{3}{2}n + 1\right), & n \ \underline{even} \\ \left(\frac{3}{4}n^2 + \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n - \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n + \frac{3}{4}\right), & n \ \underline{odd}. \end{cases} \tag{2}$$

Then, for every integer triple (i, j, k), not all 0, with $i, j, k \geq 0$ and $i + j + k \leq m_n = 2n$, we have the property that $ia_n \neq jb_n + kc_n$, $jb_n \neq ia_n + kc_n$, $kc_n \neq ia_n + jb_n$.
Moreover, $m_n = 2n$ is maximal, in the sense that there exists a triple (i^*, j^*, k^*), $i^* + j^* + k^* = 2n + 1$, that does not satisfy the property.

On admissible curves follows

Proposition

Consider the Lissajous curves in $[-1,1]^3$ *defined by*

$$\boldsymbol{\ell}_n(\theta) = (\cos(a_n\theta), \cos(b_n\theta), \cos(c_n\theta)) \ , \quad \theta \in [0,\pi] \ , \qquad (3)$$

where (a_n, b_n, c_n) *is the sequence of integer triples (2). Then, for every total-degree polynomial* $p \in \mathbb{P}^3_{2n}$

$$\int_{[-1,1]^3} p(\boldsymbol{x}) \, d\mu_3(\boldsymbol{x}) = \frac{1}{\pi} \int_0^\pi p(\boldsymbol{\ell}_n(\theta)) \, d\theta \ . \qquad (4)$$

Proof. It suffices to prove the identity for a polynomial basis (ex: for the tensor product basis $T_\alpha(\boldsymbol{x}), |\alpha| \le 2n$). $\square$

Corollary

Consider $p \in \mathbb{P}^3_{2n}$, $\boldsymbol{\ell}_n(\theta)$ *and* $\nu = n \cdot \max\{a_n, b_n, c_n\} = n \cdot c_n$. *Then*

$$\int_{[-1,1]^3} p(\boldsymbol{x})\, w(\boldsymbol{x}) d\boldsymbol{x} = \sum_{s=0}^{\mu} w_s\, p(\boldsymbol{\ell}_n(\theta_s)) \;, \tag{5}$$

where

$$w_s = \pi^2 \omega_s \;, \quad s = 0, \ldots, \mu \;, \tag{6}$$

with

$$\mu = \nu \;, \quad \theta_s = \frac{(2s+1)\pi}{2\mu + 2} \;, \quad \omega_s \equiv \frac{\pi}{\mu + 1} \;, \quad s = 0, \ldots, \mu \;, \tag{7}$$

or alternatively

$$\mu = \nu + 1 \;, \quad \theta_s = \frac{s\pi}{\mu} \;, \quad s = 0, \ldots, \mu \;, \; \omega_0 = \omega_\mu = \frac{\pi}{2\mu} \;, \quad \omega_s \equiv \frac{\pi}{\mu} \;, \quad s = 1, \ldots, \mu - 1 \;. \tag{8}$$

- The points set
$$\{\boldsymbol{\ell}_n(\theta_s),\ s = 0, \ldots, \mu\}$$
are a 3-dimensional rank-1 Chebyshev lattices (for cubature of degree of exactness $2n$).
- Cools and Poppe [cf. CHEBINT, TOMS 2013] wrote a search algorithm for constructing heuristically such lattices.
- Formulae (2) (together with (6), (7), (8)) provide explicit formulas for any degree.

Conjecture

An algebraic polynomial restricted to $\ell_n(\theta)$ is a trig polynomial of degree $\nu = n\,c_n$.
$\longrightarrow$ Complexity of interpolation and quadrature depends on ν. $\longleftarrow$

Optimality

Suppose that (a, b, c) is a triple of strictly positive integers such that $\max\{a, b, c\} < c_n$, with c_n given by (2). Then there exists a triple (i, j, k) of naturals, not all 0, and $i + j + k \leq 2n$, such that either $ia = jb + kc$, $jb = ia + kc$, or $kc = ia + jb$.

The triples (2) are optimal, that is are those satisfying the Theorem 1 having the minimum maximum.

Hyperinterpolation and interpolation

Hyperinterpolation operator

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It requires 3 main ingredients

1. a good cubature formula (positive weights and high precision);
2. a good formula for representing the reproducing kernel (accurate and efficient);
3. a slow increase of the Lebesgue constant (which is the operator norm).

