3-dimensional Weakly Admissible Meshes *

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Outline

Motivations and aims

- (Weakly) Admissible meshes, (W)AM: play a central role in the construction of multivariate polynomial approximation processes on compact sets.

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- (Weakly) Admissible meshes, (W)AM: play a central role in the construction of multivariate polynomial approximation processes on compact sets.
- Theory vs computation: 2-dimensional and (simple) 3-dimensional (W)AMs are easy to construct. What's about more general domains such as (truncated) cones or rotational sets such as toroidal domains?

Main references

1. J.P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008), 82–100.
2. L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *Computing multivariate Fekete and Leja points by numerical linear algebra*, SIAM J. Numer. Anal. 48 (2010), 1984–1999.
3. F. Piazzon and M. Vianello, *Analytic transformations of admissible meshes*, East J. Approx. 16 (2010), 313–322.
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5. S. De Marchi, M. Marchiori and A. Sommariva, *Polynomial approximation and cubature at approximate Fekete and Leja points of the cylinder*, submitted 2011.
6. M. Briani, A. Sommariva and M. Vianello, *Computing Fekete and Lebesgue points: simplex, square, disk*, submitted 2011.
7. L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *On Multivariate Newton Interpolation at Discrete Leja Points* , submitted (2011).
8. L. Bos and M. Vianello, *Subperiodic trigonometric interpolation and quadrature*, submitted (2011).

(Weakly) Admissible Meshes, (W)AM

Given a polynomial determining compact set $K \subset \mathbb{R}^d$.

Definition

An Admissible Mesh is a sequence of finite discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C\|p\|_{\mathcal{A}_n}\,, \quad \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

holds for some $C > 0$ with $\mathrm{card}(\mathcal{A}_n) \geq N := dim(\mathbb{P}_n^d(K))$ that grows at most polynomially with n.

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- These sets and inequalities are also known as: (L^∞) discrete norming sets, Marcinkiewicz-Zygmund inequalities, stability inequalities (in more general functional settings).
- Optimal Admissible Meshes the ones with $\mathcal{O}(n^d)$ cardinality and can be constructed for some classes of compact sets (cf. [Kroó 2011], [Piazzon/Vianello 2010]).

Admissible Meshes

In principle an AM of Markov compacts, i.e. $K \subset \mathbb{R}^d$ s.t.

$$\|\nabla p\|_K \leq M n^r \|p\|_K, \ \ \forall\, p \in \mathbb{P}_n^d(K)\,,$$

where $\|\nabla p\|_K = \max_{x \in K} \|\nabla p(x)\|_2$

Construction idea: take a uniform discretization of K with spacing $\mathcal{O}(n^{-r})$. The mesh will have cardinality of $\mathcal{O}(n^{rd})$ for real compacts or $\mathcal{O}(n^{2rd})$ for general complex domains.

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$r = 2$ for many (real convex) compacts: the construction and use of AM becomes difficult even for $d = 2, 3$ already for small degrees.

TOO BIG!!

Weakly Admissible Meshes: properties

P1: $C(\mathcal{A}_n)$ is invariant for affine transformations.

P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself

P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$

P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants

P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants

P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constants $C(\mathcal{A}_{ns})$ (cf. [Bos et al. 2009])

Weakly Admissible Meshes: properties

P8: any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \leq Mn^r \|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. [Calvi/Levenberg 2008])

P9: The least-squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\, \min\{\|f-p\|_K,\, p \in \mathbb{P}_n^d(K)\}$$

P10: The Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq NC(\mathcal{A}_n)$

Moreover, their asymptotic distribution is the same of the continuum Fekete points, in the sense that the corresponding discrete probability measures converge weak-$*$ to the pluripotential equilibrium measure of K (cf. [Bos et al. 2009])

2-dimensional WAMS: disk, triangle, square

It was proved in [Bos at al. 2009] that, for the disk and the triangle there are WAMs with approximately n^2 points and the growth of $C(\mathcal{A}_n)$ is the same of an AM.

