On the Lebesgue constant of the Floater-Hormann rational interpolants *

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Outline

Known things and aim

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- Michael S. Floater and Kai Hormann, *Barycentric rational interpolation with no poles and high rates of approximation*, Numer. Math. 107(2) (2007), 315–331.
- Floater and Hormann rational interpolants, FHRI, is a family of rational interpolants that perform rational interpolations on equispaced and non-equispaced points .
- From their paper... *"it seems to be perfectly stable in practice"*... but nothing was proved about its stability.
- The Lebesgue constant measures the stability of an interpolation process.
- FHRI is also on Numerical Recepies, section 3.4.1

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- The Lebesgue constant measures the stability of an interpolation process.
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Aim

What's about the growth of the Lebesgue constants for the FHRI?

Lagrange form of the interpolant

Given a function $f : [a, b] \to \mathbb{R}$, let g be its interpolant at the $n + 1$ (equispaced) interpolation points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a set of basis functions b_i which satisfy the Lagrange property

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the interpolant g can be written as $g(x) = \sum_{i=0}^{n} b_i(x) f(x_i) = \sum_{i=0}^{n} b_i(x) y_i$.

Barycentric form of the interpolant

- Interpolation of 2 data points x_0, x_1,

$$g(x) = \frac{\sum_{i=0}^{1} \lambda_i(x) y_i}{\sum_{i=0}^{1} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{x - x_i} \quad i = 0, 1$$

and

$$b_i(x) = \frac{\lambda_i(x)}{\sum_{i=0}^{1} \lambda_i(x)} .$$

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- Interpolation of $n+1$ data points

$$g(x) = \frac{\sum_{i=0}^{n} \lambda_i(x) y_i}{\sum_{i=0}^{n} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i)} .$$

$$\sum_{i=0}^{n} \lambda_i(x) = \underbrace{\frac{1}{x - x_0}}_{>0} + \underbrace{\frac{-1}{x - x_1} + \frac{1}{x - x_2}}_{>0} + \underbrace{\frac{-1}{x - x_3} + \cdots}_{>0} \quad x_0 < x < x_1$$

The Floater-Hormann Rational Interpolant (FHRI)

The construction of FHRI, is very simple.

- Let $0 \le d \le n$.
- For each $i = 0, 1, \ldots, n-d$ let p_i denote the unique polynomial of degree at most d that interpolates a function f at $d+1$ pts $x_i, \ldots, x_{i+d}$
- Then

$$g(x) = \frac{\displaystyle\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\displaystyle\sum_{i=0}^{n-d} \lambda_i(x)} \tag{1}$$

where $\lambda_i(x) = \dfrac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}$.

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where $\lambda_i(x) = \dfrac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}$.

Thus, g is a local blending of the polynomial interpolants $p_0, \ldots, p_{n-d}$ with $\lambda_0, \ldots, \lambda_{n-d}$ acting as the blending functions.

Notice: **for $d = n$ we get the classical polynomial interpolation.**

Basis functions

Assume $[a, b] = [0, 1]$ and equispaced interpolation pts $x_i = i/n, \quad i = 0, \ldots, n.$

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As **basis functions** we take

$$b_i(x) = \frac{(-1)^i \beta_i}{x - x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}, \qquad i = 0, \ldots, n \tag{2}$$

with $\beta_0, \ldots, \beta_n$ **positive weights** defined as

$$\beta_j = \begin{cases} \sum_{k=0}^{j} \binom{d}{k}, & \text{if } j \le d, \\ 2^d, & \text{if } d \le j \le n-d, \\ \beta_{n-j}, & \text{if } j \ge n-d. \end{cases} \tag{3}$$

