On the Lebesgue constant of Floater-Hormann's rational interpolant on equispaced points *

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Oslo, September 27, 2010

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Outline

Motivations and aims

- Floater and Hormann Rational Interpolant, shortly FHRI, is one of the most efficient way of constructing a rational interpolant on equispaced and non-equispaced points and, citing the paper by Floater and Hormann 2007, *it seems to be perfectly stable in practice*. How to show this stability?

Motivations and aims

- Floater and Hormann Rational Interpolant, shortly FHRI, is one of the most efficient way of constructing a rational interpolant on equispaced and non-equispaced points and, citing the paper by Floater and Hormann 2007, *it seems to be perfectly stable in practice*. How to show this stability?
- The Lebesgue constant measures the quality and stability of interpolation processes. What we know about the growth of the Lebesgue constant for the FHRI?

Main references

1. J.-P. Berrut and H. D. Mittelmann, *Lebesgue Constant Minimizing Linear Rational Interpolation of Continuous Functions over the Interval*, Computers Math. Appl. 33(6) (1997), 77–86.
2. Michael S. Floater and Kai Hormann, *Barycentric rational interpolation with no poles and high rates of approximation*, Numer. Math. 107(2) (2007), 315–331.
3. Q. Wang, P. Moin and G. Iaccarino, *A rational interpolation scheme with super-polynomial rate of convergence*, Annual Research Brief 2008, Centre for Turbulence Research, 31–54.
4. J. M. Carnicer, *Weighted interpolation for equidistant points*, Numer. Algorithms 55(2-3) (2010), 223–232.

General interpolation process

Given a function $f\colon [a,b] \to \mathbb{R}$, let g be its interpolant at the $n+1$ (equispaced) interpolation points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a set of *basis functions* b_i which satisfy the *Lagrange property*

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the interpolant g can be written as $g(x) = \sum_{i=0}^{n} b_i(x) f(x_i)$.

The Floater-Hormann Rational Interpolant (FHRI)

The construction of FHRI, say $g(x)$, is very simple.

- Choose any integer d, $0 \le d \le n$
- For each $i = 0, 1, \ldots, n-d$ let p_i denote the unique polynomial of degree at most d that interpolates a function f at $d+1$ pts $x_i, \ldots, x_{i+d}$
- Then

$$g(x) = \frac{\sum_{i=0}^{n-d} \eta_i(x) p_i(x)}{\sum_{i=0}^{n-d} \eta_i(x)} \tag{1}$$

where $\eta_i(x) = \dfrac{(-1)^i}{\prod_{j=i}^{i+d}(x - x_j)}$.

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- Then

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where $\eta_i(x) = \dfrac{(-1)^i}{\displaystyle\prod_{j=i}^{i+d}(x - x_j)}$.

Thus, g is a local blending of the polynomial interpolants $p_0, \ldots, p_{n-d}$ with $\eta_0, \ldots, \eta_{n-d}$ acting as the blending functions. Notice: for $d = n$ we get the classical polynomial interpolation.

The Floater-Hormann Rational Interpolant

Assume $[a, b] = [0, 1]$ and interpolation points $x_i = i/n, \quad i = 0, ..., n.$

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As basis functions we take

$$b_i(x) = \frac{(-1)^i \beta_i}{x - x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}, \qquad i = 0, \dots, n \tag{2}$$

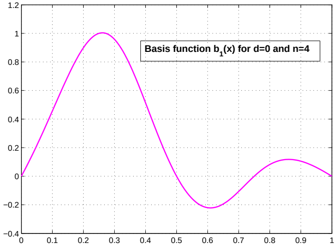
with $\beta_0, \dots, \beta_n$ that are positive weights defined as

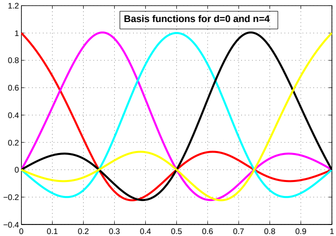
$$\beta_j = \begin{cases} \sum_{k=0}^{j} \binom{d}{k}, & \text{if } j \le d, \\ 2^d, & \text{if } d \le j \le n - d, \\ \beta_{n-j}, & \text{if } j \ge n - d. \end{cases} \tag{3}$$

