On the rescaled method for RBF approximation¹

Stefano De Marchi Department of Mathematics - University of Padova MAIA 2016 - Luminy (France) September 21, 2016



¹joint work with Andrea Idda (Verona) and Gabriele Santin (Stuttgart)



Outline

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- The rescaled kernel
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- Stability
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 - VSK
 - Lebesgue functions and constants
 - Lebesgue functions and constants
 - Future work



Introduction

RBF Approximation

- **1** Data: $\Omega \subset \mathbb{R}^n$, $X \subset \Omega$, test function *f*
 - $\blacksquare X = \{x_1, \ldots, x_N\} \subset \Omega$
 - f_1, \ldots, f_N , where $f_i = f(x_i)$
- **2** Approximation setting: kernel K_{ε} , $\mathcal{N}_{\kappa}(\Omega)$, $\mathcal{N}_{\kappa}(X) \subset \mathcal{N}_{\kappa}(\Omega)$
 - kernel K = K_ε, positive definite and radial examples:
 - globally supported: $K_{\varepsilon}(x, y) = e^{-(\varepsilon ||x-y||)^2}$ (gaussian),
 - locally supported: $K_{\varepsilon}(x, y) = (1 \varepsilon^2 ||x y||^2)_+^4 [4\varepsilon^2 ||x y||^2 + 1]$ ($C^2(\mathbb{R}^2)$ Wendland)
 - native space $N_{\kappa}(\Omega)$ (where K is the reproducing kernel)
 - finite subspace $\mathcal{N}_{\kappa}(X) = \operatorname{span}\{\mathcal{K}(\cdot, x) : x \in X\} \subset \mathcal{N}_{\kappa}(\Omega)$

Aim

Find $P_f \in \mathcal{N}_{\kappa}(X)$ s.t. $P_f \approx f$

The rescaled RBF interpolant



Deparis, Forti and Quarteroni [DFQ14, SISC 36(6)], proposed a new consistent Rescaled Localized RBF (RL-RBF) interpolant for large-scale problems based on Compactly Supported RBF (CSRBF).

Construction

On *X*, let us consider the constant function $g(\mathbf{x}) = 1$ and let $P_g(\mathbf{x})$ be the corresponding kernel-based interpolant. Letting $P_f(\mathbf{x})$ the interpolant of *f*, then the rescaled interpolant is

$$\hat{P}_f(\boldsymbol{x}) = \frac{P_f(\boldsymbol{x})}{P_g(\boldsymbol{x})} = \frac{\sum_{i=1}^N c_i K(\boldsymbol{x}, \boldsymbol{x}_i)}{\sum_{i=1}^N d_i K(\boldsymbol{x}, \boldsymbol{x}_i)}.$$
(1)

OBS: obviuosly

$$\hat{P}_f(X)=f(X).$$



Properties

- The interpolant is smoother even for small radii of the support (next slide)
- Thanks to the normalization, for each x_m they provide a strategy to select locally the shape parameter ϵ that considers the data distribution (↔ i.e the local radius of the CSRBF gives constant neighbor points) (cf. [DFQ14]).
- When points are uniformly distributed the previous strategy falls in a priori selection of the radius.
- When points are not uniformly distributed the previous strategy gives better results (using different circles diameters and neighbor points).

Remark

No theoretical study of the properties of the new interpolant were provided!



First example

As starting simple illustrative example we want to interpolate f(x) = x on the interval [0, 1] by using the W2 function at the points set $X = \{1/3, 2/3, 5/6\}, \varepsilon = 5$. Here the radius of the corresponding radial basis function is $r = 1/\epsilon$ so that on [0, 1] the rescaled interpolant never vanishes



Figure: (left) interpolants and (right) the abs error



Second example

Again, f(x) = x on [0, 1] by using W2 at the points set $X = \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1\}, \varepsilon = 5 \ (\epsilon = 1/r).$



Figure: (left) interpolants and (right) the abs error

For more results see [DFQ14, Idda15].