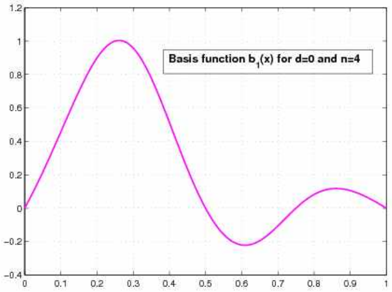
The weights β_s

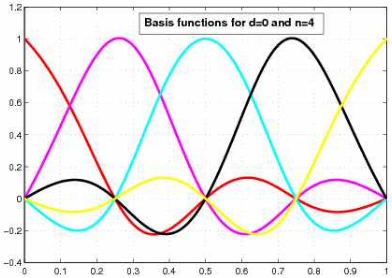
$d = 0^{\dagger}$	$1, 1, \ldots, 1, 1$
$d = 1^{\ddagger}$	$1, 2, 2 \ldots, 2, 2, 1$
$d = 2$	$1, 3, 4, 4, \ldots, 4, 4, 3, 1$
$d = 3$	$1, 4, 7, 8, 8, \ldots, 8, 8, 7, 4, 1$
$d = 4$	$1, 5, 11, 15, 16, 16, \ldots, 16, 16, 15, 11, 5, 1$

†Berrut's rational interpolant

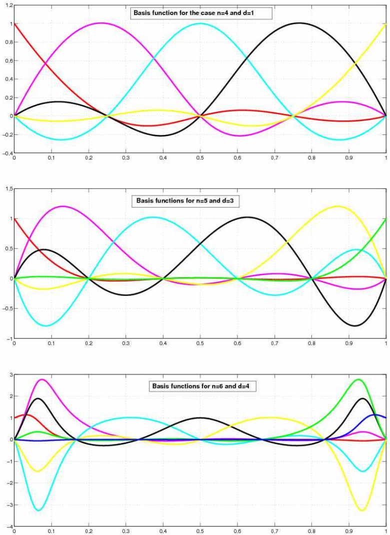
‡ $d \geq 1$ Floater-Hormann's rational interpolant

Some plots of the basis functions





Basis functions



Interpolation

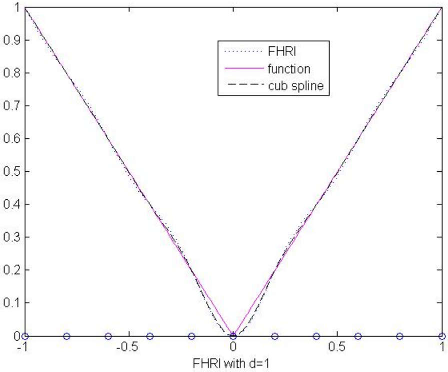


Figure: FHRI compared with a cubic spline on 11 equispaced points for the function $|x|, \ x \in [-1, 1]$

Properties of the FHRI (cf. [FH, NumMath2007])

1. The FHRI can be written in barycentric form.
   Indeed, in (1), letting $w_i = (-1)^i \beta_i$, for the numerator we have
   $$\sum_{i=0}^{n-d} \lambda_i(x) p_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k} f(x_k)$$
   where
   $$w_k = \sum_{i \in I_k} (-1)^i \prod_{j \neq k, j=i}^{i+d} \frac{1}{x_k - x_j}$$
   $I_k = \{i \in J, k - d \leq i \leq k\},\ J := \{0, ..., n - d\}$.
   Similarly for the denominator
   $$\sum_{i=0}^{n-d} \lambda_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k}$$
   It is a rational function of degree (n,n-d)

Properties of the FHRI (continue)

2. The rational interpolant $g(x)$ has no real poles. For $d = 0$ was proved by Berrut in 1998.
3. The interpolant reproduces polynomials of degree at most d, while does not reproduce rational functions (like Runge function)
4. Approximation error order $\mathcal{O}(h^{d+1})$ (for $f \in \mathcal{C}^{d+2}[0,1]$), also for non-equispaced points.

Lebesgue constant when $d = 0$

Remember: when $d = 0$, $\beta_j = 1, \forall j$.
We will derive upper and lower bounds for the Lebesgue function

$$\Lambda_n(x) = \sum_{i=0}^{n} |b_i(x)| = \sum_{i=0}^{n} \frac{\beta_i}{|x - x_i|} \Bigg/ \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j} \right|. \tag{4}$$

so that we can estimate

$$\Lambda = \max_{x \in [0,1]} \Lambda_n(x) \quad (\text{Lebesgue constant})\,.$$

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Theorem (BDeMH, JCAM11)

For any $n \geq 1$, we have

$$c_n \log(n+1) \leq \Lambda \leq 2 + \log(n).$$

where $c_n = 2n/(4 + n\pi)$ ($\lim_{n\to\infty} c_n = 2/\pi$).