The weights

$d = 0$	$1, 1, \ldots, 1, 1$
$d = 1$	$1, 2, 2 \ldots, 2, 2, 1$
$d = 2$	$1, 3, 4, 4, \ldots, 4, 4, 3, 1$
$d = 3$	$1, 4, 7, 8, 8, \ldots, 8, 8, 7, 4, 1$
$d = 4$	$1, 5, 11, 15, 16, 16, \ldots, 16, 16, 15, 11, 5, 1$

Basis functions





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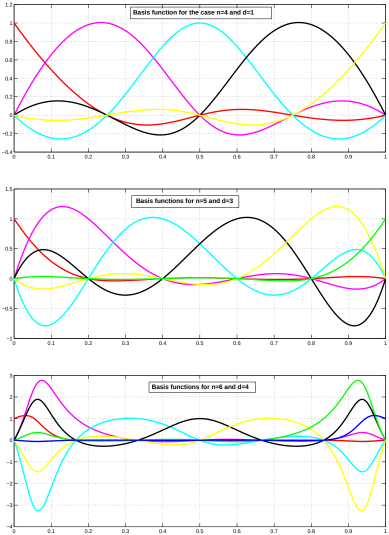

Properties of the FHRI (cf. FH's paper, 2007)

1. The FHRI can be written in barycentric form. Indeed, in (1), letting $w_i = (-1)^i \beta_i$, for the numerator we have

$$\sum_{i=0}^{n-d} \eta_i(x) p_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k} f(x_k)$$

where

$$w_k = \sum_{i \in I_k} (-1)^i \prod_{j \neq k, j=i}^{i+d} \frac{1}{x_k - x_j}$$

$I_k = \{i \in J, k - d \leq i \leq k\},\ J := \{0, ..., n - d\}$, and similarly for the denominator

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$$\sum_{i=0}^{n-d} \eta_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k}$$

2. The rational interpolant $g(x)$ has no real poles. For $d = 0$ was proved by Berrut in 1988.

Properties of the FHRI (continue)

3. The interpolant reproduces polynomials of degree at most d, while does not reproduce rational functions (like Runge function)

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3. The interpolant reproduces polynomials of degree at most d, while does not reproduce rational functions (like Runge function)
4. Approximation order $\mathcal{O}(h^{d+1})$ (for $f \in \mathcal{C}^{d+2}$, and this holds also for non-equispaced points.

Lebesgue constant. Case $d = 0$

We wish to derive an upper bound for the Lebesgue function

$$\Lambda_n(x) = \sum_{i=0}^{n} |b_i(x)| = \sum_{i=0}^{n} \frac{\beta_i}{|x - x_i|} \Bigg/ \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j} \right|. \qquad (4)$$

that is $\Lambda = \max_{x \in [0,1]} \Lambda_n(x)$.

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Main theorem

THEOREM

Let $d = 0$. *Then,*

$$\Lambda \leq 2 + \log(n).$$

The case $d = 0$: the proof

If $x = x_k$ for any k, then $\Lambda_n(x) = 1$.

The case $d = 0$: the proof

If $x = x_k$ for any k, then $\Lambda_n(x) = 1$.
So let $x_k < x < x_{k+1}$ for some k and consider the function

$$\Lambda_k(x) = \frac{(x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{|x-x_j|}}{\left|(x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{(-1)^j}{x-x_j}\right|} := \frac{N_k(x)}{D_k(x)}. \quad (5)$$

$$\begin{aligned} N_k(x) &= (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{|x-x_j|} \\ &= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\frac{1}{x-x_k}+\frac{1}{x_{k+1}-x}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right) \\ &= (x_{k+1}-x)+(x-x_k)+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right) \\ &= (x_{k+1}-x_k)+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right). \end{aligned}$$