An example on 2d



Figure: RMSE behavior at 30 values of the shape parameter in [0.1, 10] for interpolation of the 2d Franke function (stationary) on a grid 5×5 , again with W2.

The rescaled kernel



The rescaled kernel

1 Assume that $P_f(\mathbf{x})$ and $P_g(\mathbf{x})$ be constructed by using the kernel *K* (with associate native space N_K).

2 Writing

$$\hat{P}_f(\boldsymbol{x}) = \frac{P_f(\boldsymbol{x})}{P_g(\boldsymbol{x})} = \sum_{j=1}^N c_j \frac{K(\boldsymbol{x}, \boldsymbol{x}_j)}{\sum_{i=1}^N d_i K(\boldsymbol{x}, \boldsymbol{x}_i)}, \qquad (2)$$

since $P_g(\mathbf{x}_j) = 1$, we have

$$\hat{P}_{f}(\boldsymbol{x}) = \sum_{j=1}^{N} c_{j} \left[\frac{K(\boldsymbol{x}, \boldsymbol{x}_{j})}{\sum_{i=1}^{N} d_{i}K(\boldsymbol{x}, \boldsymbol{x}_{i}) \sum_{i=1}^{N} d_{i}K(\boldsymbol{x}_{j}, \boldsymbol{x}_{i})} \right]$$

But $P_g(\mathbf{x}) = \sum_{i=1}^{N} d_i K(\mathbf{x}, \mathbf{x}_i)$ then $\left\{ \frac{K(\mathbf{x}, \mathbf{x}_j)}{P_g(\mathbf{x}) \cdot P_g(\mathbf{x}_j)}, j = 1, \dots, N \right\}$ can be interpreted as a (new) basis for the rescaled interpolant.



Theorem (Aronszajn50)

Let $K : \Omega \times \Omega \to \mathbb{R}$ be a (strictly) positive definite kernel. Let $s : \Omega \to \mathbb{R}$ a continuous and nonvanishing function on Ω . Then

$$K_{s}(\boldsymbol{x},\boldsymbol{y}) = s(\boldsymbol{x})s(\boldsymbol{y})K(\boldsymbol{x},\boldsymbol{y})$$
(3)

is (strictly) positive definite.

Setting $s = 1/P_g$, which is continuous, non-vanishing on Ω , the rescaled kernel is then

$$\boldsymbol{K}_{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{P_g(\boldsymbol{x})} \frac{1}{P_g(\boldsymbol{y})} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{y})$$
(4)

which turns out to be (strictly) positive definite and then we can consider its associate native space N_{K_r} .



Native space (ideas)

Definition

A function $m \in N_K$ is called a multiplier of N_K if the product $m \cdot f \in N_K$, $\forall f \in N_K$.

This definition provide the inclusion $N_{K_r} \subset N_K$ (since $1/P_g$ is a multiplier). The other inclusion could come from the following

Theorem (Aronszajn50)

Let K_1 and K_2 be two kernels on $\Omega \times \Omega$. Then $N_{K_1} \subset N_{K_2}$ if and only if for any $N \in \mathbb{N}$, $\boldsymbol{c} \in \mathbb{R}^N$ and points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ of Ω we have

 $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{A}_{1}\boldsymbol{c} \leq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{A}_{2}\boldsymbol{c}$

with $(A_i)_{j,k} = K_i(\mathbf{x}_j, \mathbf{x}_k), i = 1, 2.$

The Theorem in our case means to show that (or the reverse inequality)

$$\boldsymbol{c}^{T}\boldsymbol{A}_{K}\boldsymbol{c} \leq \boldsymbol{c}^{T}\boldsymbol{A}_{K_{r}}\boldsymbol{c}, \qquad (5)$$



Rescaling gives a Shepard method

We start by writing the interpolant of a function $f \in N_K$ using the

cardinals
$$u_j(x_i) = \delta_{i,j}$$
, $P_f = \sum_{j=1}^N f(\mathbf{x}_j) u_j$, so that for $g \equiv 1$ we get

$$P_g = \sum_{j=1}^{N} u_j$$
. The rescaled interpolant is then

$$\hat{P}_{f} = \frac{\sum_{j=1}^{N} f(\mathbf{x}_{j}) u_{j}}{\sum_{k=1}^{N} u_{k}} = \sum_{j=1}^{N} f(\mathbf{x}_{j}) \frac{u_{j}}{\sum_{k=1}^{N} u_{k}} =: \sum_{j=1}^{N} f(\mathbf{x}_{j}) \hat{u}_{j},$$

where we introduced the new cardinal functions $\hat{u}_j := \frac{u_j}{\sum_{k=1}^{N} u_k}$.

