# New developments on rational RBF

**Stefano De Marchi**
Department of Mathematics "Tullio Levi-Civita"
University of Padova
Montecatini, 14th February 2018

DIPARTIMENTO MATEMATICA

# Outline

# From RBF to Rational RBF (RRBF)

work with A. Martinez and E. Perracchione

# Notations

1. Data: $\Omega \subset \mathbb{R}^d$, $X \subset \Omega$, test function $f$
   $X_N = \{x_1, \ldots, x_N\} \subset \Omega$, $\mathbf{f} = \{f_1, \ldots, f_N\}$, where $f_i = f(x_i)$

# Notations

1. Data: $\Omega \subset \mathbb{R}^d$, $X \subset \Omega$, test function $f$
   $X_N = \{x_1, \ldots, x_N\} \subset \Omega$, $\mathbf{f} = \{f_1, \ldots, f_N\}$, where $f_i = f(x_i)$
2. Approximation setting:
   - $\phi(\varepsilon\,\cdot)$: Conditionally Positive Definite (CPD) of order $\ell$ or Strictly Positive Definite (SPD) and radial ($\varepsilon$, shape parameter)
     - globally supported:

| *name* | $\phi$ | $\ell$ |
|---|---|---|
| Gaussian $C^\infty$ (GA) | $e^{-\varepsilon^2 r^2}$ | 0 |
| Generalized Multiquadrics $C^\infty$ (GM) | $(1 + r^2/\varepsilon^2)^{3/2}$ | 2 |

     - locally supported:

| *name* | $\phi$ | $\ell$ |
|---|---|---|
| Wendland $C^2$ (W2) | $(1 - \varepsilon r)_+^4\,(4\varepsilon r + 1)$ | 0 |
| Buhmann $C^2$ (B2) | $2r^4 \log r - 7/2r^4 + 16/3r^3 - 2r^2 + 1/6$ | 0 |

# Notations



1. Data: $\Omega \subset \mathbb{R}^d$, $X \subset \Omega$, test function $f$
   $X_N = \{x_1, \ldots, x_N\} \subset \Omega$, $\mathbf{f} = \{f_1, \ldots, f_N\}$, where $f_i = f(x_i)$
2. Approximation setting:
   - $\phi(\varepsilon\,\cdot)$: Conditionally Positive Definite (CPD) of order $\ell$ or Strictly Positive Definite (SPD) and radial ($\varepsilon$, shape parameter)
     - globally supported:

| *name* | $\phi$ | $\ell$ |
|---|---|---|
| Gaussian $C^\infty$ (GA) | $e^{-\varepsilon^2 r^2}$ | 0 |
| Generalized Multiquadrics $C^\infty$ (GM) | $(1 + r^2/\varepsilon^2)^{3/2}$ | 2 |

     - locally supported:

| *name* | $\phi$ | $\ell$ |
|---|---|---|
| Wendland $C^2$ (W2) | $(1 - \varepsilon r)_+^4 (4\varepsilon r + 1)$ | 0 |
| Buhmann $C^2$ (B2) | $2r^4 \log r - 7/2r^4 + 16/3r^3 - 2r^2 + 1/6$ | 0 |

   - kernel notation $K_\varepsilon(\cdot,\cdot)$
   - native space $\mathcal{N}_K(\Omega)$ (where $K$ is the reproducing kernel)
   - finite subspace $\mathcal{N}_K(X_N) = \operatorname{span}\{K(\cdot, x) : x \in X_N\} \subset \mathcal{N}_K(\Omega)$

# RBF Interpolation

Given $\Omega, X_N, \mathbf{f}, K$

**Aim**

Find $P_f \in \mathcal{N}_K(X_N)$ s.t. $(P_f)_{X_N} = \mathbf{f}$

# RBF Interpolation

Given $\Omega, X_N, \mathbf{f}, K$

**Aim**

Find $P_f \in \mathcal{N}_\kappa(X_N)$ s.t. $(P_f)_{X_N} = \mathbf{f}$

- Classical interpolant: $P_f(x) = \sum_{k=1}^{N} \alpha_k K(x, x_k), \quad x \in \Omega,\ x_k \in X_N.$

  [Hardy and Gofert 1975] used multiquadrics
  $K(x, y) = \sqrt{1 + \epsilon^2 \|x - y\|^2}$.

