On the Lebesgue constants of a family of rational interpolants on equispaced and non-equispaced points *

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Outline

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4. Numerical results
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Motivations and aims

- Floater and Hormann Rational Interpolant, shortly FHRI, is one of the most efficient way of constructing a rational interpolant on equispaced and non-equispaced points and, citing the paper by Floater and Hormann 2007, it seems to be perfectly stable in practice. How to show this stability?

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- Floater and Hormann Rational Interpolant, shortly FHRI, is one of the most efficient way of constructing a rational interpolant on equispaced and non-equispaced points and, citing the paper by Floater and Hormann 2007, it seems to be perfectly stable in practice. How to show this stability?
- The Lebesgue constant measures the quality and stability of interpolation processes. What we know about the growth of the Lebesgue constant for the FHRI?
- The FHRI is also on Numerical Recepies, section 3.4.1

Main references

1. J.-P. Berrut and H. D. Mittelmann, *Lebesgue Constant Minimizing Linear Rational Interpolation of Continuous Functions over the Interval*, Computers Math. Appl. 33(6) (1997), 77–86.
2. Michael S. Floater and Kai Hormann, *Barycentric rational interpolation with no poles and high rates of approximation*, Numer. Math. 107(2) (2007), 315–331.
3. Q. Wang, P. Moin and G. Iaccarino, *A rational interpolation scheme with super-polynomial rate of convergence*, Annual Research Brief 2008, Centre for Turbulence Research, 31–54.
4. J. M. Carnicer, *Weighted interpolation for equidistant points*, Numer. Algorithms 55(2-3) (2010), 223–232.
5. L. Bos, S. De Marchi and K. Hormann, *On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes* , sumitted (2010).

General interpolation process

Given a function $f : [a, b] \to \mathbb{R}$, let g be its interpolant at the $n + 1$ (equispaced) interpolation points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a set of basis functions b_i which satisfy the Lagrange property

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the interpolant g can be written as $g(x) = \sum_{i=0}^{n} b_i(x) f(x_i)$.

Barycentric interpolation

- Interpolation of 2 data points

$$g(x) = \frac{\sum_{i=0}^{1} \lambda_i(x) y_i}{\sum_{i=0}^{1} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{x - x_i}.$$

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- Interpolation of $n+1$ data points

$$g(x) = \frac{\sum_{i=0}^{n} \lambda_i(x) y_i}{\sum_{i=0}^{n} \lambda_i(x)}, \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i)}.$$

$$\sum_{i=0}^{n} \lambda_i(x) = \underbrace{\frac{1}{x - x_0}}_{>0} + \underbrace{\frac{-1}{x - x_1} + \frac{1}{x - x_2}}_{>0} + \underbrace{\frac{-1}{x - x_3} + \cdots}_{>0} \quad x_0 < x < x_1$$

The Floater-Hormann Rational Interpolant (FHRI)

The construction of FHRI, is very simple.

- Choose any integer d, $0 \leq d \leq n$
- For each $i = 0, 1, \ldots, n-d$ let p_i denote the unique polynomial of degree at most d that interpolates a function f at $d+1$ pts $x_i, \ldots, x_{i+d}$
- Then

$$g(x) = \frac{\displaystyle\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\displaystyle\sum_{i=0}^{n-d} \lambda_i(x)} \tag{1}$$

where $\lambda_i(x) = \dfrac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}$.

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where $\lambda_i(x) = \dfrac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}$.

Thus, g is a local blending of the polynomial interpolants $p_0, \ldots, p_{n-d}$ with $\lambda_0, \ldots, \lambda_{n-d}$ acting as the blending functions.

Notice: **for $d = n$ we get the classical polynomial interpolation.**

FHRI

Assume $[a, b] = [0, 1]$ and interpolation points $x_i = i/n, \quad i = 0, ..., n$.

