Trivariate polynomial approximation on Lissajous curves [1]

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[1] Joint work with Len Bos (Verona) and Marco Vianello (Padova)

Outline

1. Are parametric curves studied by Bowditch (1815) and Lissajous (1857) of the form

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$$\begin{cases} x = \cos t \\ y = -\sin\left(nt - \frac{\pi}{2}\right) \quad 0 \le t \le \pi \end{cases}$$

Lissajous curves

Properties and motivation

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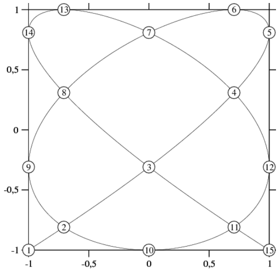
3. Padua points (of the first family) lie on $[-1,1]^2$ on the π-periodic Lissajous curve $T_{n+1}(x) = T_n(y)$ called generating curve given also as

$$\gamma_n(t) = (\cos nt, \cos(n+1)t),\ \ 0 \le t \le \pi\,,\ \ n \ge 1.$$

The generating curve of the Padua points ($n = 4$)



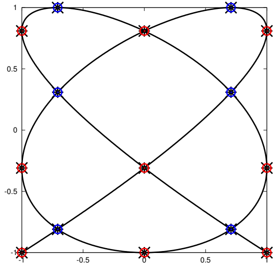


Figure : $\mathrm{Pad}_n = \mathcal{C}^{\mathrm{O}}_{n+1} \times \mathcal{C}^{\mathrm{E}}_{n+2} \cup \mathcal{C}^{\mathrm{E}}_{n+1} \times \mathcal{C}^{\mathrm{O}}_{n+2} \subset \mathcal{C}_{n+1} \times \mathcal{C}_{n+2}$

$\mathcal{C}_{n+1} = \left\{ z^n_j = \cos\left(\frac{(j-1)\pi}{n}\right),\ j = 1, \ldots, n+1 \right\}$: Chebsyhev-Lobatto points on $[-1, 1]$

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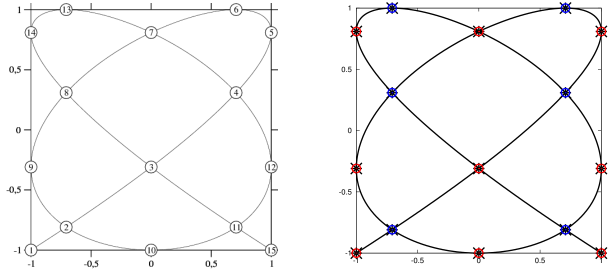

Figure : $\mathrm{Pad}_n = C^{\mathrm{O}}_{n+1} \times C^{\mathrm{E}}_{n+2} \cup C^{\mathrm{E}}_{n+1} \times C^{\mathrm{O}}_{n+2} \subset C_{n+1} \times C_{n+2}$

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Note: $|Pad_n| = \binom{n+2}{2} = dim(\mathbb{P}_n(\mathbb{R}^2))$

The generating curve and cubature

Lemma (cf. Bos at al. JAT 2006)

For all $p \in \mathbb{P}_{2n}(\mathbb{R}^2)$ *we have*

$$\frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy = \frac{1}{\pi}\int_0^\pi p(\gamma_n(t))dt.$$

Lissajous points in 2D: non-degenerate case

Erb et al. 2014 (cf. arXiv 1411.7589) in the framework of Magnetic Particle Imaging applications, considered

$$\gamma_{n,p}(t) = (\sin nt, \sin((n+p)t)) \quad 0 \leq t < 2\pi\,,$$

$n, p \in \mathbb{N}$ s.t. n and $n+p$ are relative primes.

- $\gamma_{n,p}$ is non-degenerate iff p is odd.
- Take $t_k = 2\pi k/(4n(n+p)),\ k = 1, ..., 4n(n+p)$.

$$Lisa_{n,p} := \left\{\gamma_{n,p}(t_k),\ \ k = 1, \ldots, 4n(n+p)\right\},\ |Lisa_{n,p}| = 2n(n+p)+2n+p\,.$$

Notice: $p = 1$, $|Lisa_{n,1}| = 2n^2 + 4n + 1$ while $|Pad_{2n}| = 2n^2 + 3n + 1$.

