3-dimensional Weakly Admissible Meshes: interpolation and cubature *

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Outline

Motivations and aims

- Computation of near optimal points, for polynomial interpolation in the multivariate setting, such as Fekete points, ...
- (Weakly) Admissible meshes, (W)AM: play a central role in the construction of multivariate polynomial approximation processes on compact sets.
- Theory vs computation: 2-dimensional and (simple) 3-dimensional (W)AMs are easy to construct. What's about more general domains such as (truncated) cones or rotational sets like toroidal domains?

Notation

- $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) compact set or manifold
- $\mathbf{p} = \{p_j\}_{1 \le j \le N}$, $N = \dim(\mathbb{P}_n^d(K))$ polynomial basis
- $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_N\} \subset K$ interpolation points
- $V(\boldsymbol{\xi}, \mathbf{p}) = [p_j(\xi_i)]$ Vandermonde matrix, $\det(V) \neq 0$
- $\Pi_n f(x) = \sum_{j=1}^N f(\xi_j)\,\ell_j(x)$, interpolating polynomial with ℓ_j in determinantal Lagrange formula

$$\ell_j(x) = \frac{\det(V(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_N))}{\det(V(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N))}\,, \quad \ell_j(\xi_i) = \delta_{ij}$$

Fekete points: definition and properties

1. Fekete points: $|\det(V(\xi_1,\ldots,\xi_N))|$ is max in K^N.
2. Lebesgue constant $\Lambda_n = \max_{x\in K} \sum_{j=1}^{N} |\ell_j(x)| \le N$
3. Fekete points (and Lebesgue constants) are independent of the choice of the basis
4. Fekete points are analytically known only in a few cases.
   - interval: Gauss-Lobatto points, $\Lambda_n = \mathcal{O}(\log n)$
   - complex circle: equispaced points, $\Lambda_n = \mathcal{O}(\log n)$
   - cube: for tensor-product polynomials, $\Lambda_n = \mathcal{O}(\log^d n)$
5. recent important result: Fekete points are asymptotically equidistributed w.r.t. the pluripotential equilibrium measure of K [Berman/Boucksom/Nyström 2011]
6. open problem: efficient computation, even in the univariate complex case (large scale optimization problem in $N \times d$ variables [Bos/Sommariva/Vianello 2011])

Idea

Extract Fekete points from a discretization of K

Which could be a suitable discretizaion of K?

(Weakly) Admissible Meshes , (W)AM

Definition

An Admissible Mesh for a compact $K \subset \mathbb{R}^d$ (polynomial determining), is a sequence of finite discrete subsets $\mathcal{A}_n \subset K$ s.t.

$$\|p\|_K \le C\|p\|_{\mathcal{A}_n} \,, \quad \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

holds for some $C > 0$ with $\mathrm{card}(\mathcal{A}_n) \ge N := dim(\mathbb{P}_n^d(K))$ that grows at most polynomially with n [Calvi/Levenberg 2008].

- A Weakly Admissible Mesh, or WAM, is a mesh for which $C = C(\mathcal{A}_n)$, growing also polynomially with n.
- These sets and inequalities are also known as: (L^∞) discrete norming sets, Marcinkiewicz-Zygmund inequalities, stability inequalities (in more general functional settings).
- Optimal Admissible Meshes: with cardinality $\mathcal{O}(n^d)$ (e.g. star-like domains [Kroó 2011], by analytic transf. [Piazzon/Vianello 2011]).

Weakly Admissible Meshes: properties

P1: $C(\mathcal{A}_n)$ is invariant under affine mappings

P2: good interpolation points are WAMs with $C(\mathcal{A}_n)$ being their Lebesgue constant (e.g. Chebyshev points in the interval, Padua points on the square)

P3: finite unions and products of WAMs are WAMs for the corresponding unions and products of compacts, ($C(\mathcal{A}_n)$ being the maximum or the product of constants)

P4: given a polynomial mapping π_m of degree m, then $\pi_m(\mathcal{A}_{nm})$ is a WAM for $\pi_m(K)$ with constants $C(\mathcal{A}_{nm})$ (cf. [Bos et al. 2009])

P5: Least-squares polynomial approximation of $f \in \mathcal{C}(K)$ on a WAM is near optimal in the sup norm

