Multivariate Christoffel functions and hyperinterpolation[1]

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[1] Joint work with Alvise Sommariva and Marco Vianello

Motivation

- Len Bos, Multivariate interpolation and polynomial inequalities, Ph.D. course held in 2001 at the University of Padova (unpublished notes)

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- He proved by means of the bivariate Christoffel-Darboux formula of Xu that the Lebesgue constant of the Morrow-Patterson points, $\Lambda_n^{MP} = \mathcal{O}(n^6)$.
- Morrow-Patterson points were the basis of inspiration of the Padua Points.

Outline

1. $K \subset \mathbb{R}^d$, $\mathbb{P}_n^d(K)$, $N = \dim(\mathbb{P}_n^d(K)) := \binom{n+d}{d}$;

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2. $K_n(\boldsymbol{x}, \boldsymbol{y})$: reproducing kernel of $\mathbb{P}_n^d(K)$ in $L^2_{d\mu}(K)$ (μ a positive measure on K) with representation (cf. Dunkl and Xu 2001, §3.5)

$$K_n(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{N} p_j(\boldsymbol{x}) p_j(\boldsymbol{y}) \ , \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d \ , \tag{1}$$

where $\{p_j\}$ is any orthonormal basis of $\mathbb{P}_n^d(K)$ in $L^2_{d\mu}(K)$. The function

$$K_n(\boldsymbol{x}, \boldsymbol{x}) = \sum_{j=1}^{N} p_j^2(\boldsymbol{x}) \tag{2}$$

is known as the (reciprocal of) the n-th Christoffel function of μ on K.

Definition

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Practically

It is a total-degree polynomial approximation of multivariate continuous functions, given by a truncated Fourier expansion in o.p. for the given domain

We observe the following fact.

Let $\{a_n\} \in \mathbb{R}_+$ be a sequence s.t.

$$a_n \geq C_n(d\mu, K) = \sqrt{\max_{\boldsymbol{x} \in K} K_n(\boldsymbol{x}, \boldsymbol{x})} \quad (3)$$

Let

$$\mathcal{L}_n : \left(C(K), \|\cdot\|_{L^\infty(K)}\right) \to \left(\mathbb{P}_n^d, \|\cdot\|_{L^2_{d\mu}(K)}\right) \quad (4)$$

uniformly bounded operators, i.e. $\exists\, M > 0$ s.t. for every n

$$\|\mathcal{L}_n\| = \sup_{f \neq 0} \frac{\|\mathcal{L}_n f\|_{L^2_{d\mu}(K)}}{\|f\|_{L^\infty(K)}} \leq M\,.$$

Then this estimate holds:

$$\|\mathcal{L}_n\|_\infty = \sup_{f \neq 0} \frac{\|\mathcal{L}_n f\|_{L^\infty(K)}}{\|f\|_{L^\infty(K)}} \leq a_n M\,. \quad (5)$$

Given

- cubature formula $(X, \boldsymbol{w})$ for μ, exact in $\mathbb{P}^d_{2n}(K)$, with nodes $X = X_n = \{\boldsymbol{\xi}_i(n)\,,\ i = 1, \ldots, \mathcal{V}\} \subset K$ and positive weights $\boldsymbol{w} = \boldsymbol{w}_n = \{w_i(n)\,,\ i = 1, \ldots, \mathcal{V}\}$, $\mathcal{V} \geq N = \dim(\mathbb{P}^d_n(K))$,
- $\{p_j\,,\ j = 1, \ldots, N\}$ be any orthonormal basis of $\mathbb{P}^d_n(K)$ in $L^2_{d\mu}(K)$.
- hyperinterpolation operator is the discretized orthogonal projection $\mathcal{L}_n : C(K) \to \mathbb{P}^d_n(K)$ defined as

$$\mathcal{L}_n f(\boldsymbol{x}) = \sum_{j=1}^{N} \langle f, p_j \rangle_{\ell^2_{\boldsymbol{w}}(X)}\, p_j(\boldsymbol{x})\ ,$$

where $\ell^2_{\boldsymbol{w}}(X)$ is equipped with the scalar product

$$\langle f, g \rangle = \sum_{i=1}^{\mathcal{V}} w_i f(\boldsymbol{\xi}_i) g(\boldsymbol{\xi}_i)\ .$$