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- Unit simplex: starting from the WAM of the disk for polynomials of degree $2n$ containing only even powers, by the standard quadratic transformation

$$(u, v) \longmapsto (x, y) = (u^2, v^2)\,.$$

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- Square: Chebyshev-Lobatto grid, Padua points.

Notice: by affine transformation these WAMs can be mapped to any other triangle (**P1**) or polygon (**P4**).

Polar symmetric WAMs for the disk

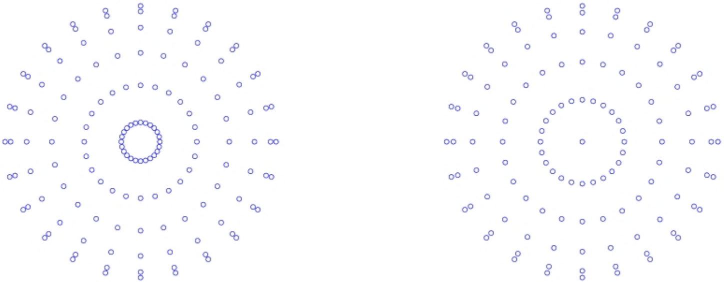


Figure: Symmetric polar WAMs for the disk: (Left) for degree $n = 11$ with $144 = (n+1)^2$ points, (Right) for $n = 10$ with $121 = (n+1)^2$ points.

WAMs for the quadrant and the triangle

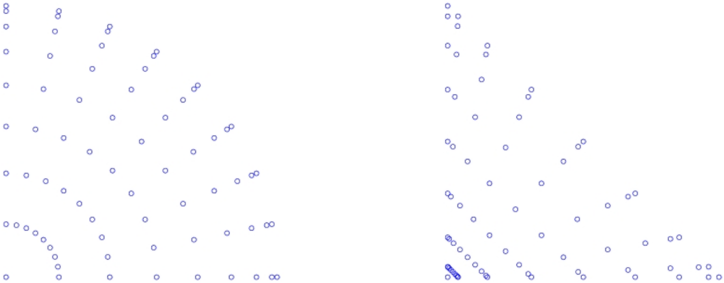


Figure: A WAM of the first quadrant for polynomial degree $n = 16$ (left) and the corresponding WAM of the simplex for $n = 8$ (right).

Optimal Lebesgue Gauss–Lobatto points on the triangle

A new set of optimal Lebesgue Gauss-Lobatto points on the simplex has recently been investigated by [Briani/Sommariva/Vianello 2011].
These points minimize the corresponding Lebesgue constant on the simplex, that grows like $\mathcal{O}(n)$.

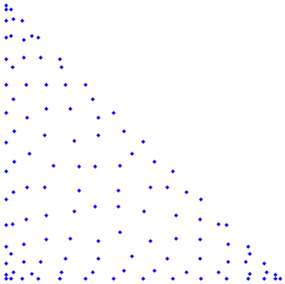


Figure: The optimal points for $n = 14$, cardinality $(n+1)(n+2)/2$.

WAMs for a quadrangle

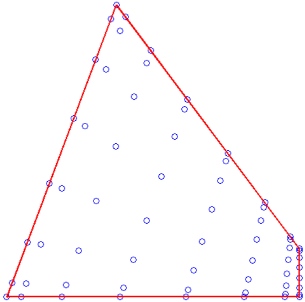
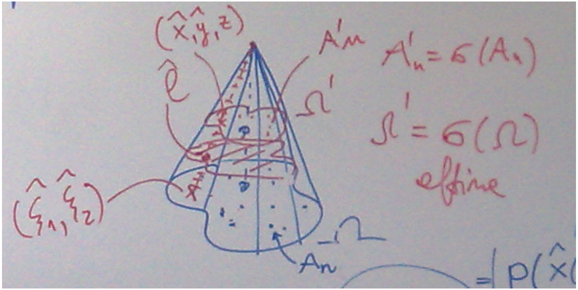


Figure: A WAM for a quadrangular domain for $n = 7$ obtained by the bilinear transformation of the Chebyshev–Lobatto grid of the square $[-1, 1]^2$

$$\frac{1}{4}[(1-u)(1-v)A+(1+u)(1-v)B+(1+u)(1+v)C+(1-u)(1+v)D]$$

WAMs for (truncated) cones

Starting from a 2-dimensional domain WAM, we "repeat" the mesh along a Chebsyhev-Lobatto grid of the z-axis, as shown here in my handwritten notes (on my whiteboard).