Lissajous points: non-degenerate case

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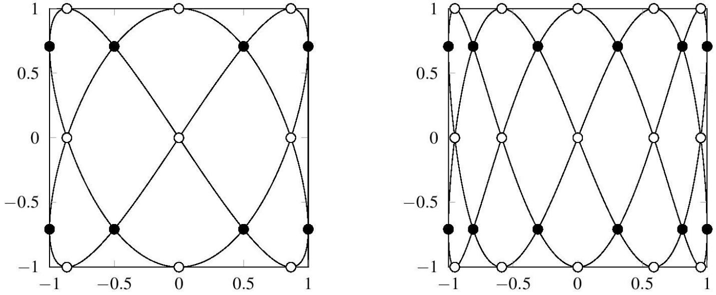

(a) Lissajous figure $\gamma_{2,1}$, $|\mathrm{Lisa}_{2,1}| = 17$. (b) Lissajous figure $\gamma_{2,3}$, $|\mathrm{Lisa}_{2,3}| = 27$.

Figure : From the paper by Erb et al. 2014 (cf. arXiv 1411.7589)

Lissajous points: degenerate case

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Erb 2015, (cf. arXiv 1503.00895) has then studied the degenerate 2π-Lissajous curves

$$\gamma_{n,p}(t) = (\cos nt, \cos((n+p)t)) \quad 0 \le t < 2\pi\,,$$

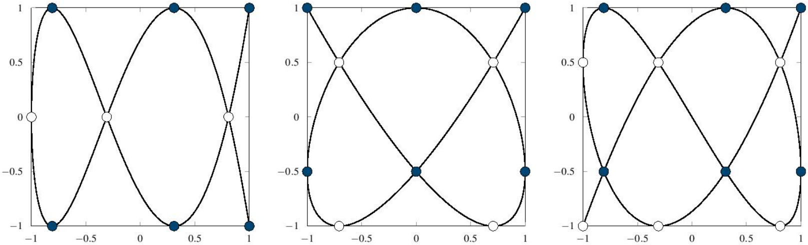
E

- Take $t_k = \pi k/(n(n+p)),\ k = 0, 1, ..., n(n+p)$.

$$LD_{n,p} := \left\{\gamma_{n,p}(t_k),\ \ k = 0, 1, \ldots, n(n+p)\right\},\ |LD_{n,p}| = \frac{(n+p+1)(n+1)}{2}\,.$$

Notice: for $p = 1$, $|LD_{n,1}| = |Pad_n| = \mathtt{dim}(\mathbb{P}_n(\mathbb{R}^2))$ and correspond to the Padua points themselves.

Lissajous points: degenerate case



(a) $\gamma_{2,3}$, $|\mathrm{LD}_{2,3}| = 9$. (b) $\gamma_{3,1}$, $|\mathrm{LD}_{3,1}| = 10$. (c) $\gamma_{3,2}$, $|\mathrm{LD}_{3,2}| = 12$.

Figure : From the paper by Erb 2015, (cf. arXiv 1503.00895)

Notation in 3d

- $\Omega = [-1, 1]^3$: the standard 3-cube
- The product Chebyshev measure

$$d\lambda = w(\boldsymbol{x})d\boldsymbol{x}\,, \quad w(\boldsymbol{x}) = \frac{1}{\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}}\,. \tag{1}$$

- $\mathbb{P}_k^3$: space of trivariate polynomials of degree k in $\mathbb{R}^3$ $(\mathtt{dim}(\mathbb{P}_k^3) = (k+1)(k+2)(k+3)/6)$.

