On Multivariate Newton Interpolation at Discrete Leja Points *

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to Robert Schaback in the occasion of his 65th birthday

Göttingen, January 14, 2011

* Joint work with L. Bos (Verona, I), A. Sommariva and M. Vianello (Padova, I)

Outline

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Motivations and aims

- (Weakly) Admissible Meshes, (W)AM: play a central role in the construction of multivariate polynomial approximation processes on compact sets.
- LU factorization with row pivoting: a tool for the construction of Newton-like interpolation (cf. de Boor 2004) and extracting Discrete Leja Points (cf. BDeMSV 2010)
- Error estimation by using the underlying (W)AM.
- Robert Schaback recent research!

Main references

1. J.P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008), 82–100.
2. L. Bos, S. De Marchi, A. Sommariva and M. Vianello: *Computing multivariate Fekete and Leja points by numerical linear algebra*, SIAM J. Numer. Anal. 48 (2010), 1984–1999.
3. R. Schaback. Newton bases. Preprint, Göttingen, 2009.
4. R. Schaback and S. De Marchi: *Nonstandard Kernels and Their Applications*, DRNA 2 (2009), 16–43
5. S. Müller and R. Schaback: *A Newton Basis for Kernel Spaces*, JAT 161 (2009), 645–655
6. C. Rieger, R. Schaback and B. Zwicknagl: *Samplying and Stability*, LNCS 5862 (2010), 347–369.

(Weakly) Admissible Meshes, (W)AM

Given a polynomial determining compact set $K \subset \mathbb{R}^d$.

Definition

An Admissible Mesh is a sequence of finite discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C\|p\|_{\mathcal{A}_n}\ , \quad \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

holds for some $C > 0$ with $\mathrm{card}(\mathcal{A}_n) \geq N := dim(\mathbb{P}_n^d(K))$ that grows at most polynomially with n.

- A Weakly Admissible Mesh, or WAM, is a mesh for which the constant C depends on n, i.e. $C = C(\mathcal{A}_n)$, growing also polynomially with n.

  Note: these sets and inequalities are also known as: (L^∞) discrete norming sets, Marcinkiewicz-Zygmund inequalities, stability inequalities (in more general functional settings).
- Optimal Admissible Meshes the ones with $\mathcal{O}(n^d)$ cardinality and can be constructed for some classes of compact sets (Kroó 2010, Piazzon and Vianello 2010).

Admissible Meshes

Markov compacts have AM (cf. CL JAT08), i.e. $K \subset \mathbb{R}^d$ s.t.

$$\|\nabla p\|_K \leq M n^r \|p\|_K, \ \ \forall\, p \in \mathbb{P}_n^d(K)\,,$$

where $\|\nabla p\|_K = \max_{x \in K} \|\nabla p(x)\|_2$

Construction idea: take a uniform discretization of K with spacing $\mathcal{O}(n^{-r})$. The mesh will have cardinality of $\mathcal{O}(n^{rd})$ for real compacts or $\mathcal{O}(n^{2rd})$ for general complex domains.

$r = 2$ for many (real convex) compacts: the construction and use of AM becomes difficult even for $d = 2, 3$ already for small degrees.

TOO BIG!!

Weakly Admissible Meshes: properties

P1: $C(\mathcal{A}_n)$ is invariant for affine transformations.

P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself

P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$

P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants

P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants

P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constants $C(\mathcal{A}_{ns})$ (cf. BCLSV Math. Comp.09)

Weakly Admissible Meshes: properties

P8: any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \leq M n^r \|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. CL JAT08)

P9: The least-squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\, \min\{\|f-p\|_K,\, p \in \mathbb{P}_n^d(K)\}$$

P10: The Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq N C(\mathcal{A}_n)$

Moreover, their asymptotic distribution is the same of the continuum Fekete points, in the sense that the corresponding discrete probability measures converge weak-$*$ to the pluripotential equilibrium measure of K (cf. BCLSV Math. Comp.09)

WAM for the disk and the triangle

In Bos at al. JCAM 09, it was proved that for the disk and the triangle there are WAMs with approximately n^2 points and the growth of $C(\mathcal{A}_n)$ is the same of an AM.