Why these are WAMs?

From the previous picture

$$|p(x,y,z)| \quad \leq \quad C(A_n)\|p\|_{A'_n(z)} \;\; C(A_n) \equiv C(A'_n(z))$$

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From the previous picture

$$
\begin{aligned}
|p(x,y,z)| &\leq C(A_n)\|p\|_{A'_n(z)} \;\; C(A_n) \equiv C(A'_n(z)) \\
\|p\|_{A'_n(z)} &= |p(\hat{x}_z,\hat{y}_z,z)| \; with \;\; (\hat{x}_z,\hat{y}_z,z) \in A'_n(z)
\end{aligned}
$$

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\begin{aligned}
|p(x,y,z)| &\leq C(A_n)\|p\|_{A_n'(z)} \quad C(A_n) \equiv C(A_n'(z)) \\
\|p\|_{A_n'(z)} &= |p(\hat{x}_z, \hat{y}_z, z)| \; with \; (\hat{x}_z, \hat{y}_z, z) \in A_n'(z) \\
&\leq C(A_n)\|p\|_{\ell(\hat{\xi}_1, \hat{\xi}_2)} \; where \; (\hat{\xi}_1, \hat{\xi}_2) \in A_n \\
&\leq C(A_n) \max_{(x,y)\in A_n} \|p\|_{\ell(x,y)} \\
&\leq \mathcal{O}(C(A_n)\log_n) \max_{(x,y)\in A_n} \|p\|_{\Gamma_n} = \mathcal{O}(C(A_n)\log_n)\|p\|_{B_n}
\end{aligned}
$$

where Γ_n are the Chebyshev-Lobatto points of $l(x,y)$ and $B_n = \bigcup_{(x,y)\in A_n} \Gamma_n(\ell(x,y))$.

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$$\begin{array}{rcl} |p(x,y,z)| & \leq & C(A_n)\|p\|_{A'_n(z)} \ \ C(A_n) \equiv C(A'_n(z)) \\ \|p\|_{A'_n(z)} & = & |p(\hat{x}_z, \hat{y}_z, z)| \ with \ \ (\hat{x}_z, \hat{y}_z, z) \in A'_n(z) \\ & \leq & C(A_n)\|p\|_{\ell(\hat{\xi}_1, \hat{\xi}_2)} \ where \ (\hat{\xi}_1, \hat{\xi}_2) \in A_n \\ & \leq & C(A_n) \max_{(x,y)\in A_n} \|p\|_{\ell(x,y)} \\ & \leq & \mathcal{O}(C(A_n)\log_n) \max_{(x,y)\in A_n} \|p\|_{\Gamma_n} = \mathcal{O}(C(A_n)\log_n)\|p\|_{B_n} \end{array}$$

where Γ_n are the Chebyshev-Lobatto points of $l(x,y)$ and $B_n = \bigcup_{(x,y)\in A_n} \Gamma_n(\ell(x,y))$.
Cardinality.