Fundamental theorem

This results shows which are the admissible 3d Lissajous curves

Theorem (cf. Bos, De Marchi at al. 2015, arXiv 1502.04114)

Let be $n \in \mathbb{N}^+$ *and* (a_n, b_n, c_n) *be the integer triple*

$$(a_n, b_n, c_n) = \begin{cases} \left(\frac{3}{4}n^2 + \frac{1}{2}n, \frac{3}{4}n^2 + n, \frac{3}{4}n^2 + \frac{3}{2}n + 1\right), & n \text{ } \underline{even} \\ \left(\frac{3}{4}n^2 + \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n - \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n + \frac{3}{4}\right), & n \text{ } \underline{odd} \end{cases} \tag{2}$$

Then, for every integer triple (i, j, k)*, not all* 0*, with* $i, j, k \geq 0$ *and* $i + j + k \leq m_n = 2n$*, we have the property that* $ia_n \neq jb_n + kc_n$*,* $jb_n \neq ia_n + kc_n$*,* $kc_n \neq ia_n + jb_n$*.*
Moreover, $m_n = 2n$ *is maximal, in the sense that there exists a triple* (i^*, j^*, k^*)*,* $i^* + j^* + k^* = 2n + 1$*, that does not satisfy the property.*

On admissible curve the integration becomes the integral on the curve.

Proposition

Consider the Lissajous curves in $[-1,1]^3$ *defined by*

$$\boldsymbol{\ell}_n(\theta) = (\cos(a_n\theta), \cos(b_n\theta), \cos(b_n\theta)) \ , \quad \theta \in [0,\pi] \ , \qquad (3)$$

where (a_n, b_n, c_n) *is the sequence of integer triples (2).*
Then, for every total-degree polynomial $p \in \mathbb{P}^3_{2n}$

$$\int_{[-1,1]^3} p(\boldsymbol{x})\, w(\boldsymbol{x}) d\boldsymbol{x} = \pi^2 \int_0^\pi p(\boldsymbol{\ell}_n(\theta))\, d\theta \ . \qquad (4)$$

Proof. It suffices to prove the identity for a polynomial basis (ex: for the tensor product basis $T_\alpha(\boldsymbol{x})$, $|\alpha| \le 2n$). □

Corollary

Let $p \in \mathbb{P}^3_{2n}$, $\boldsymbol{\ell}_n(\theta)$ *and* $\nu = n \max\{a_n, b_n, c_n\}$. *Then*

$$\int_{[-1,1]^3} p(\mathbf{x})\, w(\mathbf{x}) d\mathbf{x} = \sum_{s=0}^{\mu} w_s\, p(\boldsymbol{\ell}_n(\theta_s))\ , \tag{5}$$

where

$$w_s = \pi^2 \omega_s\ ,\quad s = 0, \ldots, \mu\ , \tag{6}$$

with

$$\mu = \nu\ ,\ \ \theta_s = \frac{(2s+1)\pi}{2\mu+2}\ ,\ \ \omega_s \equiv \frac{\pi}{\mu+1}\ ,\ \ s = 0, \ldots, \mu\ , \tag{7}$$

or alternatively

$$\mu = \nu + 1\ ,\ \ \theta_s = \frac{s\pi}{\mu}\ ,\ \ s = 0, \ldots, \mu\ ,$$

$$\omega_0 = \omega_\mu = \frac{\pi}{2\mu}\ ,\ \ \omega_s \equiv \frac{\pi}{\mu}\ ,\ s = 1, \ldots, \mu - 1\ . \tag{8}$$

Remarks

- The points set

$$\{\boldsymbol{\ell}_n(\theta_s),\ s = 0, \ldots, \mu\}$$

  are a 3-dimensional rank-1 Chebyshev lattices (for cubature of degree of exactness $2n$).
- Cools and Poppe [cf. CHEBINT, TOMS 2013] wrote a search algorithm for constructing heuristically such lattices.
- WE HAVE an explicit formula for any degree.

Hyperinterpolation operator

General definition

Definition

Hyperinterpolation of multivariate continuous functions, on compact subsets or manifolds, is a discretized orthogonal projection on polynomial subspaces [Sloan JAT1995].

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It requires 3 main ingredients

1. a good cubature formula (positive weights and high precision);
2. a good formula for representing the reproducing kernel (accurate and efficient);
3. a slow increase of the Lebesgue constant (which is the operator norm).