# RBF Interpolation

Given $\Omega, X_N, \mathbf{f}, K$

## Aim

Find $P_f \in \mathcal{N}_\kappa(X_N)$ s.t. $(P_f)_{X_N} = \mathbf{f}$

- Classical interpolant: $P_f(x) = \sum_{k=1}^{N} \alpha_k K(x, x_k), \quad x \in \Omega,\ x_k \in X_N.$
  [Hardy and Gofert 1975] used multiquadrics $K(x, y) = \sqrt{1 + \epsilon^2 \|x - y\|^2}$.
- Rescaled interpolant: $\hat{P}_f(x) = \frac{P_f(x)}{P_g(x)} = \frac{\sum_{k=1}^N \alpha_k K(x, x_k)}{\sum_{k=1}^N \beta_k K(x, x_k)}$ where $P_g$ is the kernel interpolant of $g(x) = 1,\ \forall x \in \Omega$. Localized Rescaled and exactness on constants in [Deparis et al 2014]. In [DeM et al 2017] it is shown that it is a Shepard's PU method. Linear convergence of localized rescaled interpolants [DeM and Wendland, draft 2017]

# Rational RBF

## definition

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{k=1}^{N} \alpha_k K(x, x_k)}{\sum_{k=1}^{N} \beta_k K(x, x_k)}$$

[Jackbsson et al. 2009, Sarra and Bai 2017]

$\Longrightarrow$ RRBFs well approximate data with steep gradients or discontinuites [rational with PU+VSK in DeM et al. 2017].

# Rational RBF

## Learning from rational functions, $d = 1$

- polynomial case.

$$r(x) = \frac{p_1(x)}{p_2(x)} = \frac{a_m x^m + \cdots + a_0 x^0}{x^n + b_{n-1} x^{n-1} \cdots + b_0}.$$

$M = m + n + 1$ unknowns (Padé approximation). If $M < N$ to get the coefficients we may solve the LS problem

$$\min_{p_1 \in \Pi_m^1, p_2 \in \Pi_n^1} \left( \sum_{k=1}^{N} \left| f(x_k) - r(x_k) \right|^2 \right).$$

# Rational RBF

## Learning from rational functions, $d = 1$

- polynomial case.

$$r(x) = \frac{p_1(x)}{p_2(x)} = \frac{a_m x^m + \cdots + a_0 x^0}{x^n + b_{n-1} x^{n-1} \cdots + b_0}.$$

$M = m + n + 1$ unknowns (Padé approximation). If $M < N$ to get the coefficients we may solve the LS problem

$$\min_{p_1 \in \Pi^1_m, p_2 \in \Pi^1_n} \left( \sum_{k=1}^{N} \left| f(x_k) - r(x_k) \right|^2 \right).$$

- RBF case. Let $X_m = \{x_k, \ldots, x_{k+m-1}\}, X_n = \{x_j, \ldots, x_{j+n-1}\} \subset X_N$ be non empty, such that $m + n \leq N$

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{i_1=k}^{k+m-1} \alpha_{i_1} K(x, x_{i_1})}{\sum_{i_2=j}^{j+n-1} \beta_{i_2} K(x, x_{i_2})}, \tag{1}$$

provided $R^{(2)}(x) \neq 0$, for all $x \in \Omega$.

# Rational RBF

## Find the coefficients: I

[Jackobsson et al 2009] proved the well-posedness of the interpolation on $X_N$ via

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{i=1}^{N} \alpha_i K(x, x_i)}{\sum_{k=1}^{N} \beta_k K(x, x_k)}, \tag{2}$$

# Rational RBF

## Find the coefficients: I

[Jackobsson et al 2009] proved the well-posedness of the interpolation on $X_N$ via

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{i=1}^{N} \alpha_i K(x, x_i)}{\sum_{k=1}^{N} \beta_k K(x, x_k)}, \tag{2}$$

Letting $\boldsymbol{\xi} = (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T) \in \mathbb{R}^{2N}$ and $B$ the $N \times 2N$ matrix

$$B = \begin{pmatrix} K(x_1, x_1) & \cdots & K(x_1, x_N) & -f_1 K(x_1, x_1) & \cdots & -f_1 K(x_1, x_N) \\ \vdots & & \vdots & & & \vdots \\ K(x_N, x_1) & \cdots & K(x_N, x_N) & -f_N K(x_N, x_1) & \cdots & -f_N K(x_N, x_N) \end{pmatrix}.$$

The system $B\boldsymbol{\xi} = \mathbf{0}$ can be written as $(A - DA)(\boldsymbol{\xi}) = \mathbf{0}$ with $D = \mathtt{diag}(f_1, \ldots, f_N)$, and $A_{i,j} = K(x_i, x_j)$...