The case $d = 0$: the proof

As the x_i are equally spaced $\frac{1}{x_i - x_j} = \frac{1}{h(i-j)} = \frac{n}{i-j}$ for any $i \neq j$, and $(x - x_k)(x_{k+1} - x) \leq \left(\frac{h}{2}\right)^2 = \frac{1}{4n^2}$ for $x_k < x < x_{k+1}$. Therefore,

$$\begin{aligned}
N_k(x) &\leq \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{1}{x_k - x_j} + \sum_{j=k+2}^{n}\frac{1}{x_j - x_{k+1}}\right)\\
&= \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{n}{k-j} + \sum_{j=k+2}^{n}\frac{n}{j-k-1}\right)\\
&= \frac{1}{n} + \frac{1}{4n}\left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-k-1}\right)\\
&\leq \frac{1}{n} + \frac{1}{4n}\left(\log(2k+1) + \log(2n-2k-1)\right)\\
&= \frac{1}{n} + \frac{1}{4n}\log\left((2k+1)(2n-(2k+1))\right)\\
&\leq \frac{1}{n} + \frac{1}{4n}\log\left((2n/2)^2\right)\\
&= \frac{1}{n} + \frac{1}{2n}\log(n).
\end{aligned}$$

The case $d = 0$: the proof

Let us consider the denominator $D_k(x)$.
Ignoring the absolute value and assuming, for a moment that both k and n to be even

$$\begin{aligned} D_k(x) &= (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{(-1)^j}{x - x_j} \\ &= (x - x_k)(x_{k+1} - x) \left(\sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} + \frac{1}{x - x_k} + \frac{1}{x_{k+1} - x} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \right) \\ &= h + (x - x_k)(x_{k+1} - x) \left(\sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \right). \end{aligned}$$

Pairing the positive and negative terms

$$\begin{aligned} S_k(x) &= \sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \\ &= \frac{1}{x - x_0} + \left(\frac{1}{x - x_2} - \frac{1}{x - x_1} \right) + \cdots + \left(\frac{1}{x - x_{k-2}} - \frac{1}{x - x_{k-3}} \right) - \frac{1}{x - x_{k-1}} \\ &\quad - \frac{1}{x_{k+2} - x} + \left(\frac{1}{x_{k+3} - x} - \frac{1}{x_{k+4} - x} \right) + \cdots + \left(\frac{1}{x_{n-1} - x} - \frac{1}{x_n - x} \right) \end{aligned} \tag{6}$$

The case $d = 0$: the proof

Since both the leading term and all paired terms are positive, we have

$$S_k(x) > -\frac{1}{x - x_{k-1}} - \frac{1}{x_{k+2} - x} \geq -\frac{1}{x_k - x_{k-1}} - \frac{1}{x_{k+2} - x_{k+1}} = -2n$$

and further

$$D_k(x) = h + (x - x_k)(x_{k+1} - x)S_k(x) \geq \frac{1}{n} + \frac{1}{4n^2}(-2n) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

This bound also holds if n is odd and if k is odd.

The case $d = 0$*: the proof*

Therefore, we have $|D_k(x)| \geq 1/(2n)$ regardless of the parity of k and n, and combining the bounds for numerator and denominator yields

$$\Lambda = \max_{k=0,\ldots,n} \left(\max_{x_k < x < x_{k+1}} \Lambda_k(x) \right) \leq \frac{\frac{1}{n} + \frac{1}{2n}\log(n)}{\frac{1}{2n}} = 2 + \log(n).$$

This completes the proof. □

Lebesgue constant. The case $d \geq 1$

We observe that

$$\beta_j \leq 2^d, \ \ \forall\, j\,.$$

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Then

$$\begin{aligned} N_k(x) &= (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{\beta_j}{|x - x_j|} \\ &\leq 2^d (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{1}{|x - x_j|} \\ &\leq 2^d \left(\frac{1}{n} + \frac{1}{2n} \log(n) \right), \end{aligned} \tag{7}$$

for any k.