- Unit disk: a symmetric polar WAM is made by equally spaced angles and Chebyshev-Lobatto pts along diameters.
- Unit simplex: starting from the WAM of the disk for polynomials of degree $2n$ containing only even powers, by the standard quadratic transformation

$$(u, v) \longmapsto (x, y) = (u^2, v^2) \,.$$

Notice: by affine transformation these WAMs can be mapped to any other triangle (**P1**) or polygon (**P4**).

WAMs for the disk

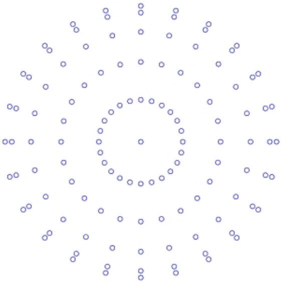


Figure: Symmetric polar WAM for the disk for degree $n = 10$ (left) and $n = 11$ (right).

WAMs for quadrant and the triangle

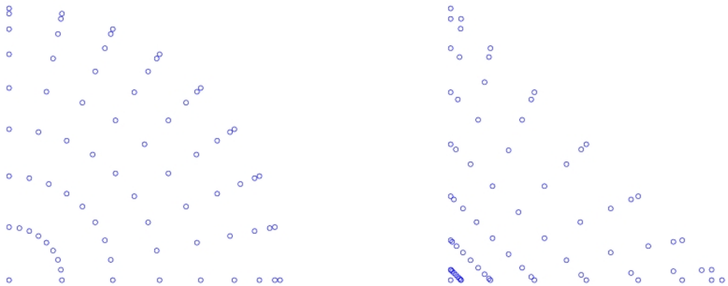


Figure: A WAM of the first quadrant for polynomial degree $n = 16$ (left) and the corresponding WAM of the simplex for $n = 8$ (right).

Notation

- Let $\mathcal{A}_n$ be an AM or WAM of $K \subset \mathbb{R}^d (or\ \mathbb{C}^d)$
- The rectangular Vandermonde-like matrix

$$V(\mathbf{a};\mathbf{p}) = V(a_1,\dots,a_M;p_1,\dots,p_N) = [p_j(a_i)] \in \mathbb{C}^{M\times N}, \quad M \geq N$$

where $\mathbf{a} = (a_i)$ is the array of the points of $\mathcal{A}_n$ and $\mathbf{p} = (p_j)$ the basis of $\mathbb{P}_n^d$.

Fekete and Approximate Fekete Points

The rows of $V(\mathbf{a};\mathbf{p})$ correspond to the points of $\mathcal{A}_n$, while the columns to the basis. Hence,

The N rows of $V(\mathbf{a};\mathbf{p})$ that maximize the volume generated by them (or maximize the absolute value of the $N \times N$ submatrix) are the Fekete points.

A greedy maximization of submatrix volumes, implemented by the QR factorization with column pivoting of $V(\mathbf{a};\mathbf{p})^t$ gives the so-called Approximate Fekete points (cf. SV CMA2009).

Discrete Leja Points

A greedy maximization of nested square submatrix determinants, implemented by the LU factorization with row pivoting of $V(\mathbf{a};\mathbf{p})$ gives the so-called Discrete Leja points (cf. BDeMSV SINUM2010 and observed in SDeM, DRNA2009).