# Rational RBF

## Find the coefficients: I

[Jackobsson et al 2009] proved the well-posedness of the interpolation on $X_N$ via

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{i=1}^N \alpha_i K(x, x_i)}{\sum_{k=1}^N \beta_k K(x, x_k)}, \tag{2}$$

Letting $\boldsymbol{\xi} = (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T) \in \mathbb{R}^{2N}$ and $B$ the $N \times 2N$ matrix

$$B = \begin{pmatrix} K(x_1, x_1) & \cdots & K(x_1, x_N) & -f_1 K(x_1, x_1) & \cdots & -f_1 K(x_1, x_N) \\ \vdots & & \vdots & & & \vdots \\ K(x_N, x_1) & \cdots & K(x_N, x_N) & -f_N K(x_N, x_1) & \cdots & -f_N K(x_N, x_N) \end{pmatrix}.$$

The system $B\boldsymbol{\xi} = \mathbf{0}$ can be written as $(A - DA)(\boldsymbol{\xi}) = \mathbf{0}$ with $D = \mathtt{diag}(f_1, \ldots, f_N)$, and $A_{i,j} = K(x_i, x_j)$... which is underdetermined!

# Rational RBF

## Find the coefficients: I

[Jackobsson et al 2009] proved the well-posedness of the interpolation on $X_N$ via

$$R(x) = \frac{R^{(1)}(x)}{R^{(2)}(x)} = \frac{\sum_{i=1}^{N} \alpha_i K(x, x_i)}{\sum_{k=1}^{N} \beta_k K(x, x_k)}, \qquad (2)$$

Letting $\boldsymbol{\xi} = (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T) \in \mathbb{R}^{2N}$ and $B$ the $N \times 2N$ matrix

$$B = \begin{pmatrix} K(x_1, x_1) & \cdots & K(x_1, x_N) & -f_1 K(x_1, x_1) & \cdots & -f_1 K(x_1, x_N) \\ \vdots & & \vdots & & & \vdots \\ K(x_N, x_1) & \cdots & K(x_N, x_N) & -f_N K(x_N, x_1) & \cdots & -f_N K(x_N, x_N) \end{pmatrix}.$$

The system $B\boldsymbol{\xi} = \mathbf{0}$ can be written as $(A - DA)(\boldsymbol{\xi}) = \mathbf{0}$ with $D = \mathtt{diag}(f_1, \ldots, f_N)$, and $A_{i,j} = K(x_i, x_j)$... which is underdetermined!

### Non trivial solution

Following [GolubReinsch1975] non-trivial solutions exist by asking $||\boldsymbol{\xi}||_2 = 1$ i.e. solving the problem $\min_{\boldsymbol{\xi} \in \mathbb{R}^N, ||\boldsymbol{\xi}||_2 = 1} ||B\boldsymbol{\xi}||_2$.

# Rational RBF

## Find the coefficients: II

**Obs:** (Since $R^{(1)}(x_i) = f_i R^{(2)}(x_i), \quad i = 1, \ldots, N$)

*find* $\mathbf{q} = (R^{(2)}(x_1), \ldots, R^{(2)}(x_N))^T$ *and, as* $\mathbf{p} = D\mathbf{q}$, *then*
$\mathbf{p} = (R^{(1)}(x_i), ..., R^{(1)}(x_N))^T$ .

# Rational RBF

## Find the coefficients: II

**Obs:** (Since $R^{(1)}(x_i) = f_i R^{(2)}(x_i), \quad i = 1, \ldots, N$)

*find* $\mathbf{q} = (R^{(2)}(x_1), \ldots, R^{(2)}(x_N))^T$ *and, as* $\mathbf{p} = D\mathbf{q}$, *then*
$\mathbf{p} = (R^{(1)}(x_i), ..., R^{(1)}(x_N))^T$ .

- If **p, q** are given then the rational interpolant is known by solving

$$A\alpha = \mathbf{p}, \quad A\beta = \mathbf{q}\,. \tag{3}$$

# Rational RBF

## Find the coefficients: II

**Obs:** (Since $R^{(1)}(x_i) = f_i R^{(2)}(x_i), \quad i = 1, \ldots, N$)

*find* $\mathbf{q} = (R^{(2)}(x_1), \ldots, R^{(2)}(x_N))^T$ *and, as* $\mathbf{p} = D\mathbf{q}$*, then*
$\mathbf{p} = (R^{(1)}(x_i), ..., R^{(1)}(x_N))^T$ .