For $f \in C([-1,1]^3)$, using (5), the hyperinterpolation polynomial of f is

$$\mathcal{H}_n f(\boldsymbol{x}) = \sum_{0 \le i+j+k \le n} C_{i,j,k}\, \hat{\phi}_{i,j,k}(\boldsymbol{x}) \,, \tag{9}$$

$\hat{\phi}_{i,j,k}(\boldsymbol{x}) = \hat{T}_i(x_1)\hat{T}_j(x_2)\hat{T}_k(x_3)$ with

$$\hat{T}_m(\cdot) = \sigma_m \cos(m \arccos(\cdot)), \quad \sigma_m = \sqrt{\frac{1+sign(m)}{\pi}}, \quad m \ge 0$$

$$C_{i,j,k} = \sum_{s=0}^{\mu} w_s\, f(\boldsymbol{\ell}_n(\theta_s))\, \hat{\phi}_{i,j,k}(\boldsymbol{\ell}_n(\theta_s)) \,. \tag{10}$$

Properties

- $\mathcal{H}_n f = f, \quad \forall f \in \mathbb{P}_n^3$ (projection operator, by construction).
- L^2-error

$$\|f - \mathcal{H}_n f\|_2 \le 2\pi^3 \, E_n(f) \;, \quad E_n(f) = \inf_{p \in \mathbb{P}_n} \|f - p\|_\infty \;. \tag{11}$$

- Lebesgue constant

$$\|\mathcal{H}_n\|_\infty = \max_{\boldsymbol{x} \in [-1,1]^3} \sum_{s=0}^{\mu} w_s \left| K_n(\boldsymbol{x}, \boldsymbol{\ell}_n(\theta_s)) \right| \tag{12}$$

$$K_n(\boldsymbol{x}, \boldsymbol{y}) = \sum_{|\mathbf{i}| \le n} \hat{\phi}_{\mathbf{i}}(\boldsymbol{x}) \hat{\phi}_{\mathbf{i}}(\boldsymbol{y}), \quad \mathbf{i} = (i, j, k) \tag{13}$$

where K_n is the reproducing kernel of $\mathbb{P}_n^3$ w.r.t. product Chebyshev measure $d\mu_3$

Hyperinterpolation operator

Norm and approximation error estimates

- Based on a conjecture stated in [DeM, Vianello & Xu, BIT 2009] and specialized in [H.Wang, K.Wang & X.Wang, CMA 2014] we get

$$\|\mathcal{H}_n\|_\infty = \mathcal{O}((\log n)^3)$$

i.e. the minimal polynomial growth.

- $\mathcal{H}_n$ is a projection, then

$$\|f - \mathcal{H}_n f\|_\infty = \mathcal{O}\left((\log n)^3 \, E_n(f)\right) \, . \tag{14}$$

Computing the hyperinterpolation coefficients

The coefficients $\{C_{i,j,k}\}$ can be computed by a single 1D discrete Chebyshev transform along the Lissajous curve.

Proposition

Letting $f \in C([-1,1]^3)$, (a_n, b_n, c_n), ν, μ, $\{\theta_s\}$, ω_s, $\{w_s\}$ *as in Corollary 1. Then*

$$C_{i,j,k} = \frac{\pi^2}{4}\,\sigma_{ia_n}\sigma_{jb_n}\sigma_{kc_n}\left(\frac{\gamma_{\alpha_1}}{\sigma_{\alpha_1}} + \frac{\gamma_{\alpha_2}}{\sigma_{\alpha_2}} + \frac{\gamma_{\alpha_3}}{\sigma_{\alpha_3}} + \frac{\gamma_{\alpha_4}}{\sigma_{\alpha_4}}\right), \qquad (15)$$

$$\alpha_1 = ia_n + jb_n + kc_n\,, \quad \alpha_2 = |ia_n + jb_n - kc_n|\,,$$

$$\alpha_3 = |ia_n - jb_n| + kc_n\,, \quad \alpha_4 = ||ia_n - jb_n| - kc_n|\,,$$

where $\{\gamma_m\}$ *are the first* $\nu + 1$ *coefficients of the discretized Chebyshev expansion of* $f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$, $t \in [-1,1]$, *namely*

$$\gamma_m = \sum_{s=0}^{\mu} \omega_s\, \hat{T}_m(\tau_s)\; f(T_{a_n}(\tau_s), T_{b_n}(\tau_s), T_{c_n}(\tau_s))\,, \qquad (16)$$

$m = 0, 1, \ldots, \nu$, *with* $\tau_s = \cos(\theta_s)$, $s = 0, 1, \ldots, \mu$.