Corollary 1

Assume that (3) holds, then

$$\|\mathcal{L}_n\|_\infty \leq a_n \sqrt{\mu(K)}\,. \qquad (6)$$

Proof. Following Sloan [JAT95], we can write by exactness in $\mathbb{P}^d_{2n}(K)$ and the Pythagorean theorem in $\ell^2_{\boldsymbol{w}}(X)$

$$\|\mathcal{L}_n f\|_{L^2_{d\mu}(K)} = \|\mathcal{L}_n f\|_{\ell^2_{\boldsymbol{w}}(X)} \leq \|f\|_{\ell^2_{\boldsymbol{w}}(X)} = \sqrt{\sum_{i=1}^{\mathcal{V}} w_i f^2(\xi_i)}$$

$$\leq \sqrt{\sum_{i=1}^{\mathcal{V}} w_i}\, \|f\|_{\ell^\infty(X)} = \sqrt{\mu(K)}\, \|f\|_{\ell^\infty(X)} \leq \sqrt{\mu(K)}\, \|f\|_{L^\infty(K)}\,,$$

so that in Proposition 1 we can take $M = \sqrt{\mu(K)}$. □

Here we use the Gegenbauer measure

$$W_\lambda(\boldsymbol{x}) = (1 - |\boldsymbol{x}|^2)^{\lambda - 1/2}\,, \quad \lambda > -\frac{1}{2}\,, \tag{7}$$

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Bos in [NZJM,94] proved

$$\mathcal{C}_n(W_0(\boldsymbol{x})\,d\boldsymbol{x}, B_d) \leq \sqrt{\frac{2}{\omega_d}\left(\binom{n+d}{d} + \binom{n+d-1}{d}\right)} = \mathcal{O}(n^{d/2})\,, \tag{8}$$

ω_d being the surface area of the unit sphere $S^d \subset \mathbb{R}^{d+1}$.

Later [Bloom, Bos, Levenberg, APM12] showed that $\mathcal{C}_n$ has polynomial growth on the ball for $d\mu = W_\lambda(\boldsymbol{x})\,d\boldsymbol{x}$, $\lambda \geq 0$. No explicit bounds were provided!

Estimates

formulas for $K = B_2$

Main ingredient: Zernike polynomials (see [Carnicer, Godés NA14]), orthogonal basis on the disk w.r.t. Lebesgue measure (used in optics)

$$\hat{Z}_h^m(r,\theta) = \begin{cases} \sqrt{\frac{2(h+1)}{\alpha_m}}\, R_h^m(r)\cos(m\theta)\ , & m \geq 0 \\ \sqrt{\frac{2(h+1)}{\alpha_m}}\, R_h^m(r)\sin(m\theta)\ , & m < 0 \end{cases} \tag{9}$$

for $0 \leq h \leq n$, $|m| \leq h$, $h - m \in 2\mathbb{Z}$, where

$$\alpha_m = \begin{cases} 2\ , & m = 0 \\ 1\ , & m \neq 0 \end{cases} \tag{10}$$

$$R_h^m(r) = (-1)^{(h-m)/2} r^m P^{m,0}_{(h-m)/2}(1 - 2r^2) \tag{11}$$

and $P_j^{m,0}$ is the corresponding Jacobi polynomial of degree j.

Estimates

formulas for the disk, $K = B_2$

Relevant property: for $0 \leq h \leq n$, $|m| \leq h$, $h - m \in 2\mathbb{Z}$

$$|\hat{Z}_h^m(r,\theta)| \leq \sqrt{\frac{2h+2}{\pi}} \;, \quad \boldsymbol{x} = (r\cos(\theta), r\sin(\theta)) \in B_2 \;.$$

$$K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{h=0}^{n} \sum_{|m|\leq h, h-m\in 2\mathbb{Z}} (\hat{Z}_h^m(r,\theta))^2 \leq \frac{1}{\pi} \sum_{h=0}^{n} \sum_{|m|\leq h, h-m\in 2\mathbb{Z}} (2h+2)$$

$$= \frac{1}{\pi} \sum_{h=0}^{n} (2h+2)(n-h+1) = \frac{1}{3\pi}(n+1)(n+2)(n+3) \;,$$

and hence

$$C_n(d\boldsymbol{x}, B_2) \leq \frac{1}{\sqrt{3\pi}} \sqrt{(n+1)(n+2)(n+3)} = \mathcal{O}(n^{3/2}) \;. \qquad (12)$$