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\, E_n(f, K)$$

P6: The Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq N C(\mathcal{A}_n)$ (often much smaller)

Discrete Extremal Sets

Idea: extracting a maximum determinant $N \times N$ submatrix from the $M \times N$ Vandermonde matrix $V = V(\mathbf{a}, \mathbf{p}) = [p_j(a_i)]$

- NP-hard problem
- We look for approximate solutions
- This can be done by basic numerical linear algebra

Key asymptotic result (cf. [Bos/De Marchi et al. 2010]): Discrete Extremal Sets extracted from a WAM by the greedy algorithms below, have the same asymptotic behavior of the true Fekete points

$$\mu_n := \frac{1}{N} \sum_{j=1}^{N} \delta_{\xi_j} \xrightarrow{N \to \infty} d\mu_K$$

where μ_K is the pluripotential equilibrium measure of K

Approximate Fekete Points: algorithm

Idea: greedy maximization of submatrix volumes [Sommariva/Vianello 2009]

- core: select the largest norm row, $row_{i_k}(V)$, and remove from each row of V its orthogonal projection onto row_{i_k} onto the largest norm one (preserves volumes as with parallelograms)
- implementation: QR factorization with column pivoting [Businger/Golub 1965] applied to V^t
- Matlab script: $\mathbf{w} = V' \backslash \texttt{ones}(1:N)$; $ind = \texttt{find}(\mathbf{w} \neq \mathbf{0})$; $\xi = \mathbf{a}(ind)$

Discrete Leja Points: algorithm

Idea: greedy maximization of nested subdeterminants [Bos/DeMarchi/et al. 2010] and already observed in [Schaback/De Marchi 2009].

- core: one column step of Gaussian elimination with row pivoting (preserves the relevant subdeterminants)
- implementation: LU factorization with row pivoting
- DLP form a sequence. In one variable they correpond to the usual notion $\xi_k = \mathtt{argmax}_{z \in \mathcal{A}_n} \prod_{j=1}^{k} |z - \xi_j|, \quad k = 2, ..., N$
- Matlab script: $[L, U, \boldsymbol{\sigma}] = \mathtt{LU}(V, \text{``}\mathtt{vector}\text{''})$; $ind = \boldsymbol{\sigma}(1:N)$; $\boldsymbol{\xi} = \mathbf{a}(ind)$

AFP in one variable

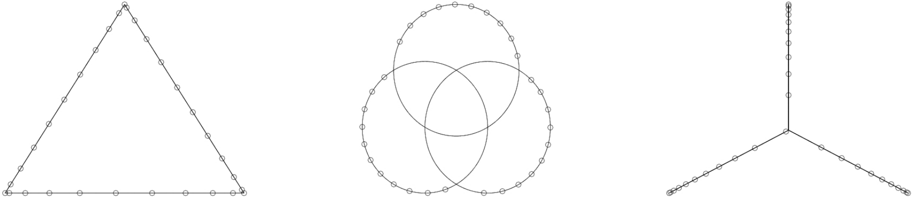


Figure: $N = 31$ AFP (deg $n = 30$) from Admissible Meshes on complex domains

Table 1. Numerically estimated Lebesgue constants of interpolation points in some 1-dimensional real and complex compacts

points	$n = 10$	20	30	40	50	60
	$N = 11$	21	31	41	51	61
equisp intv	29.9	1e+4	6e+6	4e+8	7e+9	1e+10
Fekete intv	2.2	2.6	2.9	3.0	3.2	3.3
AFP intv	2.3	2.8	3.1	3.4	3.6	3.8
AFP 2intvs	3.1	6.3	7.1	7.6	7.5	7.2
AFP 3intvs	4.2	7.9	12.6	6.3	5.8	5.3
AFP disk	2.7	3.0	3.3	3.4	3.5	3.7
AFP triangle	3.2	6.2	5.2	4.8	9.6	6.1
AFP 3disks	5.1	3.0	7.6	10.6	3.8	8.3
AFP 3branches	4.7	3.5	3.8	8.3	5.0	4.8

2-dimensional WAMS: disk, triangle, square

- Unit disk: a symmetric polar WAM (invariant by rotations of $\pi/2$) is made by equally spaced angles and Chebyshev-Lobatto points along diameters [Bos at al. 2009]

$$\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^2)\,, \quad C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$$

- Unit simplex: starting from the WAM of the disk for polynomials of degree $2n$ containing only even powers, by the standard quadratic transformation

$$(u, v) \longmapsto (x, y) = (u^2, v^2)\,.$$

- Square: Chebyshev-Lobatto grid, Padua points.