FHRI

Assume $[a,b]=[0,1]$ and interpolation points $x_i = i/n, \quad i=0,...,n$.
As **basis functions** we take

$$b_i(x) = \frac{(-1)^i \beta_i}{x - x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}, \qquad i = 0, \dots, n \tag{2}$$

with $\beta_0, \dots, \beta_n$ that are **positive weights** defined as

$$\beta_j = \begin{cases} \sum_{k=0}^{j} \binom{d}{k}, & \text{if } j \le d, \\ 2^d, & \text{if } d \le j \le n-d, \\ \beta_{n-j}, & \text{if } j \ge n-d. \end{cases} \tag{3}$$

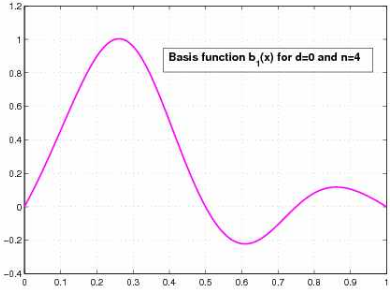
The weights β_s

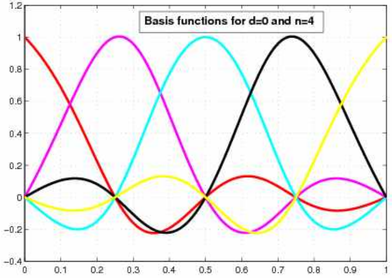
$d = 0^{\dagger}$	$1, 1, \ldots, 1, 1$
$d = 1^{\ddagger}$	$1, 2, 2 \ldots, 2, 2, 1$
$d = 2$	$1, 3, 4, 4, \ldots, 4, 4, 3, 1$
$d = 3$	$1, 4, 7, 8, 8, \ldots, 8, 8, 7, 4, 1$
$d = 4$	$1, 5, 11, 15, 16, 16, \ldots, 16, 16, 15, 11, 5, 1$

† Berrut's rational interpolant

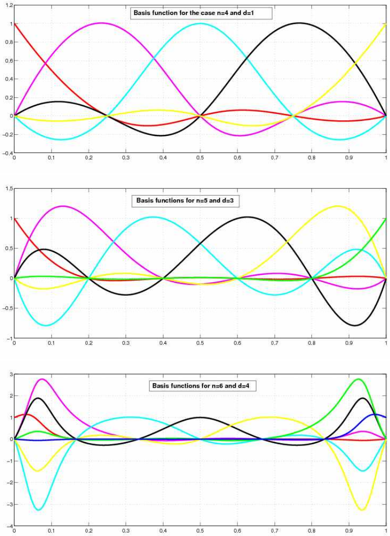
‡ $d = 1$ and weights $1/2, 1, \ldots, 1, 1/2$ in Berrut's paper and $d \geq 1$ Floater-Hormann's rational interpolant

Basis functions





Basis functions



Properties of the FHRI (cf. FH's paper, 2007)

1. The FHRI can be written in barycentric form. Indeed, in (1), letting $w_i = (-1)^i \beta_i$, for the numerator we have

$$\sum_{i=0}^{n-d} \lambda_i(x) p_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k} f(x_k)$$

where

$$w_k = \sum_{i \in I_k} (-1)^i \prod_{j \neq k, j=i}^{i+d} \frac{1}{x_k - x_j}$$

$I_k = \{i \in J, k - d \leq i \leq k\}$, $J := \{0, \ldots, n - d\}$, and similarly for the denominator

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$$\sum_{i=0}^{n-d} \lambda_i(x) = \sum_{k=0}^{n} \frac{w_k}{x - x_k}$$

2. The rational interpolant $g(x)$ has no real poles. For $d = 0$ was proved by Berrut in 1998.

Properties of the FHRI (continue)

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Properties of the FHRI (continue)

1. The interpolant reproduces polynomials of degree at most d, while does not reproduce rational functions (like Runge function)
2. Approximation order $\mathcal{O}(h^{d+1})$ (for $f \in \mathcal{C}^{d+2}[0,1]$), also for non-equispaced points.