Corollary

The rescaled interpolation method is a Shepard's method, where the weight functions are defined as $\hat{u}_j = u_j / (\sum_{k=1}^N u_k)$, $\{u_j\}_j$ being the cardinal basis of span $\{K(\cdot, x), x \in X\}$.



Stability estimate

With the usual notation, we can define the Lebesgue function and constant for the rescaled interpolant

$$\hat{\Lambda}_N(\boldsymbol{x}) := \sum_{j=1}^N |\hat{u}_j(\boldsymbol{x})|, \ \ \hat{\lambda}_N := \|\hat{\Lambda}_N\|_{\infty,\Omega},$$

getting the stability bound

$$\|\hat{P}_{f}\|_{\infty,\Omega} \leq \hat{\lambda}_{N} \|f\|_{\infty,X}.$$
(6)

Hence, to quantify the stability gain of the rescaled interpolation process over the standard one, we can then compare the behavior of $\hat{\lambda}_N$ and λ_N (\hookrightarrow see experiments)

Applications and numerical experiments



A first test

Comparison of RMSE for the classical and the rescaled interpolant on varying ε by Trial&Error



The stair behaviour of the rescaled interpolant is due to the "patching" algorithm used to avoid that $P_g(\mathbf{x}) = 0$



Application to PUM

 $\Omega = \cup_{k=1}^{n} \Omega_k$ and compactly supported functions, $supp(w_k) \subseteq \Omega_k$,

$$\sum_{i=1}^{n} w_i(\boldsymbol{x}) = 1, \quad \forall \boldsymbol{x} \in \Omega.$$
(7)

The application of the rescaling to every local interpolant gives a global rescaled interpolant of the form

$$P_{f}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{P}_{i}(\boldsymbol{x}) w_{i}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$
(8)

with

$$\hat{P}_{i}(\mathbf{x}) = \sum_{j=1}^{n_{i}} c_{j}^{(i)} \frac{\mathcal{K}^{(i)}(\mathbf{x}, \mathbf{x}_{j})}{\mathcal{P}_{g}^{(i)}(\mathbf{x})} = \sum_{j=1}^{n_{i}} c_{j}^{(i)} \frac{\mathcal{K}^{(i)}(\mathbf{x}, \mathbf{x}_{j})}{\sum_{l=1}^{n_{i}} d_{l}^{(i)} \mathcal{K}^{(i)}(\mathbf{x}, \mathbf{x}_{l})}, \text{ with } n_{i} = \#(X \cap \Omega_{i})$$

where the coefficients $d_l^{(i)}$ are chosen so that $\sum_{l=1}^{n_l} d_l^{(i)} \mathcal{K}^{(i)}(\boldsymbol{x}, \boldsymbol{x}_l) = 1, \quad \forall \boldsymbol{x} \in \Omega_i.$



Rescaled PU (RPU): example

Consider the 2d Askley's test function [R15]

$$f(x,y) = -20 \ e^{-0.2 \sqrt{0.5(x^2 + y^2)}} - e^{-0.5(\cos(2\pi x) + \cos(2\pi y))} + 20 + e \quad (9)$$

interpolated on 1000 Halton points on the disk centered in (0.5, 0.5) and radius 0.5 with W2. As evalution points we took 10000 uniformly distributed points of the disk.



Figure: RMSE (Left) and MAXERR (Right) for the classical and the rescaled PU for $\varepsilon \in [0.01, 2]$.