Greedy Algorithm (GA): Discrete Leja Points (DLP):
- $V_0 = V(\mathbf{a},\mathbf{p})$; $\mathbf{i} = [\,]$;
- `for` $k = 1 : N$

"select i_k s.t. $|\det V_0([\mathbf{i}, i_k]\,,\, 1:k)|$ is maximum"; $\mathbf{i} = [\mathbf{i}, i_k]$;

`end`
- $\xi = \mathbf{a}(i_1, \ldots, i_N)$

- GA depends on the ordering of the polynomial basis. In 1-d it produces the **Leja points**.
- DLP form a sequence. Once we have computed the points for degree n, we have automatically at hand (nested) interpolation sets for all lower degrees.

Greedy Algorithm for Discrete Leja Points

The core in the GA

select i_k : $|\det V([\mathbf{i}, i_k]\,, 1:k)|$ is maximum

can be implemented as

one column elimination step of the Gaussian elimination process with standard row pivoting.

This process is then equivalent to the **LU factorization with row pivoting** as clear in the following Matlab-like script

GA-DLP:

- $V_0 = V(\mathbf{a}, \mathbf{p})$; $\mathbf{i} = 1 : M$;
- $[L_0, U_0, P_0] = \mathtt{lu}(V_0)$; $\mathbf{i} = P_0\,\mathbf{i}$;
- $\xi = \mathbf{a}(i_1, \ldots, i_N)$

Remarks

- DLP form a sequence. Indeed, suppose $\{\mathbf{q}_1, \mathbf{q}_2\}$ is a ordered basis, the first $m_1 = dim(\mathbf{q}_1)$ points are exactly the DLP for $\mathbf{q}_1$.
- If the basis $\mathbf{p}$ is s.t.

$$\mathtt{span}\{p_1, \ldots, p_{N_s}\} = \mathbb{P}_s^d, \quad N_s = \mathtt{dim}(\mathbb{P}_s^d), \quad 0 \le s \le n \qquad (2)$$

then, the first N_s DLP are a unisolvent set for interpolation in $\mathbb{P}_s^d$.

DLP and Multivariate Newton Interpolation

1. Consider the square Vandermonde matrix at the DLP $\boldsymbol{\xi}$ and basis $\mathbf{p}$

$$V = V(\boldsymbol{\xi}, \mathbf{p}) = (P_0 V_0)_{1 \le i,j, \le N} := LU$$

   where $L = (L_0)_{1 \le i,j \le N}$ and $U = U_0$.

2. The polynomial interpolating a function f at $\boldsymbol{\xi}$, $\mathbf{f} = f(\boldsymbol{\xi}) \in \mathbb{C}^N$ is

$$\mathcal{L}_n f(x) = \mathbf{c}^t \mathbf{p}(x) = (V^{-1}\mathbf{f})^t \mathbf{p}(x) = (U^{-1} L^{-1} \mathbf{f})^t \mathbf{p}(x) = \mathbf{d}^t \phi(x) \qquad (3)$$

   where $\mathbf{d}^t = (L^{-1}\mathbf{f})^t, \;\; \phi(x) = U^{-t}\mathbf{p}(x)$.

Remarks

- Formula (3) is a type of Newton interpolant.
  Since U^{-t} is lower triangular, by (2) the basis ϕ is s.t.

$$\mathtt{span}\{\phi_1, \ldots, \phi_{N_s}\} = \mathbb{P}_s^d, \;\; 0 \le s \le n$$

- $$V(\boldsymbol{\xi}; \phi) = V(\boldsymbol{\xi}; \mathbf{p})U^{-1} = LUU^{-1} = L$$

  Hence, $\phi_j(\xi_j) = 1$ and $\phi_j(\xi_i) = 0, \; i = 1, \ldots, j-1, \;$ when $j > 1$.

- Case $d = 1$. Since $\phi_j \in \mathbb{P}^1_{j-1}$, then
  $\phi_j(x) = \alpha_j(x - x_1)\cdots(x - x_{j-1}), \; 2 \le j \le N = n + 1$ with
  $\alpha_j = ((x_j - x_1)\cdots(x_j - x_{j-1}))^{-1}$, i.e. the classical Newton basis with d_j the classical divided differences up to $1/\alpha_j$.