Case $d = 0$: lower bound

We assume that the interpolation interval is $[0, 1]$, so that the nodes are equally spaced $x_j = jh = j \cdot 1/n, \;\; j = 0, \ldots, n$.

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Our goal is bounding below

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|x - j/n|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x - j/n}\right|} = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|2nx - 2j|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{2nx - 2j}\right|} := \frac{N(x)}{D(x)}. \tag{5}$$

by bounding $N(x)$ from below and $D(x)$ from above

The Lebesgue function for $d = 0$ on equispaced points

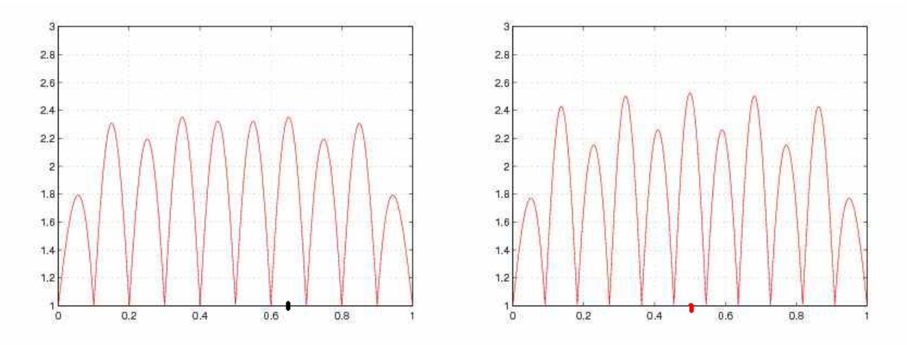

Figure: Lebesgue functions on [0,1]: n=10 (11 pts) (left) and n=11 (right).
The **black** and **red** signs indicate the points where the max is taken

Case $d = 0$: lower and upper bounds for $N(x)$ and $D(x)$

Assume $n = 2k$ and let $x^* = (n+1)/2n = 1/2 + 1/(2n)$.

Bounds [JCAM2011]

$$
\begin{aligned}
N(x^*) &\geq \frac{1}{2}\left(\ln(2k+3) + \ln(2k+1)\right) \geq \ln(2k+1) = \ln(n+1) \\
D(x^*) &\leq \frac{\pi}{2} + \frac{2}{2k+1} = \frac{\pi}{2} + \frac{2}{n+1}\,.
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Hence,

$$\Lambda_n(x^*) \geq \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}}\,.$$

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The same is true when n is odd considering $x^* = 1/2$, instead.

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In summary, for any $n \in \mathbb{N}$

$$\Lambda = \max_{0 \le x \le 1} \Lambda_n(x) \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}} \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n}} = c_n \ln(n+1)\,.$$

where $c_n = \frac{2\,n}{4+\pi\,n}$.

Case $d = 0$: upper and lower bounds for $N(x)$ and $D(x)$

Let $x_k < x < x_{k+1}$ for some k and let $N_k(x)$ and $D_k(x)$, N, D on the interval.

Bounds on the k-th interval [JCAM2011]

$$\begin{aligned} N_k(x) &\leq \frac{1}{n} + \frac{1}{2n}\ln(n) \\ D_k(x) &\geq \frac{1}{2n}\,. \end{aligned}$$

These bounds hold regardless the parity of n and k.

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These bounds hold regardless the parity of n and k. Combining the bounds for numerator and denominator yields

$$\Lambda = \max_{k=0,\ldots,n}\left(\max_{x_k<x<x_{k+1}} \Lambda_k(x)\right) \leq \frac{\frac{1}{n} + \frac{1}{2n}\log(n)}{\frac{1}{2n}} = 2 + \log(n).$$

The Lebesgue constant for $d = 0$ on uniform pts

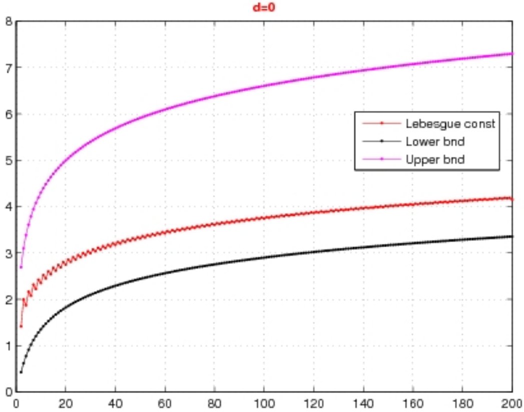

Figure: Lebesgue constant compared with its lower and upper bounds.