Implementation details: I

From previous Prop., hyperinterpolation on $\ell_n(t)$ can be done by a single 1-dimensional FFT $\longrightarrow$ Chebfun package [Chebfun 2014].

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- The polynomial interpolant of a function g can be written

$$\pi_\mu(t) = \sum_{m=0}^{\mu} c_m T_m(t) \tag{17}$$

where

$$c_m = \frac{2}{\mu} \sum_{s=0}^{\mu}{}'' \, T_m(\tau_s)\, g(\tau_s) \;, \quad m = 1, \ldots, \mu - 1 \;,$$

$$c_m = \frac{1}{\mu} \sum_{s=0}^{\mu}{}'' \, T_m(\tau_s)\, g(\tau_s) \;, \quad m = 0, \mu \;, \tag{18}$$

Note: $\sum_{s=0}^{\mu}{}''$ means first and last terms are halved

Implementation details: II

- If $g(t) = f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$ and comparing with the discrete Chebyshev expansion coefficients (16) we get

$$\frac{\gamma_m}{\sigma_m} = \begin{cases} \frac{\pi}{2}\, c_m \,, & m = 1, \ldots, \mu - 1 \\ \\ \pi\, c_m \,, & m = 0, \mu \end{cases} \tag{19}$$

i.e., the 3D hyperinterpolation coefficients (15) can be computed by the $\{c_m\}$ and (19).

... practically ...

A single call of the function `chebfun` on $f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$, truncated at the $(\mu + 1)$th-term, produces all the relevant coefficients $\{c_m\}$ in an extremely fast and stable way.

Example

Take $n = 100$ and the functions

$$f_1(\boldsymbol{x}) = \exp(-c\|\boldsymbol{x}\|_2^2)\ ,\ c > 0\ ,\quad f_2(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^\beta\ ,\ \beta > 0\ , \tag{20}$$

To compute the $\mu = \frac{3}{4}n^3 + \frac{3}{2}n^2 + n + 2 = 765102$ coefficients from which we get, by (15), the $(n+1)(n+2)(n+3)/6 = 176851$ coefficients of trivariate hyperinterpolation,

it took about 1 sec by using Chebfun 5.1 on a Athlon 64 X2 Dual Core 4400+ 2.4 GHz processor.

Example

hyperinterpolation errors

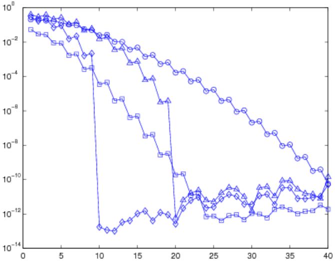

Figure : Left: Hyperinterpolation errors for the trivariate polynomials $\|\boldsymbol{x}\|_2^{2k}$ with $k = 5$ (diamonds) and $k = 10$ (triangles), and for the trivariate function f_1 with $c = 1$ (squares) and $c = 5$ (circles). Right: Hyperinterpolation errors for the trivariate function f_2 with $\beta = 5$ (squares) and $\beta = 3$ (circles).

Software for Lissajous sampling

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- The sampling set (Chebyshev lattice) $\mathcal{A}_n = \{\boldsymbol{\ell}_n(\theta_s)\,,\ s = 0, \ldots, \mu\}$ has been used as a Weakly Admissible Mesh (WAM) from which we extracted the Approximate Fekete Points (AFP) and the Discrete Leja Points (DLP).
  Notice: DLP form a sequence, i.e., its first $N_r = \dim(\mathbb{P}_r^d)$ elements span $\mathbb{P}_r^d$, $1 \leq r \leq n$.
- The extraction of $N = \mathtt{dim}(\mathbb{P}_n^3)$ points has been done by the software available at
  `www.math.unipd.it/~marcov/CAAsoft`.
  We wrote the package `hyperlissa`, a Matlab code for hyperinterpolation on 3d Lissajous curves.