Notice: by affine transformation these WAMs can be mapped to any other triangle (**P1**) or polygon (**P4**).

Polar symmetric WAMs for the disk

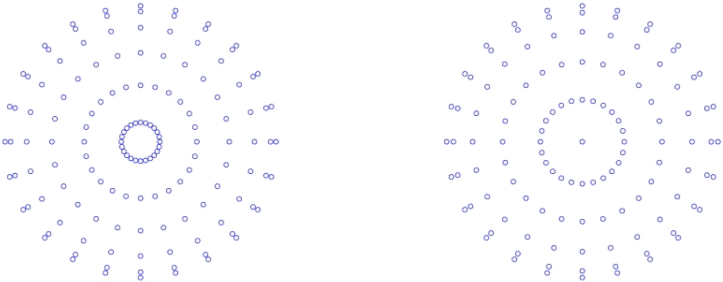


Figure: Left: for degree $n = 11$ with $144 = (n+1)^2$ points. Right: for $n = 10$ with $121 = (n+1)^2$ points.

WAMs for the quadrant and the triangle

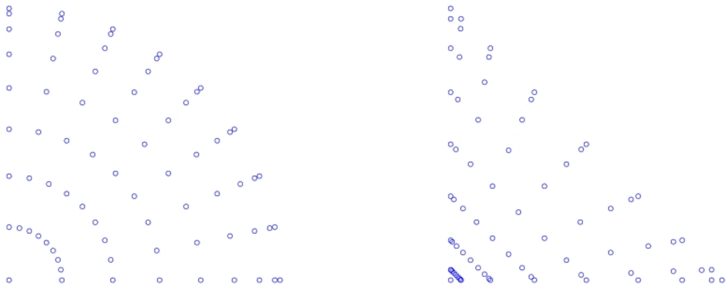


Figure: A WAM of the first quadrant for polynomial degree $n = 16$ (left) and the corresponding WAM of the simplex for $n = 8$ (right).

WAMs for a quadrangle

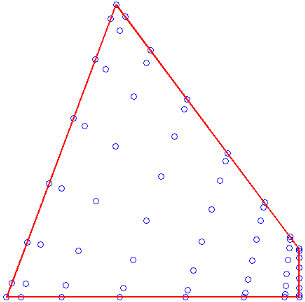


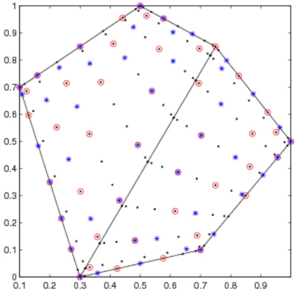
Figure: A WAM for a quadrangular domain for $n = 7$ obtained by the bilinear transformation of the Chebyshev–Lobatto grid of the square $[-1, 1]^2$

$$\frac{1}{4}[(1-u)(1-v)A+(1+u)(1-v)B+(1+u)(1+v)C+(1-u)(1+v)D]$$

WAMs for general polygons

Polygon WAMs: by triangulation/quadrangulation

$$\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^2)\,, \quad C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$$



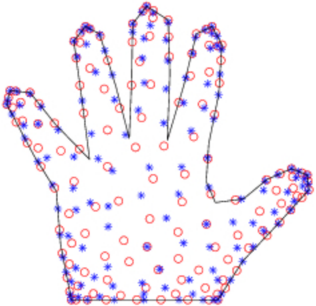


Figure: Left: $N = 45$ AFP ($\circ$) and DLP ($*$) of an hexagon for $n = 8$ from the WAM (dots) obtained by bilinear transformation of a 9×9 product Chebyshev grid on two quadrangle elements ($M = 153$ pts); Right: $N = 136$ AFP ($\circ$) and DLP ($*$) for degree $n = 15$ in a hand shaped polygon with 37 sides and a 23 element quadrangulation ($M \approx 5500$).