- If **p, q** are given then the rational interpolant is known by solving

$$A\alpha = \mathbf{p}, \quad A\beta = \mathbf{q}\,. \tag{3}$$

- Existence & Uniqueness of (3): $K$ is SPD

Using the native space norms the above problem is equivalent

**Problem 1**

$$\min_{\substack{R^{(1)}, R^{(2)} \in \mathcal{N}_K, \\ 1/\|\mathbf{f}\|_2^2 \|\boldsymbol{p}\|_2^2 + \|\boldsymbol{q}\|_2^2 = 1, \\ R^{(1)}(x_k) = f_k R^{(2)}(x_k).}} \left( \frac{1}{\|\boldsymbol{f}\|_2^2} \|R^{(1)}\|_{\mathcal{N}_K}^2 + \|R^{(2)}\|_{\mathcal{N}_K}^2 \right). \tag{4}$$

# Rational RBF

## Find the coefficients: III

$$\|R^{(1)}\|^2_{\mathcal{N}_K} = \boldsymbol{\alpha}^T A \boldsymbol{\alpha}, \quad \text{and} \quad \|R^{(2)}\|^2_{\mathcal{N}_K} = \boldsymbol{\beta}^T A \boldsymbol{\beta}.$$

Then, from (3) and symmetry of $A$

$$\|R^{(1)}\|^2_{\mathcal{N}_K} = \boldsymbol{p}^T A^{-1} \boldsymbol{p}, \quad \text{and} \quad \|R^{(2)}\|^2_{\mathcal{N}_K} = \boldsymbol{q}^T A^{-1} \boldsymbol{q}.$$

Therefore, the **Problem 1** reduces to solve

### Problem 2

$$\min_{\substack{\boldsymbol{q} \in \mathbb{R}^N, \\ 1/\|\boldsymbol{f}\|_2^2 \|D\boldsymbol{q}\|_2^2 + \|\boldsymbol{q}\|_2^2 = 1.}} \left( \frac{1}{\|\boldsymbol{f}\|_2^2} \boldsymbol{q}^T D^T A^{-1} D \boldsymbol{q} + \boldsymbol{q}^T A^{-1} \boldsymbol{q} \right).$$

# Rational RBF

## Find the coefficients: IV

[Jackbsson 2009] show that this is equivalent to solve the following generalized eigenvalue problem

### Problem 3

$$\Sigma \boldsymbol{q} = \lambda \Theta \boldsymbol{q},$$

with

$$\Sigma = \frac{1}{||\boldsymbol{f}||_2^2} D^T A^{-1} D + A^{-1}, \quad \text{and} \quad \Theta = \frac{1}{||\boldsymbol{f}||_2^2} D^T D + I_N,$$

where $I_N$ is the identity matrix.

# Rational RBF

## Find the coefficients: IV

[Jackbsson 2009] show that this is equivalent to solve the following generalized eigenvalue problem

### Problem 3

$$\Sigma \boldsymbol{q} = \lambda \Theta \boldsymbol{q},$$

with

$$\Sigma = \frac{1}{||\boldsymbol{f}||_2^2} D^T A^{-1} D + A^{-1}, \quad \text{and} \quad \Theta = \frac{1}{||\boldsymbol{f}||_2^2} D^T D + I_N,$$

where $I_N$ is the identity matrix.

$\hookrightarrow$ **q** is the eigenvector associated to the smallest eigenvalue! $\hookleftarrow$

# The new eigen-rational interpolant

work with M. Buhmann and E. Perracchione

# New class of rational RBF

$$\hat{P}_f(\boldsymbol{x}) = \frac{\sum_{i=1}^{N} \alpha_i K(\boldsymbol{x}, \boldsymbol{x_i}) + \sum_{m=1}^{L} \gamma_m p_m(\boldsymbol{x})}{\sum_{k=1}^{N} \beta_k \bar{K}(\boldsymbol{x}, \boldsymbol{x_k})} := \frac{P_g(\boldsymbol{x})}{P_h(\boldsymbol{x})} \tag{5}$$

Ratio of a CPD $K$ of order $\ell$ and an associate PD $\bar{K}$ .... $\Longrightarrow$ two kernel matrices, $\Phi_K$ and $\Phi_{\bar{K}}$.