$$\#B_n = (n+1)\#A_n - \#A_n + 1 = 1 + n\#A_n = \mathcal{O}(n^3)$$

WAMs for a cone

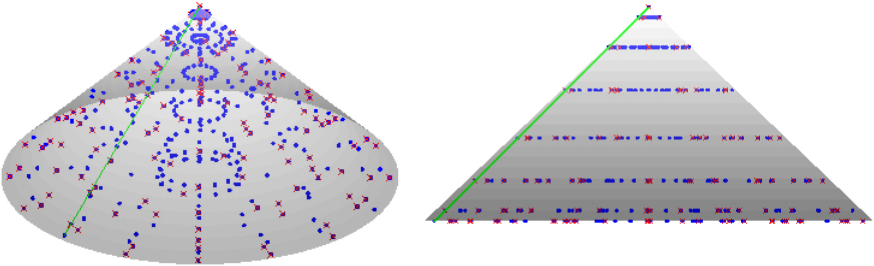
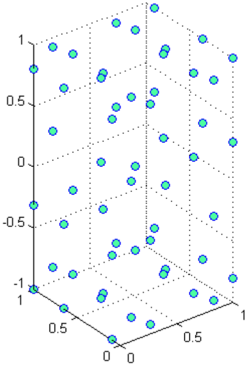


Figure: A WAM for the rectangular cone for $n = 7$

Here $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3)$

A low dimension WAM for the cube

The cube can be considered as a *cylinder with square basis*. WAMs for the cube with dimension $\mathcal{O}(n^3/4)$ were studied in [DeMarchi/Vianello/Xu 2009] in the framework of cubature and hyperinterpolation.
A WAM for the cube that for n even has $(n+2)^3/4$ points and for n odd $(n+1)(n+2)(n+3)/4$ points, is show here for a parallelpiped with $n = 4$ (here $\#A_n = 54$)



WAMs for a pyramid

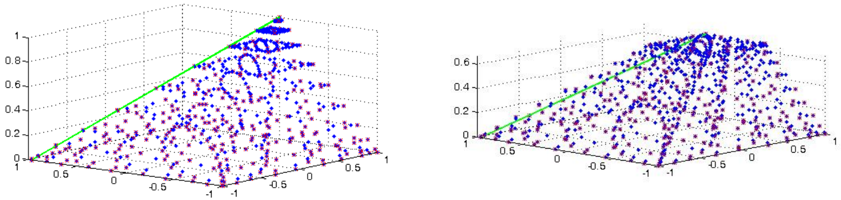
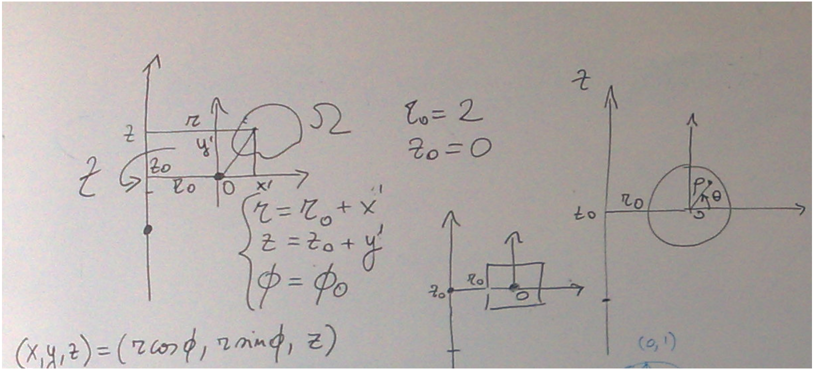


Figure: A WAM for a non-rectangular pyramid and a truncated one, made by using Padua points for $n = 10$. Notice the generating curve of Padua points that becomes a spiral

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3/2)$

WAMs for toroidal sections

Starting from a 2-dimensional WAM, A_n, by rotation around a vertical axis sampled at the $2n+1$ Chabyshev-Lobatto points of the arc of circumference, we get WAMs for the torus, sections of the torus and in general toroids.
The resulting cardinality will be $(2n+1) \times \#A_n$



Why these are WAMs?