Optimal Lebesgue Gauss–Lobatto points on the triangle

A new set of optimal Lebesgue Gauss-Lobatto points on the simplex has recently been investigated by [Briani/Sommariva/Vianello 2011].
These points minimize the corresponding Lebesgue constant on the simplex, that grows like $\mathcal{O}(n)$.

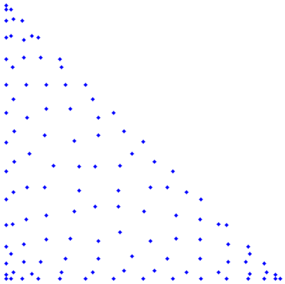
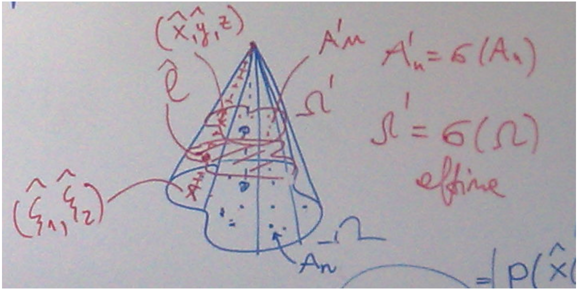


Figure: The optimal points for $n = 14$, cardinality $(n+1)(n+2)/2$.

WAMs for (truncated) cones

Starting from a 2-dimensional domain WAM, we "repeat" the mesh along a Chebsyhev-Lobatto grid of the z-axis, as shown here in my handwritten notes (on my whiteboard).



Why these are WAMs?

From the previous picture

$$
\begin{aligned}
|p(x,y,z)| &\le C(A_n)\|p\|_{A'_n(z)} \;\; C(A_n) \equiv C(A'_n(z)) \\
\|p\|_{A'_n(z)} &= |p(\hat{x}_z, \hat{y}_z, z)| \; with \;\; (\hat{x}_z, \hat{y}_z, z) \in A'_n(z) \\
&\le C(A_n)\|p\|_{\ell(\hat{\xi}_1, \hat{\xi}_2)} \; where \; (\hat{\xi}_1, \hat{\xi}_2) \in A_n \\
&\le C(A_n) \max_{(x,y)\in A_n} \|p\|_{\ell(x,y)} \\
&\le \mathcal{O}(C(A_n)\log_n) \max_{(x,y)\in A_n} \|p\|_{\Gamma_n} = \mathcal{O}(C(A_n)\log_n)\|p\|_{B_n}
\end{aligned}
$$

where Γ_n are the Chebyshev-Lobatto points of $l(x,y)$ and
$B_n = \bigcup_{(x,y)\in A_n} \Gamma_n(\ell(x,y))$.
Cardinality.

$$\#B_n = (n+1)\#A_n - \#A_n + 1 = 1 + n\#A_n = \mathcal{O}(n^3)$$

WAMs for a cone

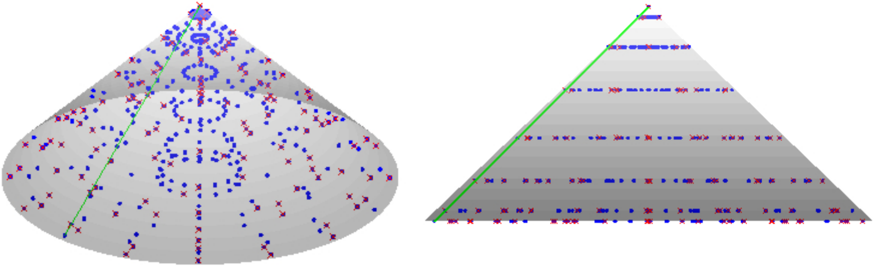
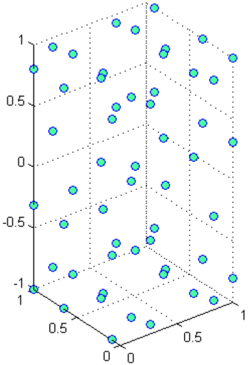


Figure: A WAM for the rectangular cone for $n = 7$

Here $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3)$

A low dimension WAM for the cube

The cube can be considered as a *cylinder with square basis*. WAMs for the cube with dimension $\mathcal{O}(n^3/4)$ were studied in [DeMarchi/Vianello/Xu 2009] in the framework of cubature and hyperinterpolation.
A WAM for the cube that for n even has $(n+2)^3/4$ points and for n odd $(n+1)(n+2)(n+3)/4$ points, is shown here for a parallelpiped with $n = 4$ (here $\#A_n = 54$)