- The connection between LU factorization and Newton Interpolation was recognized by de Boor in SINUM2004 and in a more general way by R. Schaback et al. 2008 and 2009.

Error estimation

$$\mathcal{L}_n f(x) = \mathbf{d}^t \phi(x) = \delta_0(x) + \cdots + \delta_n(x)$$

where $\delta_k \in \mathbb{P}_k^d,\ 0 \le k \le n$ are of the form

$$\delta_k = (\mathbf{d})_{j\in J_k}^t (\phi)_{j\in J_k}, \quad J_k = \{N_{k-1}+1, \ldots, N_k\}\,. \tag{4}$$

i.e. the multivariate version of the Newton interpolation formula.

Error estimation

- In the case of continuum Leja points, if f sufficiently regular to ensure uniform convergence of the interp. poly.

$$f(x) = \sum_{k=0}^{\infty} \delta_k(x)$$

then

$$f(x) - \mathcal{L}_{s-1}(x) = \sum_{k=s}^{\infty} \delta_k(x)$$

$$\|f - \mathcal{L}_{s-1}\|_K \approx \|\delta_s\|_K \le C\|\delta_s\|_{\mathcal{A}_n}, \;\; s \le n\,. \tag{5}$$

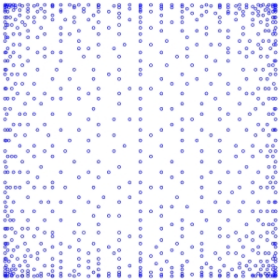
with δ_s polynomial error indicator

- For DLP we may also apply (5), but now δ_k depend on n via $\mathcal{A}_n$.
  Idea: take n sufficiently large so that $\mathcal{A}_n$ good model for K.

Points and error estimation

- $K = [-1,1]^2$.
- The Leja points are extracted from the grid of $(2n+1)\times(2n+1)$ Chebyshev-Lobatto points, which is an Admissible Mesh of low cardinality, with $C = 2$ (cf. BV MJI2011).
- The polynomial basis is the tensor product one.
- The Lebesgue constant and the interpolation error has been computed on 100×100 uniform grid of control points.

Points and Lebesgue constant



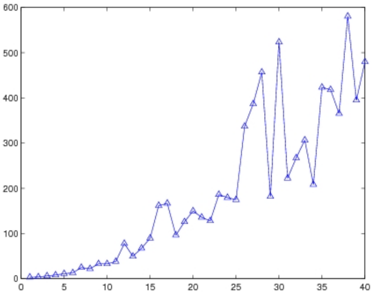


Figure: Left: $N = 861$ DLP for degree $n = 40$ on the square, extracted from a 81×81 Chebyshev-Lobatto grid. Right: Lebesgue constants of DLP on the square for $n = 1, \ldots, 40$

Errors

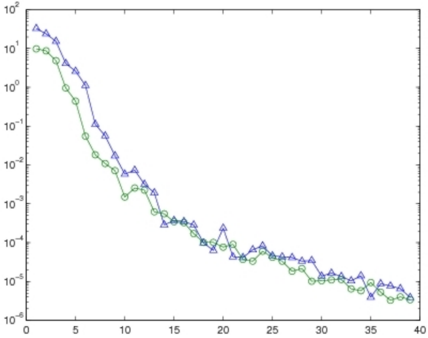


Figure: Uniform error (circles) and estimate (5) (triangles) of Newton interpolation at DLP on the square for $s = 2, \ldots, 40$. Left: $f(x_1, x_2) = |(x_1 - 1/3)^2 + (x_2 - 1/3)^2|^{5/2}$. Right: $f(x_1, x_2) = \cos(5(x_1 + x_2))$

Thanks Robert!

Robert's family

Robert thinking

THANK YOU ROBERT

ENJOY YOUR FAMILY AND ... MATHEMATICS !