Lebesgue constant: case $d \geq 1$

Notice that

$$\boxed{\beta_j \leq 2^d, \ \forall\, j}$$

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For $x_k < x < x_{k+1}$ the numerator can be bound as follows

$$\begin{aligned} N_k(x) &= (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{\beta_j}{|x - x_j|} \\ &\leq 2^d (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{1}{|x - x_j|} \\ &\leq 2^d \left(\frac{1}{n} + \frac{1}{2n} \log(n) \right), \end{aligned} \tag{6}$$

$\Longrightarrow$ that holds for **any** k.

The denominator

Fundamental observation

$$\boxed{(-1)^j \beta_j = w_j \, d! \, h^d} \tag{7}$$

Then,

$$D_k(x) = (x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right| d! h^d \; .$$

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Since [FH, NumerMath2007],

$$\sum_{j=0}^{n} \frac{w_j}{x - x_j} = \sum_{i=0}^{n-d} \lambda_i(x) \Longrightarrow \left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right| \geq |\lambda_k(x)| \, .$$

Then,

$$D_k(x) = (x - x_k)(x_{k+1} - x)|\lambda_k(x)| d! h^d = \frac{d! h^d}{\prod_{l=k+2}^{k+d} (x_l - x)} \geq \frac{d! h^d}{\prod_{l=k+2}^{k+d} (x_l - x_k)} = \frac{1}{n} .$$

The lower bound

Theorem (Klein, Dec. 2010)

Let $d \geq 2$ then,

$$\Lambda \geq \frac{(2d+1)!!}{4(d+1)!} \log\left(\frac{n}{d} - 1\right).$$

Theorem (Bos, Dec. 2010)

Let $d \geq 1$ then,

$$\Lambda \geq \frac{2}{\pi} \log(n + 2 - 2d).$$

Note: this latter is better for $d = 1$.

The Lebesgue constant bounds for $d \geq 1$

Theorem

Let $d > 1$ Then,

$$\frac{(2d+1)!!}{4(d+1)!} \log\left(\frac{n}{d} - 1\right) \leq \Lambda \leq 2^{d-1}\left(2 + \log(n)\right).$$

while for $d = 1$

$$\frac{2}{\pi} \log(n) \leq \Lambda \leq 2 + \log(n).$$

Lebesgue functions

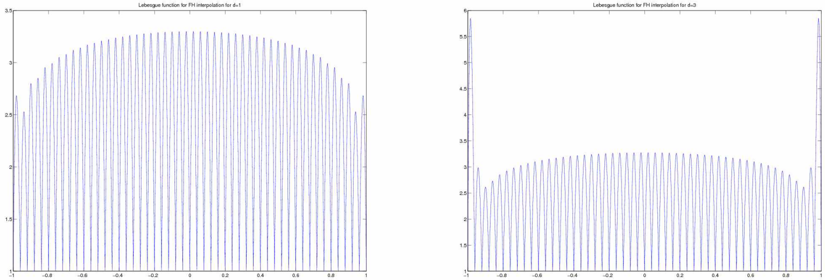
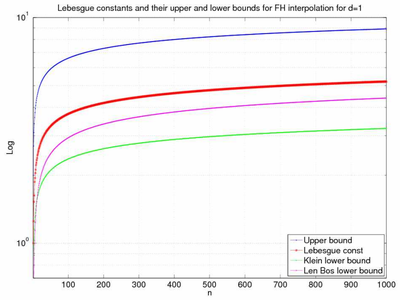


Figure: Lebesgue function for $d = 1$ (left) and $d = 3$ (right).