Some numerical evidences

- RPU reaches the same precision of PU, but using a "thinner" point set X
- The evaluation time heuristically is $T_{RPU} < cT_{PU}$ with $c \approx 1.05$
- P_f and P_g share the same collocation matrix, so that the linear systems differ only by constant terms. Hence one can use specific algorithms to speed-up the evaluation step



- VSK, introduced in [BLRS, IMA JNA15], are intended to give more flexibility to RBF approximations.
- Let $c : \mathbb{R}^d \to (0, \infty)$ be a scale function. A VSK on \mathbb{R}^d is

$$K_c(\boldsymbol{x}, \boldsymbol{y}) = K((\boldsymbol{x}, c(\boldsymbol{x})), (\boldsymbol{y}, c(\boldsymbol{y}))$$

If Φ is radial, the new kernel takes the form

$$K_c(\boldsymbol{x}, \boldsymbol{y}) := \Phi(\|\boldsymbol{x} - \boldsymbol{y}\|^2 + (c(\boldsymbol{x}) - c(\boldsymbol{y}))^2)$$



VSK (cont')

The scale function transforms a problem with data $\mathbf{x}_j \in \mathbb{R}^d$ to data location $(\mathbf{x}_j, c(\mathbf{x}_j)) \in \mathbb{R}^{d+1}$ and then use a fixed-scale kernel on \mathbb{R}^{d+1} .

Letting $\sigma : \mathbf{x} \to (\mathbf{x}, c(\mathbf{x}))$ map from \mathbb{R}^d into a *d*-dimensional submanifold $\sigma(\mathbb{R}^d)$ of \mathbb{R}^{d+1}

$$\mathcal{K}_{c}(\mathbf{x}, \mathbf{y}) = \mathcal{K}(\underbrace{(\mathbf{x}, c(\mathbf{x}))}_{\sigma(\mathbf{x})}, \underbrace{(\mathbf{y}, c(\mathbf{y}))}_{\sigma(\mathbf{y})})$$

Hence the interpolant satisfies

$$P_{\sigma,f,X}(\boldsymbol{x}) = P_{1,f,\sigma(X)}(\boldsymbol{x}, \boldsymbol{c}(\boldsymbol{x})) = P_{1,f,\sigma(X)}(\sigma(\boldsymbol{x})).$$
(10)

Example: Classical vs VSK vs Rescaled

Franke test function, W2 sampled on 200 e.s.points of half unit sphere centered in (0,0,0). The nodes in \mathbb{R}^2 are the projections on the unit disk, i.e. $c(\mathbf{x}) = \sqrt{1 - x_1^2 - x_2^2}$. The evaluation points in \mathbb{R}^2 are obtained by restricting the grid 100 × 100 of the square $[-1, 1]^2$ to the unit disk, while the points in \mathbb{R}^3 are obtained by the map $\sigma(\mathbf{x}) = (\mathbf{x}, c(\mathbf{x}))$ (in Figure 5 we show only 100 points). The shape parameter is $\varepsilon = 5$



Figure: The points of the VSK example



Example (cont')



Figure: Classical, VSK and Rescaled method for W2 on varying the shape parameter

	Standard	+R	Cond(A)
Standard	2.2 <i>e</i> – 01	5.1 <i>e</i> – 02	$\approx 10^2$
+VS	1.5 <i>e</i> + 00	9.6 <i>e</i> - 02	pprox 10 ⁰

Table: RMSE with/without rescaling applied to VSK



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Comparison to accurate PUM

- In the recent work [CDeRP, DRNA16] a accurate PUM (A-PUM), combined with an optimal local RBF approximation via a *priori* error estimates (cf. DRWA15 poster) has been presented.
- The a method enables to select both suitable sizes of the different PU subdomains and shape parameters, i.e. the optimal couple (r_i^*, ϵ_i^*) for the subdomain Ω_j .
- the method uses a Bivariate LOOCV strategy.
- the method is suitable for data with non-homogeneous density.