For the **denominator**,

$$D_k(x) = (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j},$$

it will turn out that $|D_k(x)| \geq 1/n$, but the ideas from the proof of THEOREM can be generalized only for a limited range of k.

The case $d \geq 1$: the proof

The proof is based on some technical Lemmas and *Propositions*

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Lemma

Let $d \geq 1$ and $d \leq k \leq n-d-1$. Then,

$$|D_k(x)| \geq \frac{1}{n}$$

for $x_k < x < x_{k+1}$.

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for $x_k < x < x_{k+1}$.

It remains to handle the case $0 \leq k < d$, since the case $n-d \leq k < n$ follows by symmetry, which is harder (for many reasons).

This requires some *Propositions*

The case $d \geq 1$: the proof

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$$\sum_{j=2}^{n} (-1)^j \frac{j-1}{j^p} \beta_j > 0.$$

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Notice: For this Proposition we have a complete proof for $p = 1, \ \forall \, d$ while for $p \geq 2$ we proved up to $d \leq 4$.

But, all numerical experiments confirm the claim!

The case $d \geq 1$: the proof

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Lemma

Let $d \geq 1$. Then,

$$D_0(x) = (x - x_0)(x_1 - x) \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j} \geq \frac{1}{n}$$

for $x_0 \leq x \leq x_1$

The main idea of this proof can also be applied to handle the remaining case $0 < k < d$. Note that this range of k is empty for $d = 1$, hence we assume $d \geq 2$.

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The main idea of this proof can also be applied to handle the remaining case $0 < k < d$. Note that this range of k is empty for $d = 1$, hence we assume $d \geq 2$.
To this aim we proved other three properties of the weights β_j

Further properties of the weights β_j

1. Let $d \geq 2$ and $0 \leq k \leq n-2$, then

$$\sum_{j=0}^{n} (-1)^j (j+1)\beta_{k-j} \geq 1 \,.$$

Further properties of the weights β_j

1. Let $d \geq 2$ and $0 \leq k \leq n-2$, then

$$\sum_{j=0}^{n} (-1)^j (j+1) \beta_{k-j} \geq 1 \,.$$

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2. Let $d \geq 2$, then

$$\sum_{j=1}^{n} (-1)^j j\, \beta_j = 0\,.$$

3. Let $d \geq 1$, $2 \leq k \leq n$ and $p \geq 1$, then

$$\sum_{j=2}^{n} (-1)^j \frac{j-1}{j^p}\, \beta_{k-j} > 0\,.$$

The case $d \geq 1$: the proof

Lemma

Let $d \geq 2$ and $0 < k < d$. Then,

$$|D_k(x)| \geq \frac{2}{n}$$

for $x_k \leq x \leq x_{k+1}$

The theorem for $d \geq 1$

Theorem

Let $d \geq 1$. Then,

$$\Lambda \leq 2^{d-1}\left(2 + \log(n)\right).$$

Proof.

Using the bound on the numerator of $\Lambda_k(x)$ in (7) and the common bound on the denominator derived in the Lemmas for all possible values of k, we conclude that

$$\Lambda = \max_{k=0,\ldots,n}\left(\max_{x_k < x < x_{k+1}} \Lambda_k(x)\right) \leq \frac{2^d\left(\frac{1}{n} + \frac{1}{2n}\log(n)\right)}{\frac{1}{n}} = 2^{d-1}\left(2 + \log(n)\right).$$

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□

Note that for $d = 1$ this is the same bound as for $d = 0$, which is consistent with the numerical experiments that shows that both cases have a similar Lebesgue constant.

The Lebesgue function

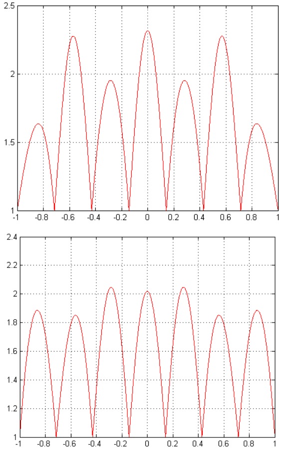

Figure: Lebesgue function for 8 uniform points for $d = 0$ (above) and $d = 1$ (below).