# New class of rational RBF

$$\hat{P}_f(\boldsymbol{x}) = \frac{\sum_{i=1}^{N} \alpha_i K(\boldsymbol{x}, \boldsymbol{x_i}) + \sum_{m=1}^{L} \gamma_m p_m(\boldsymbol{x})}{\sum_{k=1}^{N} \beta_k \bar{K}(\boldsymbol{x}, \boldsymbol{x_k})} := \frac{P_g(\boldsymbol{x})}{P_h(\boldsymbol{x})} \tag{5}$$

Ratio of a CPD $K$ of order $\ell$ and an associate PD $\bar{K}$ .... $\Longrightarrow$ two kernel matrices, $\Phi_K$ and $\Phi_{\bar{K}}$.

## Obs:

1. *Once we know the function values $P_h(\boldsymbol{x}_i) = h_i$, $i = 1, \ldots, N$, we can construct $P_g$, i.e. it interpolates $\boldsymbol{g} = (f_1 h_1, \ldots, f_N h_N)^T$. Hence $\hat{P}_f$ interpolates the given function values* **f** *at the nodes* $\mathbf{X}_N$.
2. *If $K$ is PD, we fix $\bar{K} = K$ so that we deal with the same kernel matrix for both numerator and denominator.*

# The rational interpolant is well-defined



When $P_h(\mathbf{x}) \neq 0, \ \forall \mathbf{x} \in \Omega$?

**Theorem (Perron1907)**

*All positive square matrices have a positive eigenvalue with corresponding eigenvector with all components positive (called Perron eigenpair)*

**Theorem (Perron1907)**

*All positive square matrices possess exactly one Perron eigenpair and the corresponding eigenvalue has the largest modulus.*

# Remarks

dividing the interpolant (2) by the eigenvector associated to the largest eigenvalue of $\Phi_{\bar{K}}$ makes computations more accurate and hopefully more stable.

1. hence, the coefficients $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)^T$ are the components of the eigenvector associate to the eigenvalue

$$\max_{\|\tilde{\boldsymbol{\beta}}\|_2=1} \tilde{\boldsymbol{\beta}}^T \Phi_{\bar{K}} \tilde{\boldsymbol{\beta}}, \tag{6}$$

2. This enables us to give an eigen-rational RBF expansion, independent of the function values of the approximant and depending only on the kernel $K$ (and its associate $\bar{K}$) and $\mathbf{X}_N$

# Algorithmic issues

Assume $K$ is CPD of order $\ell$ and $\bar{K}$ the associate PD kernel

1. Compute $\boldsymbol{\beta}$ and so the values $P_h(\boldsymbol{x}_i) = h_i,\ i = 1, \ldots, N$ where $\boldsymbol{h}$ is defined by using the matrix $\Phi_{\bar{K}}$ (that depends on $\mathbf{X}_N$ and $\phi$) and not on the function values.
2. Determine $\hat{P}_f$ in (5) with the function values $\boldsymbol{g} = \boldsymbol{fh}$ and $\mathbf{0}$ (of lengtth $L$) instead of $(\boldsymbol{g}, \mathbf{0})^T$.

# Cardinal functions: I

$$\hat{P}_f = \sum_{j=1}^{N} \alpha_j \frac{K(\boldsymbol{x}, \boldsymbol{x}_j)}{\sum_{i=1}^{N} \beta_i K(\boldsymbol{x}, \boldsymbol{x}_i)} = \sum_{j=1}^{N} \alpha_j \frac{h_j K(\boldsymbol{x}, \boldsymbol{x}_j)}{\sum_{i=1}^{N} \beta_i K(\boldsymbol{x}, \boldsymbol{x}_i) \sum_{i=1}^{N} \beta_i K(\boldsymbol{x}_j, \boldsymbol{x}_i)},$$

since $h_j = \sum_{i=1}^{N} \beta_i K(\boldsymbol{x}_j, \boldsymbol{x}_i)$. Then

$$\hat{P}_f = \sum_{j=1}^{N} \tilde{\alpha}_j \frac{K(\boldsymbol{x}, \boldsymbol{x}_j)}{\sum_{i=1}^{N} \beta_i K(\boldsymbol{x}, \boldsymbol{x}_i) \sum_{i=1}^{N} \beta_i \Phi(\boldsymbol{x}_j, \boldsymbol{x}_i)} =: \sum_{j=1}^{N} \tilde{\alpha}_j K_R(\boldsymbol{x}, \boldsymbol{x}_j).$$

Since $P_h$ is not vanishing, the function

$$K_R(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{P_h(\boldsymbol{x})} \frac{1}{P_h(\boldsymbol{y})} K(\boldsymbol{x}, \boldsymbol{y}),$$

is strictly positive definite [DeMIS17].