Lebesgue constant when $d = 0$

We will derive upper and lower bounds for the Lebesgue function

$$\Lambda_n(x) = \sum_{i=0}^{n} |b_i(x)| = \sum_{i=0}^{n} \frac{\beta_i}{|x - x_i|} \Bigg/ \left| \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j} \right|. \quad (4)$$

that is $\Lambda = \max_{x \in [0,1]} \Lambda_n(x)$. Remember: when $d = 0$, $\beta_j = 1, \forall j$.

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Theorem

The Lebesgue constant is bounded as

$$c_n \log(n+1) \le \Lambda \le 2 + \log(n).$$

where $c_n = 2n/(4 + n\pi)$ *(*$\lim_{n \to \infty} c_n = 2/\pi$*).*

Case $d = 0$: lower bound

We assume that the interpolation interval is $[0, 1]$, so that the nodes are equally spaced $x_j = jh = j \cdot 1/n$, $j = 0, \ldots, n$.

Case $d = 0$: lower bound

We assume that the interpolation interval is $[0, 1]$, so that the nodes are equally spaced $x_j = jh = j \cdot 1/n, \;\; j = 0, \ldots, n$. Our goal is bounding below

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|x - j/n|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x - j/n}\right|} = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|2nx - 2j|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{2nx - 2j}\right|} := \frac{N(x)}{D(x)}. \tag{5}$$

by bounding $N(x)$ from below and $D(x)$ from above!

The Lebesgue function for $d = 0$ on uniform points

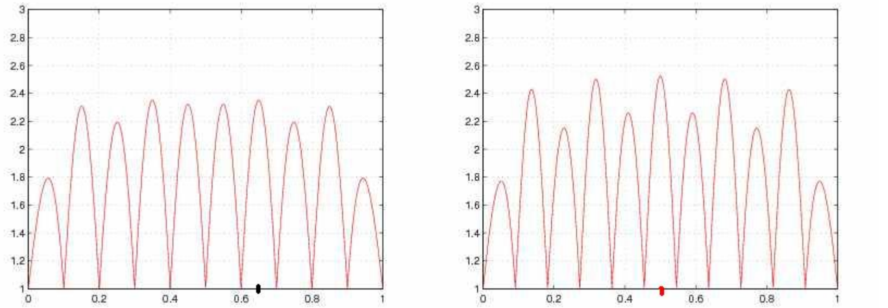

Figure: Lebesgue function on [0,1]: n=10, i.e. 11 points (left) and n=11, i.e. 12 points (right) .

Case $d = 0$: lower bound

Assume $n = 2k$ and let $x^* = (n+1)/2n = 1/2 + 1/(2n)$.

$$
\begin{aligned}
N(x^*) &= \sum_{j=0}^{n} \frac{1}{|n+1-2j|} \sum_{j=0}^{2k} \frac{1}{|2(k-j)+1|} \\
&= \sum_{j=0}^{k} \frac{1}{|2(k-j)+1|} + \sum_{j=k+1}^{2k} \frac{1}{|2(k-j)+1|} \\
&= \sum_{j=0}^{k} \frac{1}{2j+1} + \sum_{j=0}^{k-1} \frac{1}{2j+1} \\
&\geq \frac{1}{2}\left(\ln(2k+3) + \ln(2k+1)\right) \geq \ln(2k+1) = \ln(n+1)
\end{aligned}
$$

Case $d = 0$: lower bound

$$
\begin{aligned}
D(x^*) &= \left| \sum_{j=0}^{n} \frac{(-1)^j}{n+1-2j} \right| = \left| \sum_{j=0}^{2k} \frac{(-1)^j}{2(k-j)+1} \right| \\
&= \left| \sum_{j=0}^{k} \frac{(-1)^j}{2(k-j)+1} + \sum_{j=k+1}^{2k} \frac{(-1)^j}{2(k-j)+1} \right| \\
&\leq \left| (-1)^k \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \right| + \left| (-1)^k \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \right| = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} + \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \\
&\leq \left(\frac{\pi}{4} + \frac{1}{2k+3} \right) + \left(\frac{\pi}{4} + \frac{1}{2k+1} \right) \\
&\leq \frac{\pi}{2} + \frac{2}{2k+1} = \frac{\pi}{2} + \frac{2}{n+1} .
\end{aligned}
$$

Hence, $\Lambda_n(x^*) \geq \frac{2 \ln(n+1)}{\pi + \frac{4}{n+1}}$.