The Lebesgue function

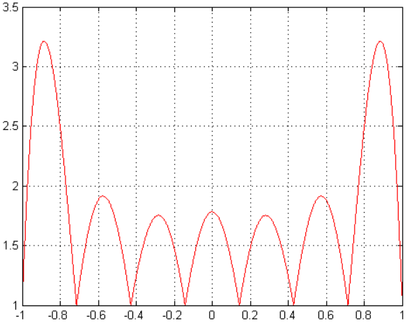

Figure: Lebesgue function for 8 uniform points for $d = 3$.

The Lebesgue function

From graphs, in $[-1, 1]$, we see that the maximum of the Lebesgue function is taken (moreless) at

$$x^* = \begin{cases} 1/n & n = 4k \\ 2/n & n = 4k+1 \\ 3/n & n = 4k+2 \\ 0 & n = 4k+3 \end{cases}$$

for some $k \in \mathbb{N}$ and so $\Lambda = \sum_{i=0}^{n} |b_i(x^*)| := \Lambda_n(x^*)$.

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for some $k \in \mathbb{N}$ and so $\Lambda = \sum_{i=0}^{n} |b_i(x^*)| := \Lambda_n(x^*)$.
In particular, for n=4k+3,

$$\Lambda_n(0) = \frac{\sum_{i=0}^{n} 1/|x_i|}{\left|\sum_{i=0}^{n} (-1)^i/|x_i|\right|} = \frac{\sum_{i=0}^{2k} 1/(n-2i)}{\left|\sum_{i=0}^{2k} (-1)^i/(n-2i)\right|}$$

$$= \frac{\left(\sum_{i=0}^{2k} 1/(2i+1) - 1/(2n)\right)}{\left|\sum_{i=0}^{2k} (-1)^i/(2i+1) - 1/(2n)\right|}$$

The Lebesgue function

Since since

$$\sum_{i=0}^{m} \frac{1}{2i+1} \sim \frac{\log(m)}{2} \quad \text{as} \quad m \to \infty$$

and

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} = \frac{\pi}{4},$$

we get the asymptotic estimate

$$\Lambda \sim \frac{2}{\pi} \log(n) \quad \text{as} \quad n \to \infty.$$

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The same is true for the other three cases. In fact, the Lebesgue function becomes

$$\Lambda_n(x^*) = \left(\sum_{i=0}^{2k} \frac{1}{1+2i} - \frac{a_n}{4} \right) \Big/ \left(\sum_{i=0}^{2k} \frac{(-1)^i}{1+2i} - \frac{b_n}{4} \right),$$

where

$$a_n = \begin{cases} \frac{1}{n-1} + \frac{3}{n+1}, & \text{if } n = 4k, \\ \frac{2}{n}, & \text{if } n = 4k+1, \\ \frac{1}{n-1} - \frac{1}{n+1}, & \text{if } n = 4k+2, \\ \frac{1}{n-2} - \frac{2}{n} - \frac{1}{n+2}, & \text{if } n = 4k+3, \end{cases} \quad \text{and} \quad b_n = \begin{cases} -\frac{1}{n-1} + \frac{3}{n+1}, & \text{if } n = 4k, \\ \frac{2}{n}, & \text{if } n = 4k+1, \\ \frac{1}{n-1} + \frac{1}{n+1}, & \text{if } n = 4k+2, \\ \frac{1}{n-2} + \frac{2}{n} - \frac{1}{n+2}, & \text{if } n = 4k+3. \end{cases}$$