From the previous "picture" Given a polynomial $p(x, y, z) \in \mathbb{P}_n^3$ we can write it in cylindrical coordinates getting

$$p(x, y, z) = q(r, z, \phi) = s(x', y', \phi) \in \mathbb{P}_n^{2,(x',y')} \otimes \mathbb{T}_n^{\phi}$$

since

$$x^i y^j x^k = (r\cos\phi)^i (r\sin\phi)^j z^k (r_0 + x')^i \cos^i\phi (r_0 + y')^j \sin^j\phi (r_0 + y')^k$$

WAMs for toroidal sections: points on the disk

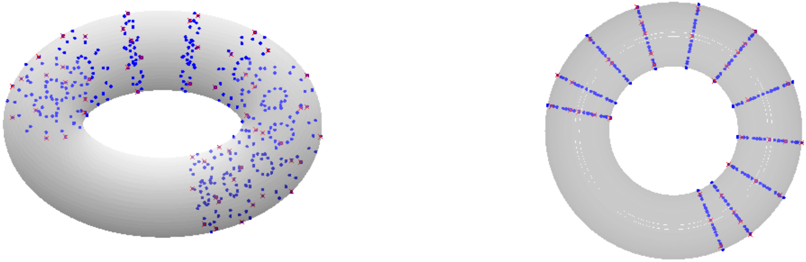


Figure: WAM for $n = 5$ on the torus centered in $z_0 = 0$ of radius $r_0 = 3$, with $-2/3\pi \leq \theta \leq 2/3\pi$.

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(2n^3)$

WAMs for toroidal sections: Padua points

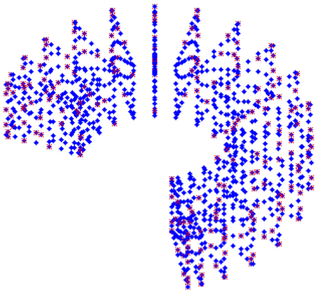


Figure: Padua points on the toroidal section with $z_0 = 0$, $r_0 = 3$ and opening $-2/3\pi \leq \theta \leq 2/3\pi$.

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3)$.

WAMs for toroidal sections: simplex, GLL points

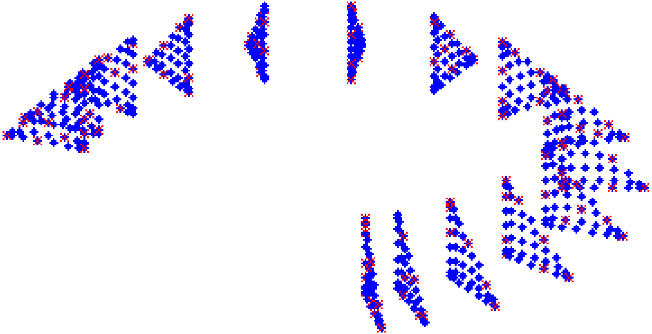


Figure: GLL points for $n = 7$ on the torus section

Cardinality is $\mathcal{O}(n^3)$

WAMs for toroidal sections: equilateral triangle, GLL points

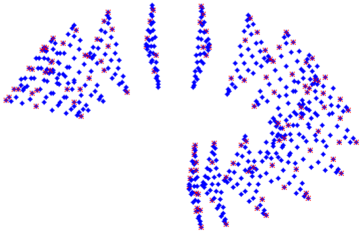


Figure: GLL points for $n = 7$ on the torus section for an equilateral triangle

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- The rectangular Vandermonde-like matrix

$$V(\mathbf{a};\mathbf{p}) = V(a_1,\dots,a_M;p_1,\dots,p_N) = [p_j(a_i)] \in \mathbb{C}^{M\times N}, \quad M \geq N$$

where $\mathbf{a} = (a_i)$ is the array of the points of $\mathcal{A}_n$ and $\mathbf{p} = (p_j)$ the basis of $\mathbb{P}_n^d$.

AFP and DLP

A greedy maximization of submatrix volumes, implemented by the QR factorization with column pivoting of $V(\mathbf{a};\mathbf{p})^t$ gives the so-called Approximate Fekete points [Sommariva/Vianello 2009].

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A greedy maximization of submatrix volumes, implemented by the QR factorization with column pivoting of $V(\mathbf{a};\mathbf{p})^t$ gives the so-called Approximate Fekete points [Sommariva/Vianello 2009].