Case $d = 0$: lower bound

The same is true when n is odd considering $x^* = 1/2$, instead.

Case $d = 0$: lower bound

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In summary, for any $n \in \mathbb{N}$

$$\Lambda_n = \max_{0 \le x \le 1} \Lambda_n(x) \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}} \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n}} = c_n \ln(n+1)\,.$$

where $c_n = \frac{2n}{4+\pi n}$.

Case $d = 0$: upper bound

If $x = x_k$ for any k, then $\Lambda_n(x) = 1$.

Case $d = 0$: upper bound

If $x = x_k$ for any k, then $\Lambda_n(x) = 1$.
So let $x_k < x < x_{k+1}$ for some k and consider the function

$$\Lambda_k(x) = \frac{(x - x_k)(x_{k+1} - x)\sum_{j=0}^{n} \frac{1}{|x - x_j|}}{\left|(x - x_k)(x_{k+1} - x)\sum_{j=0}^{n} \frac{(-1)^j}{x - x_j}\right|} := \frac{N_k(x)}{D_k(x)}. \qquad (6)$$

$$\begin{aligned}
N_k(x) &= (x - x_k)(x_{k+1} - x)\sum_{j=0}^{n} \frac{1}{|x - x_j|} \\
&= (x - x_k)(x_{k+1} - x)\left(\sum_{j=0}^{k-1} \frac{1}{x - x_j} + \frac{1}{x - x_k} + \frac{1}{x_{k+1} - x} + \sum_{j=k+2}^{n} \frac{1}{x_j - x}\right) \\
&= (x_{k+1} - x) + (x - x_k) + (x - x_k)(x_{k+1} - x)\left(\sum_{j=0}^{k-1} \frac{1}{x - x_j} + \sum_{j=k+2}^{n} \frac{1}{x_j - x}\right) \\
&= (x_{k+1} - x_k) + (x - x_k)(x_{k+1} - x)\left(\sum_{j=0}^{k-1} \frac{1}{x - x_j} + \sum_{j=k+2}^{n} \frac{1}{x_j - x}\right).
\end{aligned}$$

Case $d = 0$: upper bound

As the x_i are equally spaced $\frac{1}{x_i - x_j} = \frac{1}{h(i-j)} = \frac{n}{i-j}$ for any $i \neq j$, and $(x - x_k)(x_{k+1} - x) \le \left(\frac{h}{2}\right)^2 = \frac{1}{4n^2}$ for $x_k < x < x_{k+1}$. Therefore,

$$\begin{aligned}
N_k(x) &\le \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{1}{x_k - x_j} + \sum_{j=k+2}^{n}\frac{1}{x_j - x_{k+1}}\right)\\
&= \frac{1}{n} + \frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{n}{k - j} + \sum_{j=k+2}^{n}\frac{n}{j - k - 1}\right)\\
&= \frac{1}{n} + \frac{1}{4n}\left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-k-1}\right)\\
&\le \frac{1}{n} + \frac{1}{4n}\left(\log(2k+1) + \log(2n-2k-1)\right)\\
&= \frac{1}{n} + \frac{1}{4n}\log\left((2k+1)(2n-(2k+1))\right)\\
&\le \frac{1}{n} + \frac{1}{4n}\log\left((2n/2)^2\right)\\
&= \frac{1}{n} + \frac{1}{2n}\log(n).
\end{aligned}$$