Lebesgue constant growth

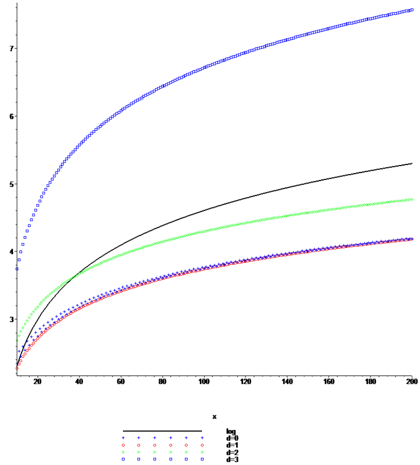
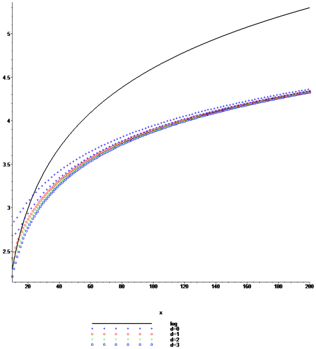


Figure: Lebesgue constant for uniformly distributed points.

Lebesgue constant growth



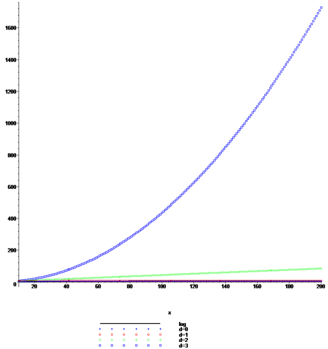


Figure: Lebesgue constant on Chebyshev points. **Left**: Chebyshev points with weights $(-1)^i \beta_i$. **Right**: here the weights are the ones constructed on non-equispaced points, garanteeing the approximation order $d+1$

Lebesgue constant growth

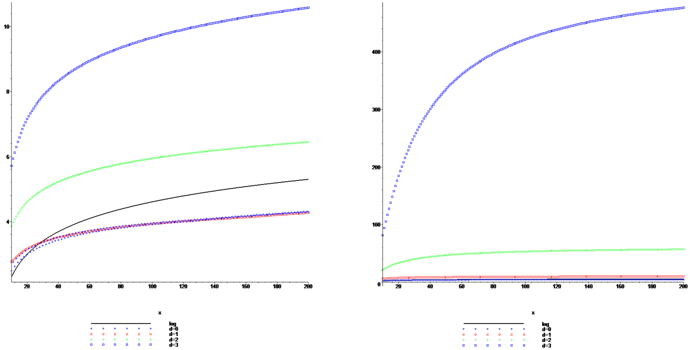


Figure: Lebesgue constant on logarithmically distributed points. **Left**: with weights $(-1)^i\beta_i$. **Right**: here the weights are the ones constructed on non-equispaced points, garanteeing the approximation order $d+1$

Points equally spaced w.r.t. a distribution

Consider the interval $I = [0, 1]$ and a distribution function $F \in \mathcal{C}^1(I)$

- $F(0) = 0, \; F(1) = 1$ and F is strictly increasing.

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- $F(x) = \int_0^x w(x)dx$ for a certain $w \in \mathcal{C}[0, 1],\ w(x) > 0,\ x \in I$

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The points

$$x_j := F^{-1}\left(\frac{j}{n}\right),\ \ 0 \le j \le n$$

are said to be *equally spaced according to F*.

Points equally spaced w.r.t. a distribution

Lemma

If $f \in \mathcal{C}[0,1]$ *then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} f(x_j) = \int_0^1 f(x)w(x)dx$$

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Proof The key observation is that $\frac{1}{n}\sum_{j=0}^{n} f(x_j) = \frac{1}{n}\sum_{j=0}^{n} f(F^{-1}(j/n))$ is a Riemann sum for $f \circ F^{-1} \in \mathcal{C}[0,1]$ and hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} f(x_j) = \int_0^1 f(F^{-1}(t))dt\ .$$

But $x = F^{-1}(t)$, then $dx = \left(\frac{d}{dt}F^{-1}(t)\right)dt = \frac{dt}{F'(F^{-1}(t))} = \frac{dt}{w(t)}$. Then, $dt = w(x)dx\,.$ $\square$

Points equally spaced w.r.t. a distribution

Lemma

Suppose that $k, n \to \infty$ in such a way that $x_k = F^{-1}(k/n)$ and $x_{k+1} = F^{-1}((k+1)/n)$ both tend to $x = F^{-1}(a)$. Then,

$$\lim_{n\to\infty} nh_k = (F^{-1})'(x) = \frac{1}{w(x)} .$$

where $h_k = x_{k+1} - x_k$.