#DataP	#EvalP	Method	RMSE	CPU time (sec)
289	1600	A-PUM	5.72e-4	3.44
		PU	4.34e-2	0.32
		RPUM	1.50e-2	0.32
1024	2500	A-PUM	1.32e-4	13.66
		PU	1.54e-2	0.63
		RPUM	7.55e-3	0.66
2500	6400	A-PUM	6.67e-5	32.42
		PU	6.14e-3	1.32
		RPUM	2.89e-3	1.30

Table: Comparison between A-PUM, PUM and RPUM with $\epsilon = 5$ with W2, on various grids on the square $[0, 1]^2$, for interpolation of $f(x_1, x_2) = (x_1^2 + y_2^2 - 1)^9$



Lebesgue functions

Domain: cardiod contained in $[-1, 1]^2$. Data set: grid of 5×5 points.



Figure: Comparison between the Lebesgue function with standard basis (Left) and the rescaled one (right) for the C^2 Wendland kernel on the cardiod with $\varepsilon = 3$.



Lebesgue functions (cont')



Figure: As in the previous slide with $\varepsilon = 3.85$.

Notice: for values of $\varepsilon \ge 2\varepsilon_M$ (in the example $\epsilon_M \approx 2$) the cardinal functions have disjoint supports.



Lebesgue constants



Figure: Comparison between the standard Lebesgue function (solid line) and the rescaled Lebesgue function (dotted line) for the C^2 Wendland kernel. From top left to bottom right, $\varepsilon = 0.5, 1, 2, 4$.



An heuristic comparison of the "classical wrt Rescaled" and "classical PUM wrt RPUM"

- If $\epsilon \to 0$ or " $\epsilon \to \infty$ ", both methods give same results.
- There exists $\overline{\epsilon}$ s.t., defining $I = (\overline{\epsilon} \delta_1, \overline{\epsilon} + \delta_2)$, we have
 - if ε ∈ (ε − δ₁, ε), the standard approach behaves better (sometimes δ₁ = 0)
 - if $\epsilon \in (\overline{\epsilon}, \overline{\epsilon} + \delta_2)$, the rescaled is better. In general $\delta_1 \ll \delta_2$.
- The rescaled is much less sensitive to the shape parameter.



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Generalized Rescaled Interpolant

 $K_r(\mathbf{x},\mathbf{y}) := s(\mathbf{x})s(\mathbf{y})K(\mathbf{x},\mathbf{y})$

preserves the properties of the original Kernel. Define *s* such that

$$s(\mathbf{x}) = s_{\gamma}(\mathbf{x}) := rac{1}{P_g(\mathbf{x})^{\gamma}} = rac{1}{(\sum_{j=1}^N c_j K(\mathbf{x}, \mathbf{x}_j))^{\gamma}}$$

 γ is an oscillation parameter.

The generalized rescaled kernel takes the form

$$\begin{split} \mathcal{K}_{r,\gamma}(\mathbf{x},\mathbf{y}) &:= \frac{1}{(\sum_{j=1}^{N} c_j \mathcal{K}(\mathbf{x},\mathbf{x}_j))^{\gamma}} \frac{1}{(\sum_{j=1}^{N} c_j \mathcal{K}(\mathbf{y},\mathbf{x}_j))^{\gamma}} \mathcal{K}(\mathbf{x},\mathbf{y}) = \\ &= \frac{1}{\left(\sum_{j=1}^{N} c_j \mathcal{K}(\mathbf{x},\mathbf{x}_j) \sum_{j=1}^{N} c_j \mathcal{K}(\mathbf{y},\mathbf{x}_j)\right)^{\gamma}} \mathcal{K}(\mathbf{x},\mathbf{y}) \quad . \end{split}$$

Notice: $K_{r,0} = K$.



Figure: $X = \{\frac{1}{5}, \frac{1}{2}, \frac{5}{6}\}, y = x$ (green), Classic Interpolant (Blue), Generalized Interpolants (Red)



Polynomial reproduction ?



Figure: For $y = x^k$, $k \le 50$, $\epsilon = 3$, W2, X = linspace(0, 1, 10), we take 100 values for $\gamma \in [k/2 - 1, k/2 + 1]$. Similar results for different ϵ



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merci pour votre attention! thanks for your attention!