**Obs:**

*The same argument applies when $K$ is only CPD of order $\ell$ giving $K_R$ CPD of order $\ell$.*

# Cardinal form of the interpolant and stability

## Proposition (BDeMP18)

*Suppose $K$ is CPD of order $\ell$ in $\mathbb{R}^d$, $\bar{K}$ is the associated PD kernel. Suppose $\mathbf{X}_N \subset \Omega$ is $(\ell - 1)$-unisolvent, then there exist $N$ functions $\hat{u}_k$ so that the interpolant is*

$$\hat{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} f_j \hat{u}_j(\boldsymbol{x}).$$

# Cardinal form of the interpolant and stability

## Proposition (BDeMP18)

*Suppose $K$ is CPD of order $\ell$ in $\mathbb{R}^d$, $\bar{K}$ is the associated PD kernel. Suppose $\mathbf{X}_N \subset \Omega$ is $(\ell-1)$-unisolvent, then there exist $N$ functions $\hat{u}_k$ so that the interpolant is*

$$\hat{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} f_j \hat{u}_j(\boldsymbol{x}).$$

$\Longrightarrow$ If $K = \bar{K}$ is PD, the $\hat{u}_k$, $k = 1, \ldots, N$, form a partition of unity [DeMIS, AT15 (2017)].

$\Longrightarrow$ Stability bound

$$\left|\sum_{i=1}^{N} f(\boldsymbol{x}_i)\hat{u}_i(\boldsymbol{x})\right| \leq \left(\sum_{i=1}^{N} |\hat{u}_i(\boldsymbol{x})|\right) \|f\|_\infty =: \hat{\lambda}_N(\boldsymbol{x})\|f\|_\infty \leq \hat{\Lambda}_N\|f\|_\infty,$$

where $\Lambda_N$ the Lebesgue constant

$$\hat{\Lambda}_N := \max_{\boldsymbol{x}\in\Omega} \hat{\lambda}_N(\boldsymbol{x}).$$

# Error bound

## Proposition (BDeMP18)

*Let $\Omega \subseteq \mathbb{R}^d$ be open. Suppose $K \in C(\Omega \times \Omega)$ be CDP of order $\ell$ and $\bar{K}$ the associate PD kernel. Assume that $\mathbf{X}_N \subset \Omega$ is $(\ell-1)$-unisolvent. Then, for $\boldsymbol{x} \in \Omega$, the pointwise error*

$$|f(\boldsymbol{x}) - \hat{P}_f(\boldsymbol{x})| \leq \frac{1}{|P_h(\boldsymbol{x})|}\left(\mathcal{P}_{\bar{K},\mathbf{X}_N}(\boldsymbol{x})|h|_{\mathcal{N}_{\bar{K}}(\Omega)}|f(\boldsymbol{x})| + \mathcal{P}_{K,\mathbf{X}_N}(\boldsymbol{x})|g|_{\mathcal{N}_K(\Omega)}\right).$$

with $\mathcal{P}_{K,\mathbf{X}_N}$ the power function for the kernel $K$ and point set $\mathbf{X}_N$ and $|\cdot|_{\mathcal{N}_K(\Omega)}$ the semi-norm.

$\Longrightarrow$ Similar bounds are derived using the fill-distance, $h_{\mathbf{X}_N,\Omega}$.

## Numerical tests:

rational eigen-basis vs the standard one

# Numerical tests: Lebesgue constants I

| $K$ | $\Lambda_N$ ($\varepsilon = 0.5$) | $\Lambda_N$ ($\varepsilon = 3$) | $\hat{\Lambda}_N$ ($\varepsilon = 0.5$) | $\hat{\Lambda}_N$ ($\varepsilon = 3$) |
|---|---|---|---|---|
| GM | 4.98 | 8.69 | 5.14 | 9.13 |
| GA | 9.81 | 2.58 | 12.6 | 3.59 |
| M6 | 10.6 | 2.20 | 11.1 | 2.60 |
| W6 | 2.10 | 9.58 | 2.73 | 7.90 |
| B2 | 30.6 | – | 29.1 | – |

Table: Lebesgue constants $\Lambda_N$ and $\hat{\Lambda}_N$ for classical and eigen-rational interpolants, respectively. They are computed on $N = 10$ equally spaced points on $\Omega = [-1, 1]$. Note: Buhmann's function is independent of the shape parameter.