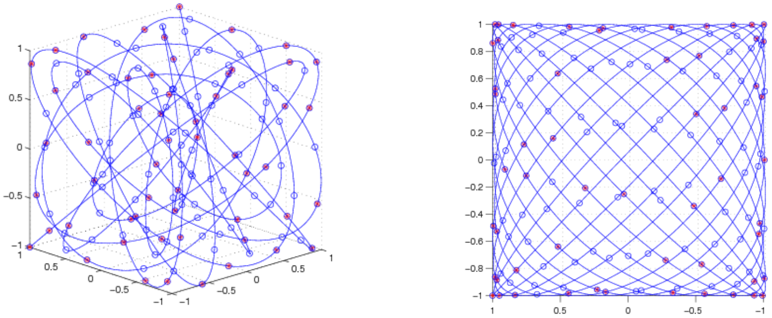

Figure : Left: the Chebyshev lattice (circles) and the extracted AFP (red asterisks), on the Lissajous curve for polynomial degree $n = 5$. Right: A face projection of the curve and the sampling nodes

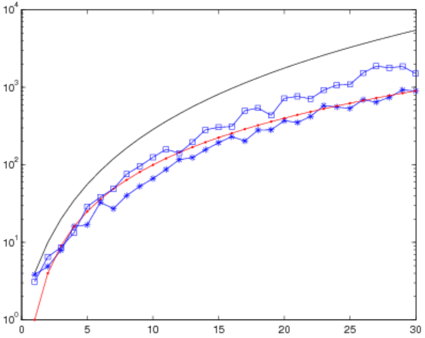

Figure : Lebesgue constants (log scale) of the AFP (asterisks) and DLP (squares) extracted from the Chebyshev lattices on the Lissajous curves, for degree $n = 1, 2, \ldots, 30$, compared with $\dim(\mathbb{P}_n^3) = (n+1)(n+2)(n+3)/6$ (upper solid line) and n^2 (dots).

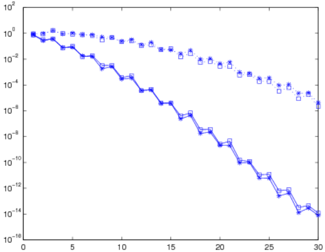

Figure : Interpolation errors on AFP (asterisks) and DLP (squares) for the trivariate functions f_1 (Left) with $c = 1$ (solid line) and $c = 5$ (dotted line), and f_2 (Right) with $\beta = 5$ (solid line) and $\beta = 3$ (dotted line).

The general approach

Some notation

- $\mathbb{P}_m^d$, the space of polynomials of total degree at most m (in $\mathbb{R}^d$)
- $\mathbb{P}_m^{\otimes d}$, the d ordered tensor product of $\mathbb{P}_m^1$

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- $V = \mathbb{P}_m^d$ and $\alpha \in \mathbb{Z}^d$ we set $|\alpha|_V := \sum_{i=1}^d |\alpha_i|$
- $V = \mathbb{P}_m^{\otimes d}$ and $\alpha \in \mathbb{Z}^d$ we set $|\alpha|_V := \max_{1\le i\le d} |\alpha|_i$

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Take $\mathbf{a} \in \mathbb{Z}^d_{>0}$: $\ell_{\mathbf{a}}(t) := (\cos(a_1 t), \cos(a_2 t), \cdots, \cos(a_d t))$ the Lissajous curve.

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Problem

Among the curves $\ell_{\mathbf{a}}(t)$ select the ones s.t.

$$\max_{p \in V} \deg(p(\ell_{\mathbf{a}}(t))) \to \min$$

Definition

$\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d_{>0}$ is V-admissible (of order m) if

$$\left(\nexists\ 0 \neq \mathbf{b} \in \mathbb{Z}^d,\ |\mathbf{b}|_V \leq m\right) \ \ s.t. \ \ \sum_{i=1}^d b_i a_i = 0.$$

We denote this set by $\mathcal{A}(V)$.

$\longrightarrow$ $\mathbf{a} \in \mathcal{A}(V)$ means that there are no “small” solution of the diophantine equation $\sum_{i=1}^d x_i a_i = 0$.