WAMs for a pyramid

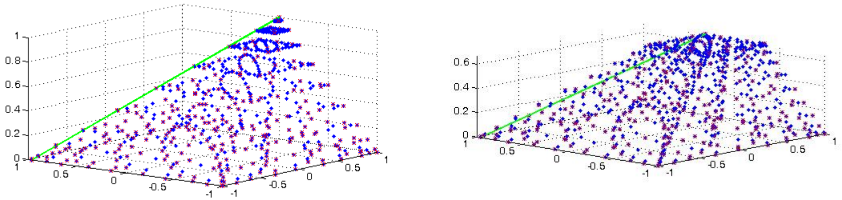
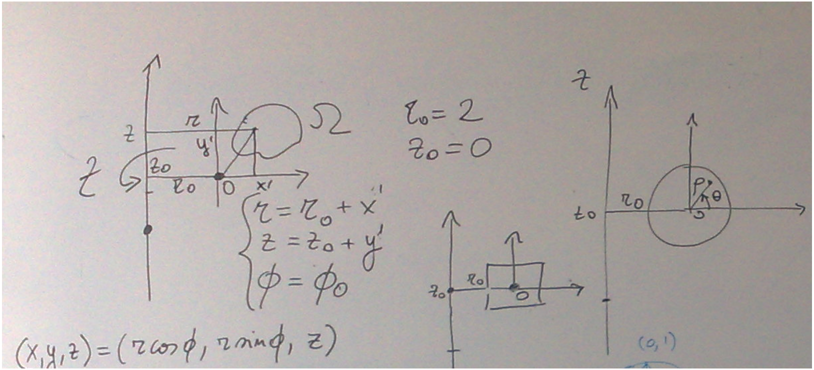


Figure: A WAM for a non-rectangular pyramid and a truncated one, made by using Padua points for $n = 10$. Notice the generating curve of Padua points that becomes a spiral

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3/2)$

3-dimensional WAMs: toroidal sections

Starting from a 2-dimensional WAM, A_n, by rotation around a vertical axis sampled at the $2n+1$ Chebyshev-Lobatto points of the arc of circumference, we get WAMs for the torus, sections of the torus and in general toroids.
The resulting cardinality will be $(2n+1) \times \#A_n$



Why these are WAMs?

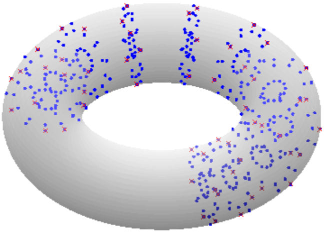
From the previous "picture" Given a polynomial $p(x, y, z) \in \mathbb{P}_n^3$ we can write it in cylindrical coordinates getting

$$p(x, y, z) = q(r, z, \phi) = s(x', y', \phi) \in \mathbb{P}_n^{2,(x',y')} \otimes \mathbb{T}_n^{\phi}$$

since

$$x^i y^j x^k = (r\cos\phi)^i (r\sin\phi)^j z^k (r_0 + x')^i \cos^i\phi (r_0 + y')^j \sin^j\phi (r_0 + y')^k$$

WAMs for toroidal sections: points on the disk



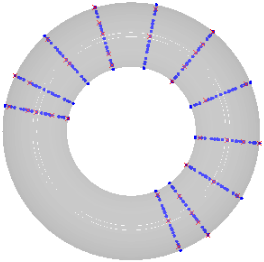


Figure: WAM for $n = 5$ on the torus centered in $z_0 = 0$ of radius $r_0 = 3$, with $-2/3\pi \leq \theta \leq 2/3\pi$.

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(2n^3)$

WAMs for toroidal sections: Padua points

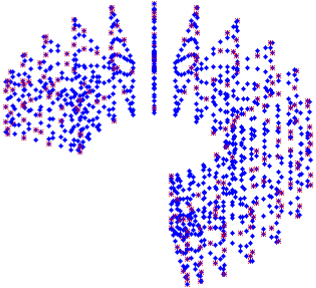


Figure: Padua points on the toroidal section with $z_0 = 0$, $r_0 = 3$ and opening $-2/3\pi \leq \theta \leq 2/3\pi$.