Estimates

formulas for the cube, $K = [-1, 1]^d$

- Jacobi measure

$$d\mu = W_{\alpha,\beta}(\boldsymbol{x})\, d\boldsymbol{x}\ ,\quad W_{\alpha,\beta}(\boldsymbol{x}) = \prod_{i=1}^{d} (1 - x_i)^{\alpha}(1 + x_i)^{\beta}\ ,\quad \alpha, \beta > -1\ , \tag{13}$$

- Total-degree *orthonormal* product basis

$$\Pi_{\boldsymbol{k}}^{\alpha,\beta}(\boldsymbol{x}) = \prod_{i=1}^{d} \hat{P}_{k_i}^{\alpha,\beta}(x_i)\ ,\quad 0 \le |\boldsymbol{k}| \le n\ , \tag{14}$$

where $\boldsymbol{k} = (k_1, \ldots, k_d)$ with $k_i \ge 0$ and $|\boldsymbol{k}| = \sum_{i=1}^{d} k_i$, and $\hat{P}_m^{\alpha,\beta}$ denotes the m-th degree polynomial of the univariate orthonormal Jacobi basis with parameters α and β.

For $\max\{\alpha,\beta\} \geq -1/2$, $\max|\hat{P}_{k_i}^{\alpha,\beta}|$ at ± 1, then

$$|\hat{P}_m^{\alpha,\beta}(t)| \leq |\hat{P}_m^{\alpha,\beta}(\mathrm{sign}(\alpha-\beta))| = \sqrt{\frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)\Gamma(m+q+1)}{2^{\alpha+\beta+1}\, m!\,\Gamma(m+\min\{\alpha,\beta\}+1)}}$$

$$\leq c(\alpha,\beta)\, m^{q+1/2}\,, \quad t \in [-1,1]\,, \quad q = \max\{\alpha,\beta\} \geq -\frac{1}{2}\,, \qquad (15)$$

Estimates

formulas for $K = [-1,1]^d$

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \max_{\boldsymbol{x}\in[-1,1]^d} \sum_{0\le|\boldsymbol{k}|\le n} \left(\Pi_{\boldsymbol{k}}^{\alpha,\beta}(\boldsymbol{x})\right)^2$$

$$= \sum_{0\le|\boldsymbol{k}|\le n} \prod_{i=1}^{d} \left(\hat{P}_{k_i}^{\alpha,\beta}(\text{sign}(\alpha-\beta))\right)^2 \le (c(\alpha,\beta))^{2d} \sum_{0\le|\boldsymbol{k}|\le n} \prod_{i=1}^{d} k_i^{2q+1}$$

$$= (c(\alpha,\beta))^{2d} \sum_{k_1=0}^{n} k_1^{2q+1} \sum_{k_2=0}^{n-k_1} k_2^{2q+1} \cdots \sum_{k_d=0}^{n-\sum_{j=1}^{d-1} k_j} k_d^{2q+1} = \mathcal{O}(n^{(2q+2)d}) \,,$$

which gives the qualitative bound

$$C_n(W_{\alpha,\beta}(\boldsymbol{x})\, d\boldsymbol{x}, [-1,1]^d) = \mathcal{O}(n^{(q+1)d}) \,. \tag{16}$$

Special cases

$\alpha = \beta = 0$, *Legendre polynomials*

$$|\hat{P}_m^{0,0}(t)| \leq \hat{P}_m^{0,0}(1) = \sqrt{\frac{2m+1}{2}}\,, \quad t \in [-1,1]\,,$$

from which we have

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{0\leq|\boldsymbol{k}|\leq n} \prod_{i=1}^{d} \left(\hat{P}_{k_i}^{0,0}(1)\right)^2 = \frac{1}{2^d} \sum_{k_1=0}^{n} (2k_1+1) \sum_{k_2=0}^{n-k_1} (2k_2+1) \cdots \sum_{k_d=0}^{n-\sum_{j=1}^{d-1} k_j} (2k_d+1)\,, \qquad (17)$$

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$$C_n(dx,[-1,1]) = \frac{1}{\sqrt{2}}\,(n+1)\ , \tag{18}$$

$$C_n(d\boldsymbol{x},[-1,1]^2) = \frac{1}{2\sqrt{6}}\ \sqrt{(n+1)(n+2)(n^2+3n+3)}\ \ , \tag{19}$$

$$C_n(d\boldsymbol{x},[-1,1]^3) = \frac{1}{12\sqrt{10}}\ \sqrt{(n+1)(n+2)^2(n+3)(2n^2+8n+15)} \tag{20}$$