Case $d = 0$: upper bound

Let us consider the denominator $D_k(x)$.
Ignoring the absolute value and assuming, for a moment that both k and n to be even

$$\begin{aligned} D_k(x) &= (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{(-1)^j}{x - x_j} \\ &= (x - x_k)(x_{k+1} - x) \left(\sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} + \frac{1}{x - x_k} + \frac{1}{x_{k+1} - x} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \right) \\ &= h + (x - x_k)(x_{k+1} - x) \left(\sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \right) . \end{aligned}$$

Pairing the positive and negative terms

$$\begin{aligned} S_k(x) &= \sum_{j=0}^{k-1} \frac{(-1)^j}{x - x_j} - \sum_{j=k+2}^{n} \frac{(-1)^j}{x_j - x} \\ &= \frac{1}{x - x_0} + \left(\frac{1}{x - x_2} - \frac{1}{x - x_1} \right) + \cdots + \left(\frac{1}{x - x_{k-2}} - \frac{1}{x - x_{k-3}} \right) - \frac{1}{x - x_{k-1}} \\ &\quad - \frac{1}{x_{k+2} - x} + \left(\frac{1}{x_{k+3} - x} - \frac{1}{x_{k+4} - x} \right) + \cdots + \left(\frac{1}{x_{n-1} - x} - \frac{1}{x_n - x} \right) \end{aligned} \tag{7}$$

Case $d = 0$: upper bound

Since both the leading term and all paired terms are positive, we have

$$S_k(x) > -\frac{1}{x - x_{k-1}} - \frac{1}{x_{k+2} - x} \geq -\frac{1}{x_k - x_{k-1}} - \frac{1}{x_{k+2} - x_{k+1}} = -2n$$

and further

$$D_k(x) = h + (x - x_k)(x_{k+1} - x)S_k(x) \geq \frac{1}{n} + \frac{1}{4n^2}(-2n) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

This bound also holds if n is odd and if k is odd.

Case $d = 0$: upper bound

Therefore, we have $|D_k(x)| \geq 1/(2n)$ regardless of the parity of k and n, and combining the bounds for numerator and denominator yields

$$\Lambda = \max_{k=0,\dots,n} \left(\max_{x_k < x < x_{k+1}} \Lambda_k(x) \right) \leq \frac{\frac{1}{n} + \frac{1}{2n}\log(n)}{\frac{1}{2n}} = 2 + \log(n).$$

This completes the proof. □

The Lebesgue constant for $d = 0$ on uniform pts

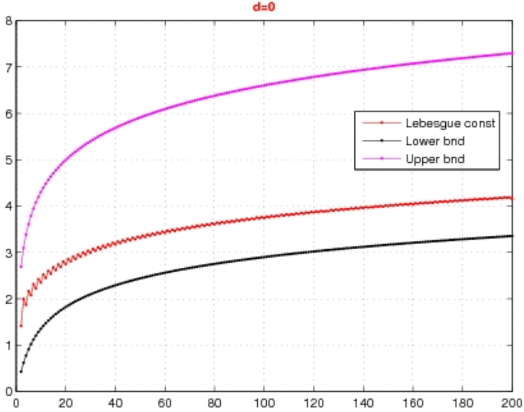

Figure: Lebesgue constant compared with its lower and upper bounds.

Lebesgue constant: case $d \geq 1$

We observe that

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Then

$$\begin{aligned} N_k(x) &= (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{\beta_j}{|x - x_j|} \\ &\leq 2^d (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{1}{|x - x_j|} \\ &\leq 2^d \left(\frac{1}{n} + \frac{1}{2n} \log(n) \right), \end{aligned} \tag{8}$$

for any k.