Lebesgue constants growth: I



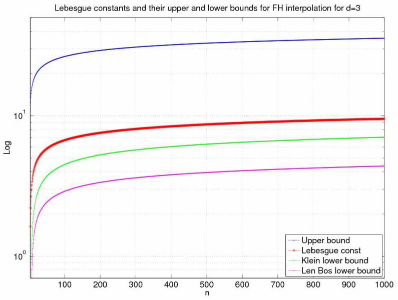
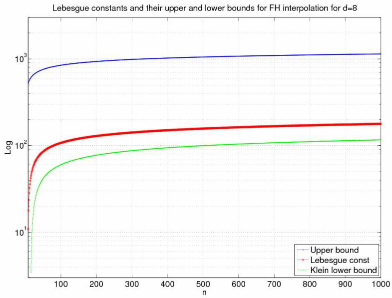


Figure: Lebesgue constant growth $d = 1$ (left) and $d = 3$ (right).

Lebesgue constant growth: II



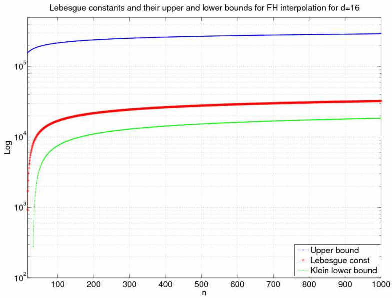


Figure: Lebesgue constant growth $d = 8$ (left) and $d = 16$ (right).

Quasi-equidistant nodes

- Equispaced nodes perturbed by a randomly chosen $\delta \in (0, 1/2)$, that is

$$x_j = j + \delta_j, \quad j = 0, \dots, n\,.$$

- We also assume that there exist $M \geq 1$ (independent on n) s.t. set $h := \max_{0 \leq j \leq n-1}(x_{j+1} - x_j)$ and $h^* := \min_{0 \leq j \leq n-1}(x_{j+1} - x_j)$ then

$$\frac{h}{h^*} \leq M\,.$$

Quasi-equidistant nodes

Lemma (Bounds on the weights, HKDeM2012)

$$W_k \leq |w_k| \leq M^d W_k$$

where

$$W_k = \frac{1}{h^d d!} \sum_{i \in J_k} \binom{d}{k-i}, k = 0, 1, \ldots, n$$

Moreover

$$W_k \leq \frac{2^d}{h^d d!} := W$$

with equality iff $d \leq k \leq n - d$.

Quasi-equidistant nodes

Theorem (Bounds on the Λ_n, HKDeM2012)

$$\Lambda_n \geq \frac{1}{2^{d+2}M^{d+1}} \binom{2d+1}{d} \cdot \begin{cases} (2+\log(2n+1)) & d=0 \\ \log(\frac{n}{d}-1) & d \geq 1 \end{cases}$$

$$\Lambda_n \leq (2+M\log(n)) \cdot \begin{cases} \frac{3}{4}M, & d=0 \\ 2^{d-1}M^d, & d \geq 1 \end{cases}$$

Lebesgue functions for quasi-equidistant points

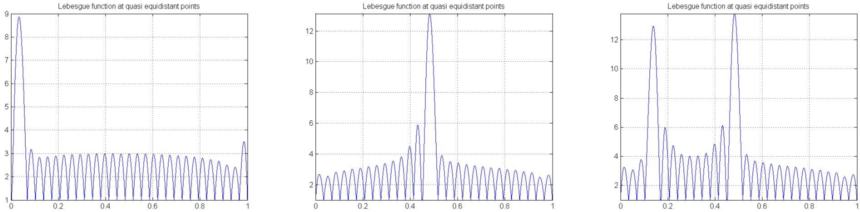


Figure: Lebesgue functions for 30 quasi-equidistant points perturbed at first, central, 5th and central points respectively, for $d = 2$

Definition

The set $X = (X_n)_{n\ge 0}$ *is a family of* *well-spaced* nodes, *if there exist* $R, C \ge 1$ *(independent on* n*) so that*

$$\frac{x_{k+1}-x_k}{x_{k+1}-x_j} \le \frac{C}{k+1-j}, \quad j=0,\ldots,k \quad k=0,\ldots,n-1,$$

$$\frac{x_{k+1}-x_k}{x_j-x_k} \le \frac{C}{j-k}, \quad j=k+1,\ldots,n \quad k=0,\ldots,n-1,$$

$$\frac{1}{R} \le \frac{x_{k+1}-x_k}{x_k-x_{k-1}} \le R, \quad k=1,\ldots,n-1,$$

hold for each set X_n.