# Numerical tests: Lebesgue functions (1d)

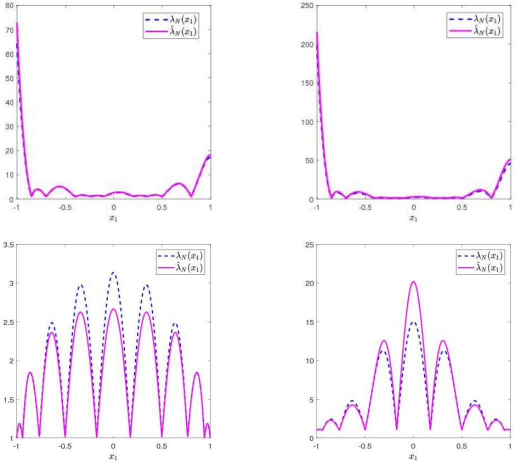



Figure: Top: 10 Halton points, GM kernel with $\varepsilon = 0.5$ (left) and $\varepsilon = 3$ (right). Bottom: 10 Chebyshev points, GA kernel with $\varepsilon = 0.5$ (left) and $\varepsilon = 3$ (right).

# Numerical tests: Lebesgue functions (2d)

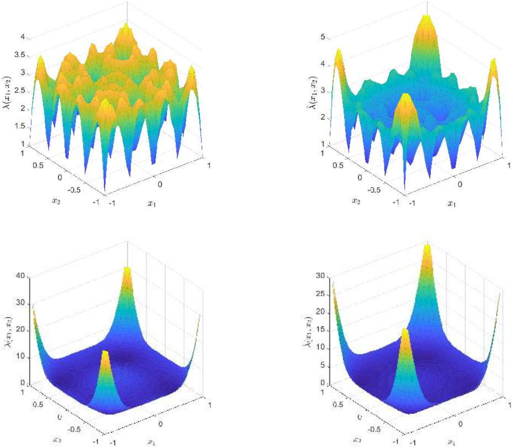



Figure: Top: Lebesgue functions computed via the W6 kernel with $\varepsilon = 0.5$ for standard (left) and eigen-rational (right) interpolants. Bottom: Lebesgue functions computed via the B2 kernel for standard (left) and eigen-rational (right) interpolants.

# Error and max-abs error comparison

$$f_1(x_1) = \mathrm{sinc}(x_1), \quad x_1 \in [-1, 1]$$

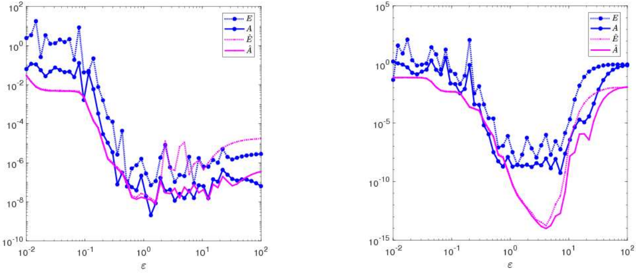
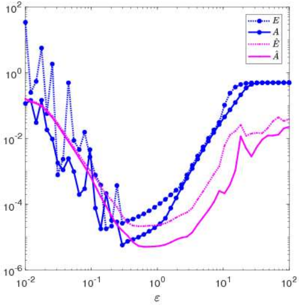



Figure: Error estimates $\hat{E}$ and $E$ via LOOCV of max. abs. errors $\hat{A}$ and $A$ for eigen-rational and classical interpolants, respectively. Here we consider $f_1$ on 81 Chebyshev points on $\Omega = [-1, 1]$ via GM (left) and GA (right) kernels.

$$f_2(x_1) = \frac{x_1^8}{\tan(1 + x_1^2) + 0.5}, \quad x_1 \in [-1, 1]$$



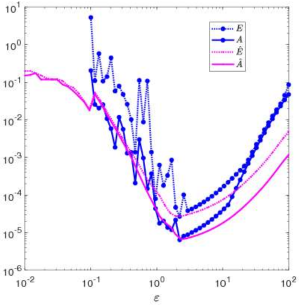




Figure: Error estimates $\hat{E}$ and $E$ via LOOCV of max. abs. errors $\hat{A}$ and $A$ for eigen-rational and classical interpolants, respectively. Result are computed with $f_2$ and 81 Random points on $\Omega = [-1, 1]$ via W6 (left) and M6 (right) kernels.