In this case $C(A_n) = \mathcal{O}(\log^2 n)$ and the cardinality is $\mathcal{O}(n^3)$.

Interpolation and Least-Squares

1. **Positive aspects**:
   1. AFP (and DLP) are near optimal interpolation points. We know them also on subarc-based product WAMs on sections of disk,sphere,ball, torus, such as circular sectors and lenses, spherical caps, quandrangles, lunes, slices,.... with $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^d)$ and $C(\mathcal{A}_n) = \mathcal{O}(\log^k n)$, ($k = 2$ for surfaces, $k = 3$ for solids) [Bos/Vianello 2011, Bos/DeMarchi et al 2011]
   2. so far, we can construct, by linear algebra approach, polynomial interpolation and least-square approximation on AFP or DLP up to degree 30 on solid cones and torus.
2. **Negative aspects**.
   1. Extracting AFP or DLP, is costly!
   2. Find good polynomial basis (for cylinders [Wade 2010, De Marchi/Marchioro/Sommariva 2011]).

Cubature

1. For (generalized) solid cones, in literature there exist cubature product rules with $\mathcal{O}(n^3/8)$ points, e.g. [Stround 1971]. For the torus with circular and square cross-section the software `STROUD` (Matlab and C) by J. Burkardt 2004-2009, implements Stroud formulas for cubature on the solid torus.
2. Our approach, uses as cubature points for torus with square, circular or triangular cross-sections, the AFP/DLP. Cardinality $\mathcal{O}(n^3/6)$ instead of $\mathcal{O}(n^3/2)$ (tensor product formulas of Burkardt). As polynomial basis we use tensor product Chebsyhev polynomials.

Conic section: disk

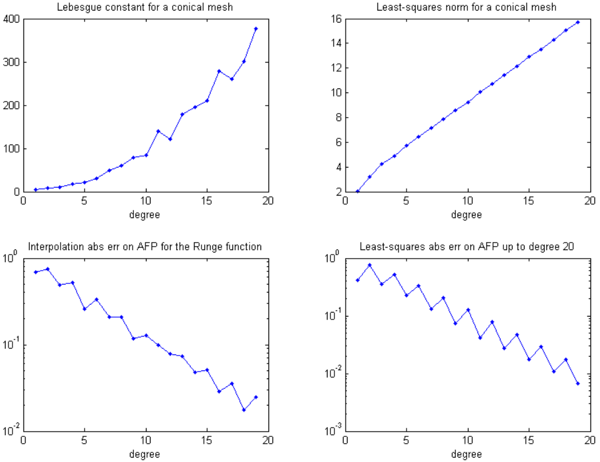
K is the solid cone. Given an n, then

- The AFP are extracted from a WAM having $\mathcal{O}(n^3)$ points
- The polynomial basis is the tensor product Chebyshev polynomial basis.
- The Lebesgue constant and the interpolation error has been computed on a mesh of control points (consisting of the original WAM with $2n$ instead of n).

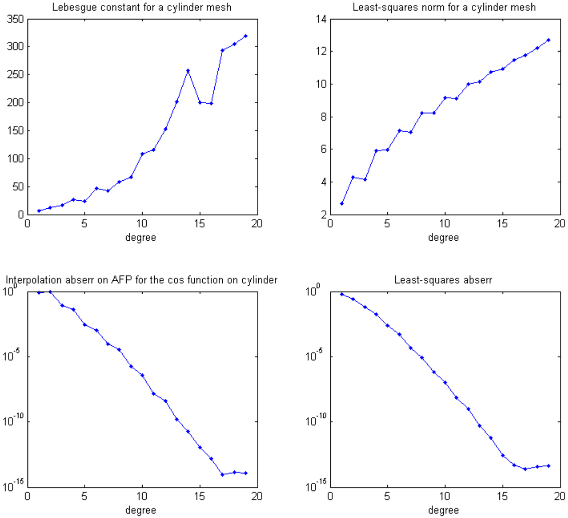
We also computed the

1. least-square operator norm, $\|L_{A_n}\| = \max_{x\in K} \sum_{i=1}^{M} |g_i(x)|$ where $g_i,\ i=1,\dots,M$ are a set of generators and $M \ge N = \dim \mathbb{P}_n^3$ (cf. [Bos/De Marchi et al. 2010])
2. interpolation sup error $\|f - p_n(f)\|_\infty$
3. least-square sup error $\|f - L_{A_n}(f)\|_\infty$