Special cases

$\alpha = \beta = -1/2$, *Chebyshev polynomials* of first kind

$$|\hat{P}_m^{-\frac{1}{2},-\frac{1}{2}}(t)| = |\hat{T}_m(t)| \leq \hat{T}_m(1) = \sqrt{\frac{2-\delta_{0,m}}{\pi}} \;, \;\; t \in [-1,1] \,,$$

which entails by a little algebra

$$\pi^d \max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \pi^d \sum_{0\leq|\boldsymbol{k}|\leq n} \prod_{i=1}^{d} \left(\hat{T}_{k_j}(1)\right)^2 = \sum_{k_1=0}^{n} (2-\delta_{0,k_1}) \sum_{k_2=0}^{n-k_1} (2-\delta_{0,k_2}) \cdots \sum_{k_2=0}^{n-\sum_{j=1}^{d-1} k_j} (2-\delta_{0,k_d})$$

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$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(x)\,dx,[-1,1]) = \frac{1}{\sqrt{\pi}}\ \sqrt{2n+1} \qquad (21)$$

(observe that (21) coincides with the bound in (8) for $d=1$),

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^2) = \frac{1}{\pi}\ \sqrt{2n^2+2n+1}\ , \qquad (22)$$

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^3) = \frac{1}{\sqrt{3\pi^3}}\ \sqrt{4n^3+6n^2+8n+3}\ . \qquad (23)$$

Special cases

$\alpha = \beta = 1/2$, *Chebyshev polynomials* of second kind

$$|\hat{P}_m^{\frac{1}{2},\frac{1}{2}}(t)| = |\hat{U}_m(t)| \le \hat{U}_m(1) = \sqrt{\frac{2}{\pi}}\,(m+1)\,, \quad t \in [-1,1]\,,$$

which leads to

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{0\le|\boldsymbol{k}|\le n} \prod_{i=1}^{d} \left(\hat{U}_{k_i}(1)\right)^2 = \left(\frac{2}{\pi}\right)^d \sum_{k_1=0}^{n} (k_1+1)^2 \sum_{k_2=0}^{n-k_1} (k_2+1)^2 \cdots \sum_{k_d=0}^{n-\sum_{j=1}^{d-1} k_j} (k_d+1)^2\,, \tag{24}$$

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$$C_n(W_{\frac{1}{2},\frac{1}{2}}(x)\,dx, [-1,1]) = \frac{1}{\sqrt{3\pi}} \sqrt{(n+1)(n+2)(2n+3)}\ , \qquad (25)$$

$$C_n(W_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x}, [-1,1]^2) = \frac{1}{3\pi\sqrt{10}} \sqrt{P_6(n)}\ , \qquad (26)$$

$P_6(n) = (n+1)(n+2)(n+3)(n+4)(2n^2+10n+15)$,

$$C_n(W_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x}, [-1,1]^3) = \frac{1}{18\sqrt{35\pi^3}} \sqrt{P_9(n)}\ , \qquad (27)$$

$P_9(n) = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(2n+7)(n^2+7n+18)$.

By Corollary 1, we can give upper bounds for the “Lebesgue constant”, $\|\mathcal{L}_n\|_\infty$, independent of the underlying cubature formula, whenever we are able to estimate the maximum of $K_n(\boldsymbol{x}, \boldsymbol{x})$.

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- Wade in [JMAA13] provided this bound in the d-ball

$$a_{d,\lambda} n^{(d-1)/2+\lambda} \leq \|\mathcal{L}_n\|_\infty \leq b_{d,\lambda} n^{(d-1)/2+\lambda} \;,\;\; n \text{ even} \;,\;\; d > 1 \;,$$

  $a_{d,\lambda}$ and $b_{d,\lambda}$ positive constants. This improves the bound $\mathcal{O}(n \log n)$ for hyperinterpolation w.r.t the Lebesgue measure on the disk ($\lambda = 1/2$, $d = 2$), by [Hansen et al. IMA JNA09].

- When $\lambda = 0$, Corollary 1 and (8) gives $\|\mathcal{L}_n\|_\infty = \mathcal{O}(n^{d/2})$, an overestimate by a factor $\sqrt{n}$ (for any d).

- For the Lebesgue measure on the disk ($\lambda = 1/2$, $d = 2$), by (6) and (12) we get $\|\mathcal{L}_n\|_\infty = \mathcal{O}(n^{3/2})$, again an overestimate by a factor $\sqrt{n}$.