# Image registration: landmarks [CDeR2018, C et al 2015]

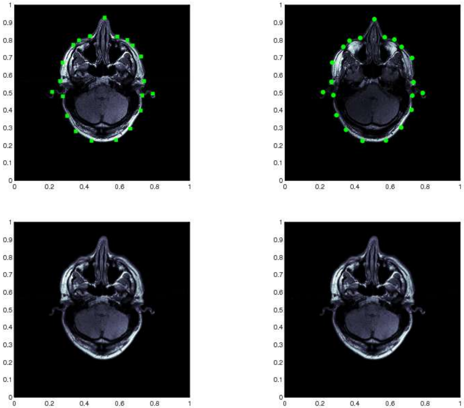



Figure: (Above) 21 landmarks are plotted with squares on the source image (left) and with dots on the target image (left). (Below) The registered image via eigen-rational interpolants computed with the W2 (left) and M2 (right) kernels and shape parameter $\varepsilon = 0.1$.

# Image registration: mean error comparison

$$M = \left( \frac{\sum_{\boldsymbol{s} \in \mathcal{S}} \|\boldsymbol{s} - \boldsymbol{F}(\boldsymbol{s})\|_2^2}{\#\mathcal{S}} \right)^{1/2}.$$

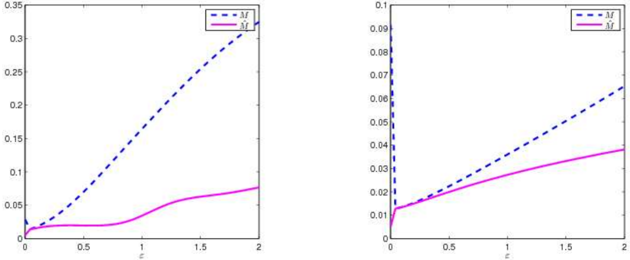



Figure: Mean errors $M$ (standard) and $\hat{M}$ (rational), varying the shape parameter.

Breve resoconto del progetto GNCS 2016-17

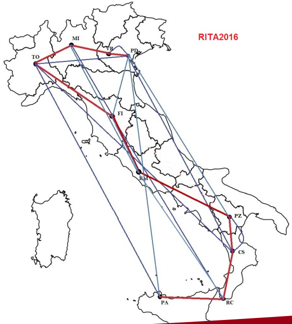
# Alcuni dati

1. Finanziamento richiesto/ricevuto: 9.0K/7.8K euro
2. Partecipanti: 15 strutturati, 9 non strutturati;
3. Quote individuali: circa 300 euro
4. Organizzati dai componenti i seguenti meetings: Bernried17, MATAA17 (Torino), CMMSE (Cadiz), DRWA17 (Canazei), SMART17(Gaeta), AMTA17 (Palermo).
5. Pubblicazioni relative al progetto: CAA Padova+Verona(22+13), Milano(3), Torino(10+2), Firenze(4+1), Potenza(9+2), Cosenza(4+?), Reggio Calabria(4), Palermo(5).

# RITA

## RITA (Rete ITaliana di Approssimazione): sito

https://sites.google.com/site/italianapproximationnetwork/




# Some references


- R. Cavoretto, S. De Marchi, A. De Rossi, E. Perracchione, G. Santin, *Partition of unity interpolation using stable kernel-based techniques*, Appl. Numer. Math. **116** (2017), pp. 95–107
- R. Cavoretto, A. De Rossi, E. Perracchione, *Optimal selection of local approximants in RBF-PU interpolation*, to appear on J. Sci. Comput. (2017).
- S. De Marchi, G. Santin, *Fast computation of orthonormal basis for RBF spaces through Krylov space methods*, BIT **55** (2015), pp. 949–966.
- Stefano De Marchi, Andrea Idda and Gabriele Santin: A rescaled method for RBF approximation. Springer Proceedings on Mathematics and Statistics, Vol. 201, pp.39–59. (2017).
- S. De Marchi, E. Perracchione, A. Martinez Calomardo and M. Rossini: RBF-based partition of unity method for elliptic PDEs: adaptivity and stability issues via VSKs, submitted to J. Sci. Comput. 2017
- S. De Marchi, E. Perracchione, A. Martinez Calomardo : Rational Radial Basis Functions interpolation , submitted to J. Comput. Appl. Math. 2017
- M. Buhmann, S. De Marchi, E. Perracchione : Analysis of a new class of rational RBF expansions, darft, 2018
- E. Perracchione: Rational RBF-based partition of unity method for efficiently and accurately approximating 3D objects arXiv preprint arXiv:1802.01842, 2018. To appear in Comput. Appl. Math.

grazie per la vostra attenzione!
thanks for your attention!