The **denominator**

$$D_k(x) = (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j},$$

it will turn out that $|D_k(x)| \geq 1/n$

The **denominator**

Fundamental observation

$$(-1)^j \beta_j = w_j \, d! \, h^d \tag{9}$$

Then,

$$D_k(x) = (x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right| d! h^d \ .$$

The **denominator**

Moreover, in Floater-Hormann's paper 2007,

$$\sum_{j=0}^{n} \frac{w_j}{x - x_j} = \sum_{i=0}^{n-d} \lambda_i(x)$$

showing also that

$$\left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right| \geq |\lambda_k(x)| \,.$$

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Then,

$$D_k(x) = (x - x_k)(x_{k+1} - x)|\lambda_k(x)| d! h^d = \frac{d! h^d}{\prod_{l=k+2}^{k+d}(x_l - x)}$$

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Then,

$$D_k(x) = (x - x_k)(x_{k+1} - x)|\lambda_k(x)| d! h^d = \frac{d! h^d}{\prod_{l=k+2}^{k+d} (x_l - x)}$$

Maximizing over k we get

$$D_k(x) \geq \frac{d! h^d}{\prod_{l=k+2}^{k+d} (x_l - x_k)} = h = \frac{1}{n} \, .$$

The **lower bound**

Theorem

(Klein, Dec. 2010) Let $d \geq 2$, then,

$$\Lambda \geq \frac{(2d+1)!!}{4(d+1)!} \log\left(\frac{n}{d} - 1\right).$$

Theorem

(Bos, Dec. 2010)

$$\Lambda \geq \frac{2}{\pi} \log(n + 2 - 2d).$$

This latter is better for $d = 1$.

The theorem for $d \geq 1$

Theorem

Let $d > 1$ Then,

$$\frac{(2d+1)!!}{4(d+1)!} \log\left(\frac{n}{d} - 1\right) \leq \Lambda \leq 2^{d-1}(2 + \log(n)).$$

while for $d = 1$

$$\frac{2}{\pi} \log(n) \leq \Lambda \leq 2 + \log(n).$$

Lebesgue functions

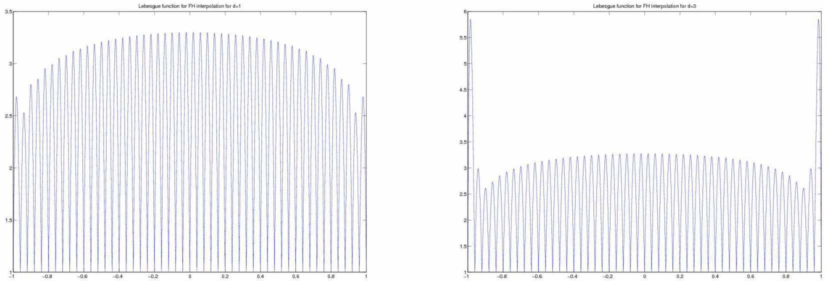
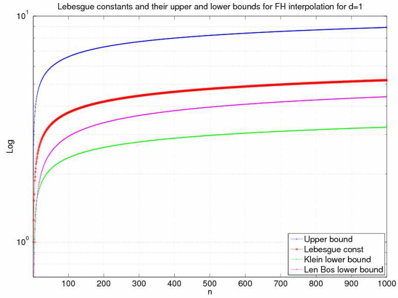


Figure: Lebesgue function for $d = 1$ (left) and $d = 3$ (right).

Lebesgue constants



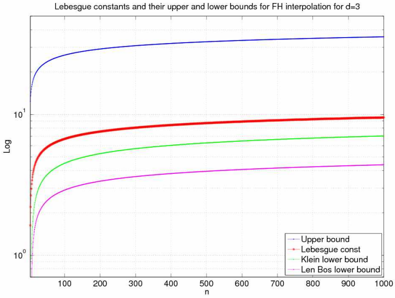


Figure: Lebesgue constant growth $d = 1$ (left) and $d = 3$ (right).

Lebesgue constant growth

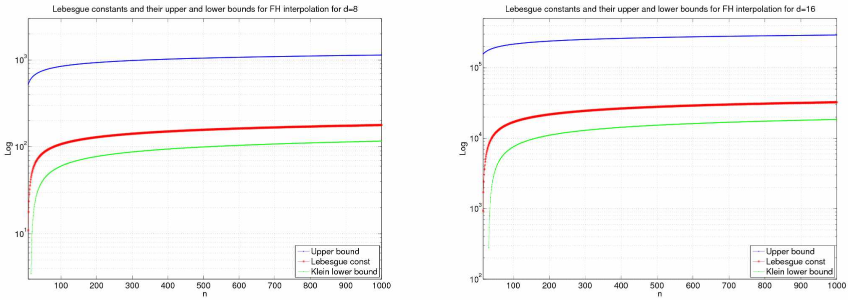


Figure: Lebesgue constant growth $d = 8$ (left) and $d = 16$ (right).