Runge function on the cone



Cosine function on the rectangular cylinder



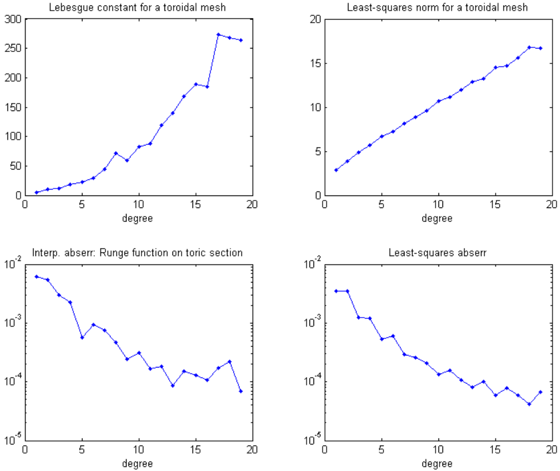
Circular and square toric sections

K is now a toric section. Given n then

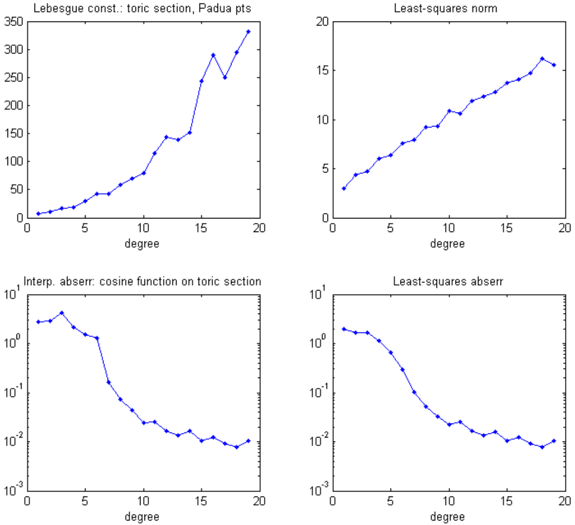
- The AFP are extracted from a WAM having $(n+1)^2(2n+1)$ points in the case of the disk and $\frac{(n+1)(n+2)}{2}(2n+1)$ in the case of the square (by using Padua points).
- The polynomial basis is the tensor product Chebyshev polynomial basis.
- The Lebesgue constant and the interpolation error has been computed on a mesh of control points (the original WAM of degree $2n$).

We computed as before least-square operator sup-norm, interpolation error $\|f - p_n(f)\|_\infty$ and least-square error $\|f - L_{A_n}(f)\|_\infty$.

Runge function on circular toric cross-section



Cosine function on the toric section using Padua points



Cubature

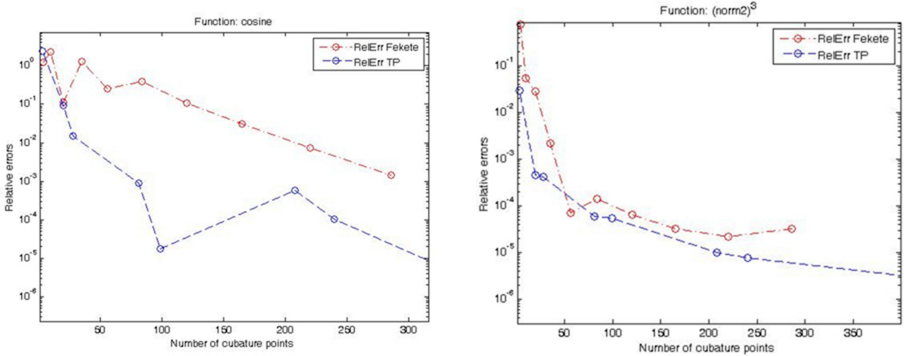


Figure: Relatives errors versus cubature points. Left: for the function $f(x, y, z) = \cos(x + y + z)$. Right: for the function $f(x, y, z) = \sqrt{((x-4)^2 + y^2 + z^2)^3}$.

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Thank you for your attention

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