For $f \in C([-1,1]^3)$, using (5), the hyperinterpolation polynomial of f is

$$\mathcal{H}_n f(\boldsymbol{x}) = \sum_{0 \le i+j+k \le n} C_{i,j,k}\, \hat{\phi}_{i,j,k}(\boldsymbol{x}) \,, \tag{9}$$

$\hat{\phi}_{i,j,k}(\boldsymbol{x}) = \hat{T}_i(x_1)\hat{T}_j(x_2)\hat{T}_k(x_3)$ with $T_s(\cdot)$ the normalized Chebyshev polynomials

$$C_{i,j,k} = \sum_{s=0}^{\mu} w_s\, f(\boldsymbol{\ell}_n(\theta_s))\, \hat{\phi}_{i,j,k}(\boldsymbol{\ell}_n(\theta_s)) \,. \tag{10}$$

Properties

- $\mathcal{H}_n f = f, \;\; \forall f \in \mathbb{P}_n^3$ (projection operator, by construction).
- L^2-error

$$\|f - \mathcal{H}_n f\|_2 \leq 2\pi^3 \, E_n(f) \,, \quad E_n(f) = \inf_{p \in \mathbb{P}_n} \|f - p\|_\infty \,. \qquad (11)$$

- Lebesgue constant

$$\|\mathcal{H}_n\|_\infty = \max_{\boldsymbol{x} \in [-1,1]^3} \sum_{s=0}^{\mu} w_s \left| K_n(\boldsymbol{x}, \boldsymbol{\ell}_n(\theta_s)) \right| \,, \; K_n(\boldsymbol{x}, \boldsymbol{y}) = \sum_{|\mathbf{i}| \leq n} \hat{\phi}_{\mathbf{i}}(\boldsymbol{x}) \hat{\phi}_{\mathbf{i}}(\boldsymbol{y}), \qquad (12)$$

where K_n is the reproducing kernel of $\mathbb{P}_n^3$ w.r.t. product Chebyshev measure

- Based on a conjecture stated in [DeM, Vianello & Xu, BIT 2009] and specialized in [H.Wang, K.Wang & X.Wang, CMA 2014] we get

$$\|\mathcal{H}_n\|_\infty = \mathcal{O}((\log n)^3)$$

i.e. the minimal polynomial growth.

- $\mathcal{H}_n$ is a projection, then

$$\|f - \mathcal{H}_n f\|_\infty = \mathcal{O}\left((\log n)^3 \, E_n(f)\right) \, . \tag{13}$$

Computing the hyperinterpolation coefficients

The coefficients $\{C_{i,j,k}\}$ can be computed by a single 1D discrete Chebyshev transform along the Lissajous curve.

Proposition

Let be $f \in C([-1,1]^3)$, (a_n, b_n, c_n), ν, μ, $\{\theta_s\}$, ω_s, $\{w_s\}$ as in Corollary 1. Then

$$C_{i,j,k} = \frac{\pi^2}{4}\,\sigma_{ia_n}\sigma_{jb_n}\sigma_{kc_n}\left(\frac{\gamma_{\alpha_1}}{\sigma_{\alpha_1}} + \frac{\gamma_{\alpha_2}}{\sigma_{\alpha_2}} + \frac{\gamma_{\alpha_3}}{\sigma_{\alpha_3}} + \frac{\gamma_{\alpha_4}}{\sigma_{\alpha_4}}\right), \tag{14}$$

$$\alpha_1 = ia_n + jb_n + kc_n\,, \quad \alpha_2 = |ia_n + jb_n - kc_n|\,,$$

$$\alpha_3 = |ia_n - jb_n| + kc_n\,, \quad \alpha_4 = ||ia_n - jb_n| - kc_n|\,,$$

where $\{\gamma_m\}$ are the first $\nu + 1$ coefficients of the discretized Chebyshev expansion of $f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$, $t \in [-1,1]$, namely

$$\gamma_m = \sum_{s=0}^{\mu} \omega_s\, \hat{T}_m(\tau_s)\; f(T_{a_n}(\tau_s), T_{b_n}(\tau_s), T_{c_n}(\tau_s))\,, \tag{15}$$

$m = 0, 1, \ldots, \nu$, with $\tau_s = \cos(\theta_s)$, $s = 0, 1, \ldots, \mu$.