Note. When the nodes are equispaced $R = C = 1$. The definition include also Chebyshev and extended Chebyshev nodes.

Lebesgue constant growth for $d = 0$

Theorem (Bounds on the Λ and $d = 0$, BDeMHS2013)

If $X = (X_n)_{n \geq 0}$ is a family of well-spaced nodes then

$$\Lambda(X_n) \leq (R+1)(1+2C\log(n)) = c\log(n), \quad n \geq 2\,.$$

How to get well-spaced nodes?

Definition

We say that a function $F \in \mathcal{C}[0,1]$ is a distribution function if it is a strictly increasing bijection on the interval $[0,1]$

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$\Longrightarrow$ To generate points that realize the bound of the previous Theorem, F has to be as follows

Definition

We say that a distribution is regular, if $F \in \mathcal{C}'[0,1]$ and F' has a finite number of zeros $T = \{t_1, \ldots, t_l\} \subset [0,1]$ with finite multiplicities.

Properties of regular distributions

Proposition

Let F be a regular distribution function. Then there exists a constant $C > 0$ such that

$$\frac{F[x,y]}{F[x,z]} \le C$$

for all $x, y, z \in [0,1]$ s.t. $x > y \ge z$.

Proposition

Let F be a regular distribution function. Then there exist an $\epsilon > 0$ and $R > 0$ such that

$$\frac{1}{R} \le \frac{F[x,x+s]}{F[x-s,x]} \le R$$

for all $s \in [0,\epsilon]$ and $x \in [s, 1-s]$.

Examples of regular distributions

1. $F_1(x) = x$, that generates equispaced points.
2. $F_2(x) = \log(1 + x(e-1))$, that generates logarithmically distributed points. Note that F_2 is regular since $F_2' > 0$.
3. $F_3(x) = (1 - \cos(\pi x))/2$, that is regular since F_3' has 2 simple zeros at $x_1 = 0$, $x_2 = 1$. This generates the Chebyshev extrema (or Chebyshev-Lobatto points) mapped in $[0, 1]$.

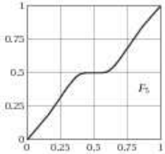
   In this case, for $\delta = 1/2$, $\epsilon = 1/4$, we get $C = 2\pi$ and $R = 9\pi/2$ and

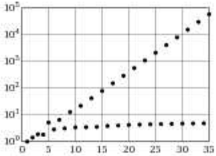
$$\Lambda(X_n) \leq (9\pi/2 + 1)(1 + 4\pi \log(n))) .$$

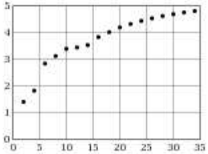
Non-regular distributions

1.
$$F_4(x) = \begin{cases} 0 & x = 0 \\ exp(1 - 1/x) & 0 < x \le 1 \end{cases}$$
which is non-regular since $\mathcal{C}^\infty$ at $x = 0$

2.
$$F_5(x) = \frac{1}{2} \begin{cases} 1 - exp(1 + 1/(2x - 1)) & 0 \le x < 1/2 \\ 1 & x = 1/2 \\ 1 + exp(1 - 1/(2x - 1)) & 1/2 \le x \le 1 \end{cases}$$
$F'(x) = 0$ in $x = 1/2$ with infinite multiplicity.
The last is non regular for odd n while for even n seems to growth logaritmically







Other non-regular nodes

We can also verify directly if a given family of nodes is well-spaced even if it is not generated by a distribution.