Lebesgue constant growth for non uniform pts

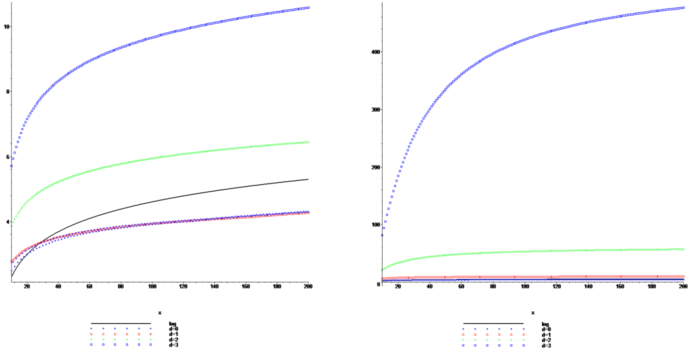


Figure: Lebesgue constant on logarithmically distributed points. **Left**: with weights $(-1)^i\beta_i$. **Right**: here the weights are the ones constructed on non-equispaced points, garanteeing the approximation order $d+1$

Points equally spaced w.r.t. a distribution

Consider the interval $I = [0, 1]$ and a distribution function $F \in \mathcal{C}^1(I)$

- $F(0) = 0,\ F(1) = 1$ and F is strictly increasing.

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- $F(x) = \int_0^x w(t)dt$ for a certain $w \in \mathcal{C}[0,1],\ w(x) > 0,\ x \in I$
- Moreover, if $F'(x) > 0$ on I, we say F is non-singular.

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$$x_j := F^{-1}\left(\frac{j}{n}\right),\ \ 0 \le j \le n$$

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Examples:

- $F(x) = x$ usual equally spaced pts,
- $F(x) = (1 - cos(\pi x))/2,\ x \in [0, 1]$ the extended Chebyshev points for which $F'(0) = F'(1) = 0$, i.e. they form a singular distribution

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Lemma

If $f \in \mathcal{C}[0,1]$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} f(x_j) = \int_0^1 f(x)w(x)dx$$

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Proof The key observation is that $\frac{1}{n}\sum_{j=0}^{n} f(x_j) = \frac{1}{n}\sum_{j=0}^{n} f(F^{-1}(j/n))$ is a Riemann sum for $f \circ F^{-1} \in \mathcal{C}[0,1]$ and hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} f(x_j) = \int_0^1 f(F^{-1}(t))dt \, .$$

Since $x = F^{-1}(t)$, then $dx = \left(\frac{d}{dt}F^{-1}(t)\right) dt = \frac{dt}{F'(F^{-1}(t))} = \frac{dt}{w(x)}$. Then, $dt = w(x)dx \, .$ $\square$

Points equally spaced w.r.t. a distribution

Lemma

Suppose that $k, n \to \infty$ in such a way that $x_k = F^{-1}(k/n)$ and $x_{k+1} = F^{-1}((k+1)/n)$ both tend to $x = F^{-1}(a)$. Then,

$$\lim_{n\to\infty} nh_k = (F^{-1})'(x) = \frac{1}{w(x)} \ .$$

where $h_k = x_{k+1} - x_k$.