Remarks

1. Within the emerging 3d MPI (Magnetic Particle Imaging) technology, Lissajous sampling is one of the most common sampling method.
2. Hyperinterpolation polynomials on d-dimensional cubes can be done by other cubature formulas (exacteness $2n$) for the product Chebyshev measure : $\mathcal{O}(n^4)$ [DeM, Vianello, Xu BIT 2009]; [Godzina 1995] which have the lowest cardinality and used in the package CHEBINT by Cools&Poppe.
3. The set $\mathcal{A}_n = \{\boldsymbol{\ell}_n(\theta_s)\,,\ s = 0, \ldots, \mu\}$ in (7)-(8), forms a Weakly Admissible Mesh for $[-1, 1]^3$, so they provide a Lissajous sampling to 3D polynomial interpolation.

From Prop. 2, hyperinterpolation on $\ell_n(t)$ can be done by a single 1-dimensional FFT. We used the Chebfun package [Chebfun 2014].

- The polynomial interpolant of a function g can be written

$$\pi_\mu(t) = \sum_{m=0}^{\mu} c_m T_m(t) \tag{16}$$

where

$$c_m = \frac{2}{\mu} \sum_{s=0}^{\mu}{}'' \, T_m(\tau_s)\, g(\tau_s)\,, \quad m = 1, \ldots, \mu - 1\,,$$

$$c_m = \frac{1}{\mu} \sum_{s=0}^{\mu}{}'' \, T_m(\tau_s)\, g(\tau_s)\,, \quad m = 0, \mu\,, \tag{17}$$

Note: $\sum_{s=0}^{\mu}{}''$ means first and last terms are halved

- If $g(t) = f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$ and comparing with the discrete Chebyshev expansion coefficients (15)

$$\frac{\gamma_m}{\sigma_m} = \begin{cases} \frac{\pi}{2} c_m \,, & m = 1, \dots, \mu - 1 \\ \\ \pi \, c_m \,, & m = 0, \mu \end{cases} \tag{18}$$

  i.e., the 3D hyperinterpolation coefficients (14) can be computed by the $\{c_m\}$ and (18).

- A single call of the function `chebfun` on $f(T_{a_n}(t), T_{b_n}(t), T_{c_n}(t))$, truncated at the $(\mu + 1)$th-term, produces all the relevant coefficients $\{c_m\}$ in an extremely fast and stable way.

Example

Take $n = 100$ and the functions

$$f_1(\boldsymbol{x}) = \exp(-c\|\boldsymbol{x}\|_2^2)\;,\; c > 0\;,\quad f_2(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^{\beta}\;,\; \beta > 0\;, \qquad (19)$$

To compute the $\mu = \frac{3}{4}n^3 + \frac{3}{2}n^2 + n + 2 = 765102$ coefficients from which we get by (14) the $(n+1)(n+2)(n+3)/6 = 176851$ coefficients of trivariate hyperinterpolation,

it took about 1 second by using Chebfun 5.1 on a Athlon 64 X2 Dual Core 4400+ 2.4 GHz processor.

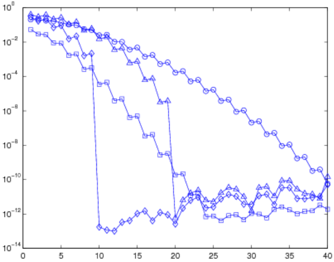

Figure : Left: Hyperinterpolation errors for the trivariate polynomials $||\boldsymbol{x}||_2^{2k}$ with $k = 5$ (diamonds) and $k = 10$ (triangles), and for the trivariate function f_1 with $c = 1$ (squares) and $c = 5$ (circles). Right: Hyperinterpolation errors for the trivariate function f_2 with $\beta = 5$ (squares) and $\beta = 3$ (circles).