V-admissible tuples and cubature

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Proposition

Let $\mathbf{a} \in \mathbb{Z}^d_{>0}$, then

$$\int_{[-1,1]^d} p(\boldsymbol{x}) d\mu_d(\boldsymbol{x}) = \frac{1}{\pi} \int_0^\pi p(\ell_{\mathbf{a}}(t)) dt \tag{21}$$

for all polynomials $p \in V$ if and only if $\mathbf{a} \in \mathcal{A}(V)$.

Proof. It suffices to prove (21) for a basis. □

$p(x) \in V$ restricted to the curve $\ell_{\mathbf{a}}(t)$ is a *univariate* trigonometric polynomial $q(t) := p(\ell_{\mathbf{a}}(t))$ whose complexity is bounded by its degree.

- For example: $p \in V = \mathbb{P}_m^d$,

$$\deg(q(t)) \leq \left(\max_{1 \leq i \leq d} a_i\right) m$$

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- For d=3 [cf. Theorem 1, Bos et al. 2016] has been indeed proved

$$\boxed{\min_{\mathbf{a}\in\mathcal{A}(V)} \max_{1\leq i\leq d} a_i = \mathcal{O}(m^2)\,.}$$

Conjecture

For $m \equiv 0(4)$ *let*

$$a_1 = \frac{3m^2+4m}{16},\ a_2 = \frac{3m^2+8m}{16},\ a_3 = \frac{3m^2+12m+16}{16}.$$

For $m \equiv 1(4)$ *let*

$$a_1 = \frac{3m^2+6m+7}{16},\ a_2 = \frac{3m^2+10m+19}{16},\ a_3 = \frac{3m^2+14m+15}{16}.$$

For $m \equiv 2(4)$ *let*

$$a_1 = \frac{3m^2+4}{16},\ a_2 = \frac{3m^2+12m-4}{16},\ a_3 = \frac{3m^2+12m+12}{16}.$$

For $m \equiv 3(4)$ *let*

$$a_1 = \begin{cases} \frac{3m^2+2m-1}{16} & m \equiv 3\,(8) \\ \frac{3m^2+6m+19}{16} & m \equiv 7\,(8), \end{cases} \quad a_2 = \begin{cases} \frac{3m^2+14m+11}{16} & m \equiv 3\,(8) \\ \frac{3m^2+10m+7}{16} & m \equiv 7\,(8), \end{cases} \quad a_3 = \frac{3m^2+14m+27}{16}.$$

The triple $(a_1, a_2, a_3) \in \mathcal{A}(V)$ is then optimal, that is

$$\boxed{a_3 = \max\{a_1, a_2, a_3\} = \min_{\mathbf{b} \in \mathcal{A}(V)} \max_{1 \le i \le d} b_i.}$$

“Optimal” triples obtained by computer search

m			
2	1	2	3
3	1	3	5
	3	4	5
4	4	5	7
	4	6	7
5	7	8	10
	7	9	10
6	7	11	12
7	7	15	17
	9	11	17
	9	15	17
	10	16	17
	13	14	17
	13	16	17
8	14	16	19
	14	17	19
9	19	21	24
	19	22	24
10	19	26	27
11	24	33	34
12	30	33	37
	30	34	37
15	41	47	57
	49	52	57
	49	54	57
31	177	191	209
	177	195	209
	184	208	209
	193	200	209
	193	202	209

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The case of $d = 4$ seems already to be more complicated

m				
2	1	2	3	4
3	1	3	5	7
	4	5	6	7
4	5	9	11	12
5	5	13	17	19
6	11	24	27	28
	15	24	27	28
7	9	31	37	39
8	34	50	54	55
9	59	61	71	74
	59	62	72	74
10	59	90	95	96
	65	90	91	96
	53	89	90	96
11	77	89	119	121
	53	109	119	121
12	105	138	150	152
13	159	167	187	188
14	177	215	229	230
15	193	219	267	273
	199	215	271	273

Remark: search complexity $\mathcal{O}(m^3)$, very expensive! No idea about the pattern (as for $d = 3$)

The tensor product case

In this setting

$$\boxed{\deg(p(\ell_{\mathbf{a}}(t)) \leq \left(\sum_{i=1}^{d} a_i\right) m}$$

so that we have to solve

$$\boxed{\min_{\mathbf{a}\in\mathcal{A}(V)} \sum_{i=1}^{d} a_i} \qquad (P)$$

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The unique solution exists

Proposition

For $V = \mathbb{P}_M^{\otimes d}$ the tuple

$$\mathbf{g} = (1, (m+1), (m+1)^2, \cdots, (m+1)^{d-1}) \in \mathcal{A}(V)$$

is the unique minimizer (up to permutation) of the problem (P).