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Proof

$$\begin{aligned} n\, h_k &= n(x_{k+1} - x_k) = n\left(F^{-1}((k+1)/n) - F^{-1}(k/n)\right) \\ &= \frac{F^{-1}((k+1)/n) - F^{-1}(k/n)}{1/n} = F^{-1}\left[\frac{k+1}{n}, \frac{k}{n}\right] = (F^{-1})'(c_n), \quad \textit{for } c_n \in \end{aligned}$$

Hence, $\lim_{n\to\infty} n\, h_k = \lim_{n\to\infty} (F^{-1})'(c_n) = (F^{-1})'(a)$ as $c_n \to a$. But $(F^{-1})'(a) = \frac{1}{F'(F^{-1}(a))} = \frac{1}{w(F^{-1}(a))} = \frac{1}{w(x)}$, $\square$

Points equally spaced w.r.t. a distribution

Note also that, as $(F^{-1})'(t) = \dfrac{1}{w(F^{-1}(t))} > 0$ and it is continuous (by assumption) then there exist two positive constants c_1, c_2 so that

$$c_1 < nh_k < c_2\,.$$

Points equally spaced w.r.t. a distribution

Theorem

F and w as above and $x_j^{(n)} = F^{-1}(j/n), \; 0 \le j \le n$. Let

$$b_i(x) = \frac{(-1)^i \beta_i}{x - x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x - x_j}, \qquad i = 0, \dots, n, \quad \Lambda_n(x) = \sum_{i=0}^{n} |b_i(x)| \,, \tag{10}$$

Then, there is a constant C such that

$$\Lambda_n(x) \le C \; \log(n), \quad x \in [0,1] \,. \tag{11}$$

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A similar result holds for the singular case (cf. Bos, De Marchi and Hormann, paper in progress).

Numerical Quadrature

On $I = [-1, 1]$

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Numerical Quadrature

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1. we computed integrals with the quadrature based on the FHRI, on equispaced points at different values of n end/or d
2. to speed up the quadrature, the quadrature weights were computed by a Gaussian quadrature rule (Gautschi software in Matlab)
3. For $d = 0$ we have proven (a week ago!) that
   (a) $b_i(x) = \mathtt{sinc}(n(x - x_i))$ normalized so that $\sum_i b_i(x) = 1$
   (b)
   $$\lim_{n \to \infty} n\,\alpha_i = 1, \;\; \alpha_i = \int_0^1 b_i(x)dx\,.$$
   that is the quadrature process asymptotically converges.

b_i e sinc

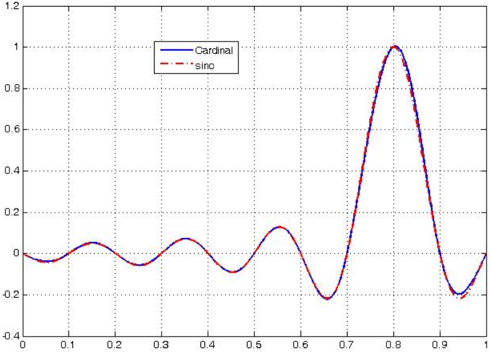

Figure: Comparison of b_{n-1} and $sinc(n(x - x_{n-1}))$ for $n = 10$

err $= 0.0101$

nw $=(1.10072,0.94272,1.03588,0.97972,1.00660,1.00660,0.97972,1.03588\ 0.94272,1.10072)$

Numerical Quadrature

The table below shows the quadrature relative errors for $d = 0$ (left) and $d = 3$ (right) at different n, for the Runge function. errS=quadrature relative error by using cubic splines

n	err (d=0)	err (d=3)	errS
10	3.5e-3	1.1e-2	7.2e-3
30	1.1e-4	1.6e-6	5.9e-5
50	7.6e-6	2.6e-8	3.2e-7
100	3.6e-7	7.9e-10	2.4e-8
150	4.9e-7	1.0e-10	1.5e-9
200	5.4e-7	2.4e-11	6.4e-11

Numerical Quadrature

About the quadrature weights: Klein and Berrut have proven numerically that the weights are all positive at least for $d \le n \le 1250$ and $0 \le d \le 5$. For other values of d and n, there might be a few negative weights, the number of which increases slowly with d and n.

THANK YOU
FOR YOUR KIND ATTENTION