Interpolation by Lissajous sampling

- The sampling set (Chebyshev lattice) $\mathcal{A}_n = \{\boldsymbol{\ell}_n(\theta_s)\,,\ s = 0, \ldots, \mu\}$ has been used as a WAM from which we have extracted the AFP and the DLP.
- The extraction of $N = \mathtt{dim}(\mathbb{P}_n^3)$ points has been done by the software available at `www.math.unipd.it/~marcov/CAAsoft`.
  DLP form a sequence, i.e., its first $N_r = \dim(\mathbb{P}_r^d)$ elements span $\mathbb{P}_r^d$, $1 \leq r \leq n$.
- We wrote the package `hyperlissa`, a Matlab code for hyperinterpolation on 3d Lissajous curves. Available at the same web page.

Example

Chebyshev lattice points

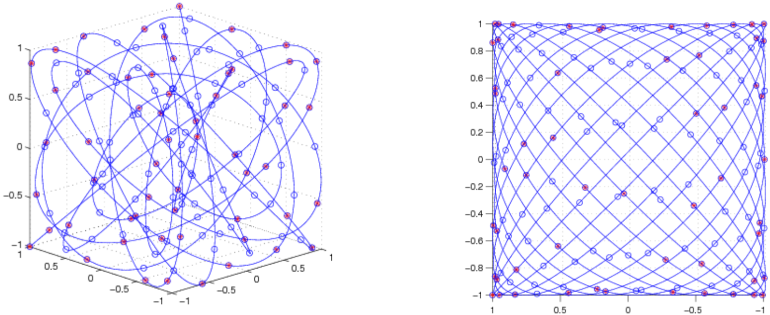

Figure : Left: the Chebyshev lattice (circles) and the extracted Approximate Fekete Points (red asterisks), on the Lissajous curve for polynomial degree $n = 5$. Right: A face projection of the curve and the sampling nodes

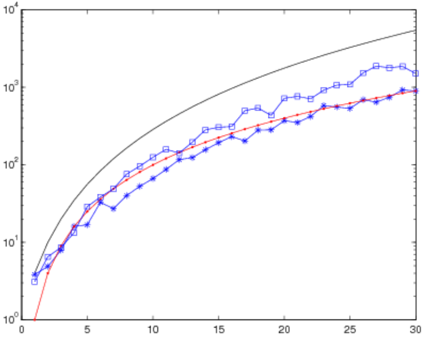

Figure : Lebesgue constants (log scale) of the Approximate Fekete Points (asterisks) and Discrete Leja Points (squares) extracted from the Chebyshev lattices on the Lissajous curves, for degree $n = 1, 2, \ldots, 30$, compared with $\dim(\mathbb{P}_n^3) = (n+1)(n+2)(n+3)/6$ (upper solid line) and n^2 (dots).

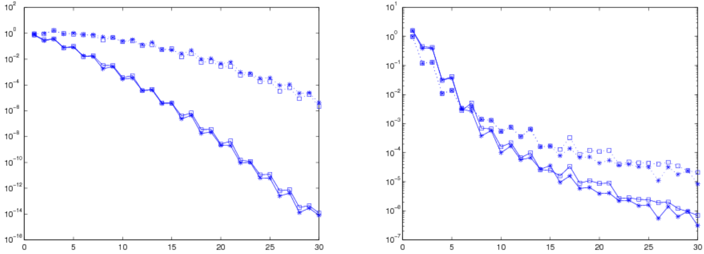

Figure : Interpolation errors on Approximate Fekete Points (asterisks) and Discrete Leja Points (squares) for the trivariate functions f_1 (Left) with $c = 1$ (solid line) and $c = 5$ (dotted line), and f_2 (Right) with $\beta = 5$ (solid line) and $\beta = 3$ (dotted line).

Future works

- A general results for generating "good" Lissajous curves for every dimension *d*.
- A faster way to extract AFP and LDP from Lissajous curves.
- Construct Padua points on the cube $[-1, 1]^3$.

Future meetings in the Dolomites

7th Research week: 5-8/9/2015, http://events.math.unipd.it/drwa15/

4th Workshop: 8-13/9/2016, http://events.math.unipd.it/dwcaa16/

#thankyou!