Proof: long and technical [Bos el al. 2016] .

Hyperinterpolation in $\mathbb{P}_m^{\otimes d}$

1. When $m = 2n$ then for $\mathbf{a} \in \mathcal{A}(V)$, we have the quadrature formula (21)

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2. Equivalently, for any $p, q \in \mathbb{P}_n^{\otimes d}$,

$$\int_{[-1,1]^d} p(x)q(x)d\mu_d(x) = \frac{1}{\pi}\int_0^\pi p(\ell_{\mathbf{a}}(t))q(\ell_{\mathbf{a}}(t))dt. \qquad (22)$$

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3. For $f \in C([-1,1]^d)$, its best least squares approximation in $L_2([-1,1]^d; d\mu_d)$ is given by

$$\pi_n(f) = \sum_{|\alpha|_\infty \le n} \langle f, \hat{T}_\alpha \rangle \hat{T}_\alpha.$$

with $\langle \cdot, \cdot \rangle$ the inner product

$$\langle f, g \rangle := \int_{[-1,1]^d} f(x)g(x)d\mu_d(x)$$

and $\hat{T}_\alpha(x) = c_\alpha \prod_{j=1}^d T_j(x_j)$ the normalized Chebyshev polynomials (orthonormal basis of $\mathbb{P}_n^{\otimes d}$)

Hyperinterpolation in $\mathbb{P}_m^{\otimes d}$ (cont)

- Define

$$\pi_n^{\mathbf{a}} := \sum_{|\alpha|_\infty \le n} \langle f, \hat{T}_\alpha \rangle_{\mathbf{a}} \hat{T}_\alpha$$

where $\langle f, g \rangle_{\mathbf{a}} := \frac{1}{\pi} \int_0^\pi f(\ell_{\mathbf{a}}(t)) g(\ell_{\mathbf{a}}(t)) dt$. By integrating along the Lissajous curve, (22), we have

$$\pi_n^{\mathbf{a}}(p) = \pi_n(p) = p, \quad \forall p \in \mathbb{P}_n^{\otimes d},$$

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- Quadrature of $\pi_n^{\mathbf{a}}$

$$\frac{1}{\pi} \int_0^\pi t(\theta) d\theta = \frac{1}{N} \left\{ \frac{1}{2} t(\theta_0) + \sum_{k=1}^{N-1} t(\theta_k) + \frac{1}{2} t(\theta_N) \right\}$$

for (at least) all even trigonometric polynomials of degree at most $2N-1$ ($\theta_k := k\pi/N$, $0 \le k \le N$, are the equally spaced angles).

Hyperinterpolation in $\mathbb{P}_m^{\otimes d}$ (cont)

Now, $p(\ell_{\mathbf{a}})$ is an even trigonometric polynomial of degree $\leq 2n(\sum_{i=1}^{d} a_i)$. Taking

$$N := 1 + n \sum_{i=1}^{d} a_i$$

letting

$$x_k := \ell_{\mathbf{a}}(\theta_k), \quad w_0 := \frac{1}{2N}, \; w_k := \frac{1}{N}, \; 1 \leq k \leq N-1, \; w_N = \frac{1}{2N},$$

we have

$$\langle p, q \rangle_{\mathbf{a}} = \sum_{k=0}^{N} w_k p(x_k) q(x_k) \tag{23}$$

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Computing $\pi_n^{\mathbf{a}}$ by means of (23) with get the hyperinterpolation operator with uniform norm of $\mathcal{O}(\log^d(n))$ (in the Chebyshev measure on the d-cube) [H. Wang, K. Wang, X. Wang 2014].

Summary

1. Lissajous curves on 2d, 3d for total degree polynomial (hyper)-interpolation and cubature
2. Lissajous "optimal" for 3d
3. Lissajous optimal for the tensor product polynomials

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Opne problem

Are these curves suitable for finding Padua-like points on the d-cube?

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Alba di Canazei (ITALY), 8-13/9/2016

http://events.math.unipd.it/dwcaa16/

#thankyou!