A greedy maximization of nested square submatrix determinants, implemented by the LU factorization with row pivoting of $V(\mathbf{a};\mathbf{p})$ gives the so-called Discrete Leja points ([Bos/DeMarchi/et al. 2010] and already observed in [Schaback/De Marchi 2009]).

DLP and Multivariate Newton Interpolation

1. Consider the square Vandermonde matrix

$$V = V(\boldsymbol{\xi}, \mathbf{p}) = (P_0 V_0)_{1 \le i,j, \le N} := LU$$

where $V_0 = V(\mathbf{a}, \mathbf{p})$, $L = (L_0)_{1 \le i,j \le N}$ and $U = U_0$.

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2. The polynomial interpolating a function f at $\boldsymbol{\xi}$, $\mathbf{f} = f(\boldsymbol{\xi}) \in \mathbb{C}^N$ is

$$\mathcal{L}_n f(x) = \mathbf{c}^t \mathbf{p}(x) = (V^{-1}\mathbf{f})^t \mathbf{p}(x) = (U^{-1}L^{-1}\mathbf{f})^t \mathbf{p}(x) = \mathbf{d}^t \phi(x) \qquad (2)$$

where $\mathbf{d}^t = (L^{-1}\mathbf{f})^t$, $\phi(x) = U^{-t}\mathbf{p}(x)$.

Remarks

- Formula (2) is indeed a Newton-type interpolant.
  Since U^{-t} is lower triangular, the basis ϕ is s.t.

$$\mathtt{span}\{\phi_1, \ldots, \phi_{N_s}\} = \mathbb{P}_s^d, \;\; 0 \le s \le n$$

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- $$V(\boldsymbol{\xi}; \boldsymbol{\phi}) = V(\boldsymbol{\xi}; \mathbf{p})U^{-1} = LUU^{-1} = L$$

  Hence, $\phi_j(\xi_j) = 1$ and $\phi_j(x_i) = 0, \; i = 1, \ldots, j-1, \;$ when $j > 1$.

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- Case $d = 1$. Since $\phi_j \in \mathbb{P}^1_{j-1}$, then
  $\phi_j(x) = \alpha_j(x - x_1) \cdots (x - x_{j-1}), \; 2 \leq j \leq N = n+1$ with
  $\alpha_j = ((x_j - x_1) \cdots (x_j - x_{j-1}))^{-1}$, i.e. the classical Newton basis with d_j the classical divided differences up to $1/\alpha_j$.

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- The connection between LU factorization and Newton Interpolation was recognized by [de Boor 2004] and in a more general way by [R. Schaback et al. 2008, 2009].

Conic sections: disk

K is the cone. Given an n, then

- The AFP are extracted from a WAM having $\mathcal{O}(n^3)$ points
- The polynomial basis is the tensor product Chebyshev polynomial basis.
- The Lebesgue constant and the interpolation error has been computed on a mesh of control points (consisting of the original WAM with $2n$ instead of n).

Conic sections: disk

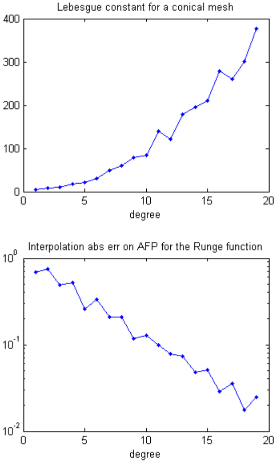
K is the cone. Given an n, then

- The AFP are extracted from a WAM having $\mathcal{O}(n^3)$ points
- The polynomial basis is the tensor product Chebyshev polynomial basis.
- The Lebesgue constant and the interpolation error has been computed on a mesh of control points (consisting of the original WAM with $2n$ instead of n).