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Proof

$$\begin{aligned} n\,h_k &= n(x_{k+1} - x_k) = n\left(F^{-1}((k+1)/n) - F^{-1}(k/n)\right) \\ &= \frac{F^{-1}((k+1)/n) - F^{-1}(k/n)}{1/n} = F^{-1}\left[\frac{k+1}{n}, \frac{k}{n}\right] = (F^{-1})'(c_n), \quad \textit{for } c \end{aligned}$$

Hence, $\lim_{n\to\infty} n\,h_k = \lim_{n\to\infty} (F^{-1})'(c_n) = (F^{-1})'(a)$ as $c_n \to a$. But $(F^{-1})'(a) = \frac{1}{F'(F^{-1}(a))} = \frac{1}{w(F^{-1}(a))} = \frac{1}{w(x)}$, $\square$

Points equally spaced w.r.t. a distribution

Note also that, as $(F^{-1})'(t) = \dfrac{1}{w(F^{-1}(t))} > 0$ and it is continuous (by assumption) then there exist two positive constants c_1, c_2 so that

$$c_1 < nh_k < c_2 \, .$$

Points equally spaced w.r.t. a distribution

Theorem

Suppose that the distribution F and the density w are as above and that, as before, $x_j^{(n)} = F^{-1}(j/n),\ \ 0 \le j \le n$. Let

$$b_i(x) = \frac{(-1)^i \beta_i}{x - x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}, \qquad i = 0, \ldots, n, \quad \Lambda_n(x) = \sum_{i=0}^{n} |b_i(x)| \ , \tag{8}$$

and, for $x_k < x < x_{k+1}$

$$\Lambda_k(x) = \frac{(x - x_k)(x_{k+1} - x) \displaystyle\sum_{j=0}^{n} \frac{1}{|x - x_j|}}{\left|(x - x_k)(x_{k+1} - x) \displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x - x_j}\right|}.$$

Then, there is a constant C such that

$$\Lambda_n(x) \le C\ \log(n), \quad x \in [0, 1]\,. \tag{9}$$

Numerical Quadrature

On $I = [-1, 1]$

1. we computed integrals with the quadrature based on the FHRI, on equispaced points at different values of n end/or d

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Numerical Quadrature

On $I = [-1, 1]$

1. we computed integrals with the quadrature based on the FHRI, on equispaced points at different values of n end/or d
2. to speed up the quadrature, the quadrature weights were computed by a Gaussian quadrature rule (Gautschi software in Matlab)
3. from numerical experiments, the weights are all positive (this has to be proved !)

Numerical Quadrature

The table below shows the quadrature relative errors for $d = 0$ (left) and $d = 3$ (right) at different n, for the Runge function. errS=quadrature relative error by using cubic splines

n	err (d=0)	err (d=3)	errS
10	3.5e-3	1.1e-2	7.2e-3
30	1.1e-4	1.6e-6	5.9e-5
50	7.6e-6	2.6e-8	3.2e-7
100	3.6e-7	7.9e-10	2.4e-8
150	4.9e-7	1.0e-10	1.5e-9
200	5.4e-7	2.4e-11	6.4e-11

Numerical Quadrature

About the quadrature weights. *Georges Klein*, a PhD student of Jean-Paul Berrut, proved *numerically* that the weights are all positive at least for $d \leq n \leq 1250$ and $0 \leq d \leq 5$. For other values of d and n, there might be a few negative weights, the number of which increases slowly with d and n.

THANK YOU
FOR YOUR KIND ATTENTION