- For $d = 3$, [Caliari at al. CMA08], showed that for *any* hyperinterpolation operator w.r.t the $d\mu = W_{-1/2,-1/2}(\boldsymbol{x})\, d\boldsymbol{x}$ (cf. (13)), the following estimate holds

$$\|\mathcal{L}_n\|_\infty = \mathcal{O}(\log^d n)\ . \tag{28}$$

An estimate of this kind was previously obtained in the case of hyperinterpolation at the Morrow-Patterson-Xu points of the square (cf. [Caliari at al. JCAM07]).

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- Corollary 1 and (21)-(23), gives $\|\mathcal{L}_n\|_\infty = \mathcal{O}(n^{d/2})$, again an ovestimate of the actual order of growth by a factor $(\sqrt{n}/\log(n))^d$.

- For other Jacobi measures, there are apparently no results in the literature for $d > 1$. By Corollary 1 we get

$$\|\mathcal{L}_n\|_\infty = \mathcal{O}(n^{(q+1)d}) \;, \tag{29}$$

  for *any* hyperinterpolation operator w.r.t. **any** Jacobi measure with $q = \max\{\alpha, \beta\} \geq -1/2$.

- Notice, for $d = 1$, the Lebesgue constant of interpolation at the zeros of $P_{n+1}^{\alpha,\beta}$ increases asymptotically like $\log(n)$ for $q \leq -1/2$, and like $n^{q+1/2}$ for $q > -1/2$, in view of a classical result by Szëgo.

  Hence, (29) is again an overestimate!

We specialized (29) to the case of the product Chebyshev measure of the second kind on the square.

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Morrow-Patterson points [SIAM JNA78]

For even degree n, the MP points are the set $\{(x_m, y_k)\} \subset (-1,1)^2$

$$x_m = \cos\left(\frac{m\pi}{n+2}\right), \quad y_k = \begin{cases} \cos\left(\frac{2k\pi}{n+3}\right) & m \text{ odd} \\ \cos\left(\frac{(2k-1)\pi}{n+3}\right) & m \text{ even} \end{cases} \qquad (30)$$

$1 \leq m \leq n+1$, $1 \leq k \leq \frac{n}{2}+1$.

The Morrow-Patterson points

plots

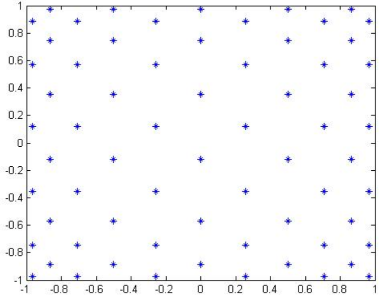

Figure: Left: MP points for $n = 10$. Right: MP points for $n = 20$

- The MP points are important for cubature on the square: minimal formulas of exactness $2n$ for Chebyshev measure of the second kind, $d\mu = W_{\frac{1}{2},\frac{1}{2}}(x_1, x_2)\, dx_1 dx_2$
- Len Bos in a manuscript of 2001, proved by means of the bivariate Christoffel-Darboux-Xu formula $\Lambda_n^{MP} = \mathcal{O}(n^6)$

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Using our approach of hyperinterpolation, we prove

The Lebesgue constant of bivariate polynomial interpolation at the Morrow-Patterson points has the following upper bound

$$\Lambda_n^{MP} \leq \frac{1}{6\sqrt{10}} \sqrt{(n+1)(n+2)(n+3)(n+4)(2n^2+10n+15)} = \mathcal{O}(n^3)\,. \tag{31}$$

Again, (31) is an overestimate. [Caliari et al. AMC05] showed that the values of Λ_n^{MP} are well-fitted by the quadratic polynomial $(0.7n + 1)^2$. Hence, it can be conjectured that the actual order of growth is $\Lambda_n^{MP} = \mathcal{O}(n^2)$.

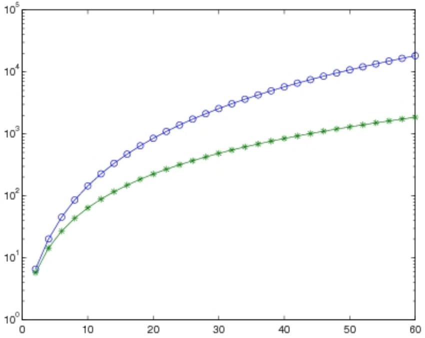

Figure: The upper bound (31) (∘) and the numerically evaluated Lebesgue constant (∗) of interpolation at the MPX points.

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#thankyou!