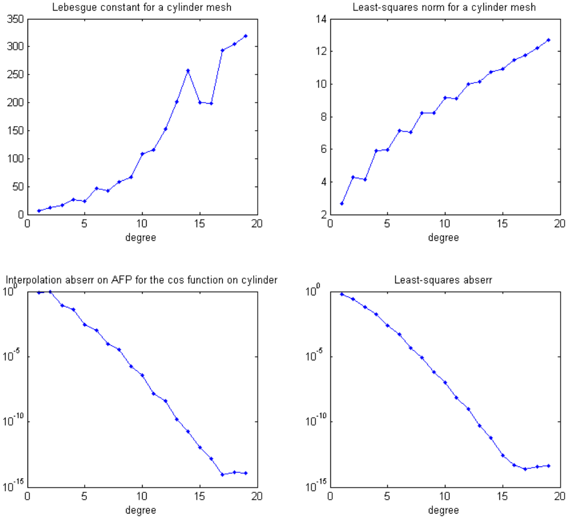
We also computed the

1. least-square operator norm, $\|L_{A_n}\| = \max_{x \in K} \sum_{i=1}^{M} |g_i(x)|$ where $g_i,\ i = 1, \ldots, M$ are a set of generators and $M \geq N = \dim \mathbb{P}_n^3$ (cf. [Bos/De Marchi et al. 2010])
2. interpolation error $\|f - p_n(f)\|_\infty$
3. least-square error $\|f - L_{A_n}(f)\|_\infty$

Runge function on the cone



Cosine function on the rectangular cylinder



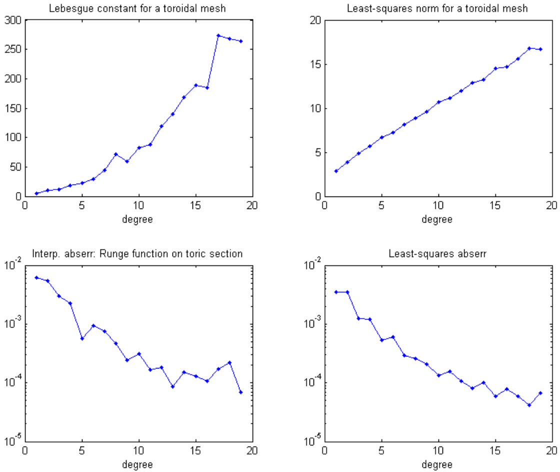
Toric sections: disk, square

K is now a toric section. Given n then

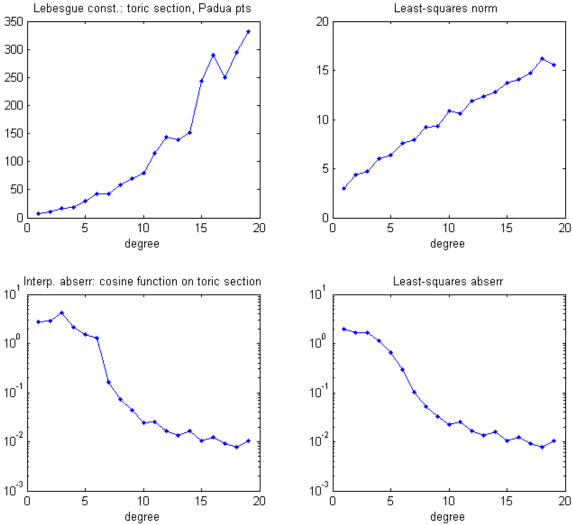
- The AFP are extracted from a WAM having $(n+1)^2(2n+1)$ points in the case of the disk and $\frac{(n+1)(n+2)}{2}(2n+1)$ in the case of the square (by using Padua points).
- The polynomial basis is the tensor product Chebyshev polynomial basis.
- The Lebesgue constant and the interpolation error has been computed on a mesh of control points (the original WAM of degree $2n$).

We computed as before least-square operator sup-norm, interpolation error $\|f - p_n(f)\|_\infty$ and least-square error $\|f - L_{A_n}(f)\|_\infty$.

Runge function on the toric section



Cosine function on the toric section using Padua points



Future works

- investigate other (general) domains
- correct polynomial basis for the domains
- improve the cputime for extraction of AFP and DLP
- applications: cubature, pdes, graphics and more
- RBF setting?
- ...

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Thank you for your attention