Extended Chebyshev nodes (on [0,1])

$$x_k = \frac{1}{2}\left[1 - \frac{\cos\left(\frac{2k+1}{2n+2}\pi\right)}{\cos\left(\frac{\pi}{2n+2}\right)}\right]\,,\ k = 0,\ldots,n$$

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Proposition

For extended Chebyshev nodes we have

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} \leq \frac{\pi^2/2}{k+1-j}, \quad j = 0, \ldots, k, \; k = 0, \ldots, n-1, \; \forall n\,.$$

$$\frac{1}{2} \leq \frac{x_{k+1} - x_k}{x_k - x_{k-1}} \leq 2, \quad k = 0, \ldots, n-1, \; \forall n\,.$$

$\Longrightarrow$ they satisfy the Definition of well-spaceness with $C = \pi^2/2$ and $R = 2$

Lebesgue constant growth for EC nodes

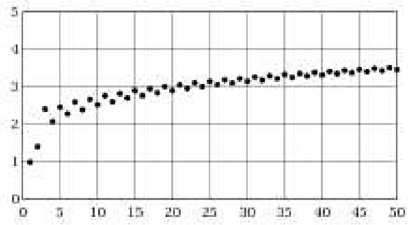

Figure: Lebesgue constant for Berrut's rational interpolant at $n+1$ extended Chebyshev nodes for $n = 1, \ldots, 50$.

giving the bound

$$\Lambda(X_n) \leq 3 + 3\pi^2 \log(n)\,.$$

Numerical quadrature

On $I = [-1, 1]$

1. we computed integrals with the quadrature based on the FHRI, on equispaced points at different values of n and d

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Numerical quadrature

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2. to speed up the quadrature, the quadrature weights were computed by a Gaussian quadrature rule (Gautschi software in Matlab)
3. For $d = 0$ we have proven that [BDeM, EJA2011]
   (a) $b_i(x) = \mathtt{sinc}(n(x - x_i))$ normalized so that $\sum_i b_i(x) = 1$
   (b)
   $$\lim_{n\to\infty} n\,\alpha_i = 1, \;\; \alpha_i = \int_0^1 b_i(x)dx\,.$$
   that is the quadrature process asymptotically converges.

b_i e sinc

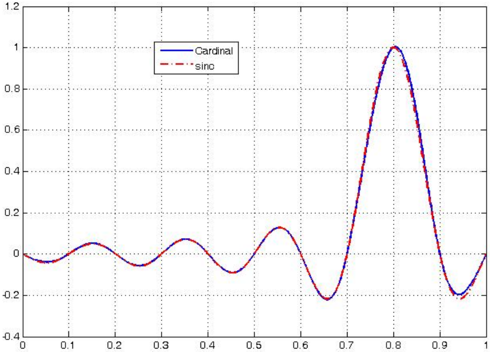

Figure: Comparison of b_{n-1} and $sinc(n(x - x_{n-1}))$ for $n = 10$

err $= 0.0101$

nw $=(0.4899, 1.1007, 0.9388, 1.0475, 0.9582, 1.0402, 0.9582, 1.0475, 0.9388, 1.1007, 0.4899)$

Numerical quadrature

The table below shows the quadrature relative errors for $d = 0$ (left) and $d = 3$ (right) at different n, for the Runge function. errS=quadrature relative error by using cubic splines

n	err (d=0)	err (d=3)	errS
10	3.5e-3	1.1e-2	7.2e-3
30	1.1e-4	1.6e-6	5.9e-5
50	7.6e-6	2.6e-8	3.2e-7
100	3.6e-7	7.9e-10	2.4e-8
150	4.9e-7	1.0e-10	1.5e-9
200	5.4e-7	2.4e-11	6.4e-11

Numerical quadrature: an open problem

About the quadrature weights: Klein and Berrut have proven numerically that the weights are all positive at least for $d \leq n \leq 1250$ and $0 \leq d \leq 5$. For other values of d and n, there might be a few negative weights, the number of which increases slowly with d and n.

A Matlab package

C. Bandiziol in her degree thesis (University of Padova, Feb. 2015) have organized all these results and applications in a Matlab package (to be available soon) that allows

- Compute FHRI
- Compute the Lebesgue constants
- Compute integrals by the Direct and Indirect Rational Quadrature method (BK, BIT 2012)
- Compute the Least-Square approximation by the FHRI

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