Weakly Admissible Meshes and Discrete Extremal Sets *

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* Joint work with L. Bos (Verona), A. Sommariva and M. Vianello (Padua)

Outline

Motivations and aims

- Nearly optimal interpolation points on compacts of $K \subset \mathbb{R}^d$: existence versus computation.
- (Weakly) Admissible Meshes for compacts: discrete sets that contain good interpolation nodes
- Computation: greedy algorithms for computing *approximate Fekete points* and *discrete Leja points*.

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Main references

1. L. Bos, S. De Marchi, A. Sommariva and M. Vianello: *Computing multivariate Fekete and Leja points by numerical linear algebra*, to appear on SIAM J. Numer. Anal. (2010).
2. L. Bos, S. De Marchi, A. Sommariva and M. Vianello: *Weakly Admissible Meshes and Discrete Extremal Sets*, to appear on Numer. Math. Theory Meth. Appl. (2010).
3. L. Bos, J.P. Calvi and N. Levenberg, A. Sommariva and M. Vianello, *Geometric Weakly Admissible Meshes, Discrete Least Squares Approximation and Approximate Fekete Points*, to appear in Math. Comp. (2009).
4. L. Bos, A. Sommariva and M. Vianello, *Least Squares Polynomial Approximation on Weakly Admissible Meshes: disk and triangle*, to appear in J. Comput. Appl. Math.(2009).
5. J.P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008), 82–100.

Weakly Admissible Meshes (WAMs)

Given a polynomial determining compact set $K \subset \mathbb{R}^d$.

Definition

A Weakly Admissible Mesh (WAM) is a sequence of discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n}\ , \ \ \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

where both $\mathrm{card}(\mathcal{A}_n) \geq N := dim(\mathbb{P}_n^d(K))$ and $C(\mathcal{A}_n)$ grow at most polynomially with n.

When $C(\mathcal{A}_n)$ is bounded we speak of an Admissible Mesh (AM).

Main properties of WAMs

P1: $C(\mathcal{A}_n)$ is invariant for affine transformations.

P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself

P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$

P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants

P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants

P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constants $C(\mathcal{A}_{ns})$ (cf. BCLSV Math. Comp.09)

Main properties of WAMs (continues)

P8: any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \leq Mn^r \|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. CL JAT08)

P9: The least-squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\, \min\{\|f - p\|_K,\ p \in \mathbb{P}_n^d(K)\}$$

P10: The Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq NC(\mathcal{A}_n)$

Moreover, their asymptotic distibution is the same of the continuum Fekete points, in the sense that the corresponding discrete probability measures converge weak-$*$ to the pluripotential equilibrium measure of K (cf. BCLSV Math. Comp.09)

AM for convex-compact sets

Assume $K \subset \mathbb{R}^2$ convex and compact. Marcov inequality holds with exponent $r = 2$.

By property **P8** we are able to construct an Admissible Mesh, $\mathcal{A}_n$ of K, with $\mathcal{O}(n^4)$ points.

Actually, it is $\mathrm{card}(\mathcal{A}_n) \approx n^4\, A(K)\, M^2$ (ex. πn^4 for the unit disk),

$A(K)$: the area of the compact;

$M = \alpha(K)/w(K)$: with $\alpha(K) \le 4$ and $w(K)$ is the minimal distance between 2 parallel supporting lines of a rectangle that covers K .

Too big to be computed!!!

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WAM for the disk and the triangle

In BSV JCAM 09, it was proved that for the disk and the triangle there are WAMs with approximately n^2 points and still the same growth of $C(\mathcal{A}_n)$.

- **UNIT DISK**: a symmetric polar WAM is made by equally spaced angles and Chebyshev-Lobatto pts along diameters.
- **UNIT SIMPLEX**: starting from the WAM of the disk for polynomials of degree $2n$ containing only even powers, by the standard quadratic transformation

$$(u, v) \longmapsto (x, y) = (u^2, v^2)\,.$$

Notice: by affine transformation these WAMs can be mapped to any other triangle (**P1**) or polygons (**P4**).

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WAMs for the disk

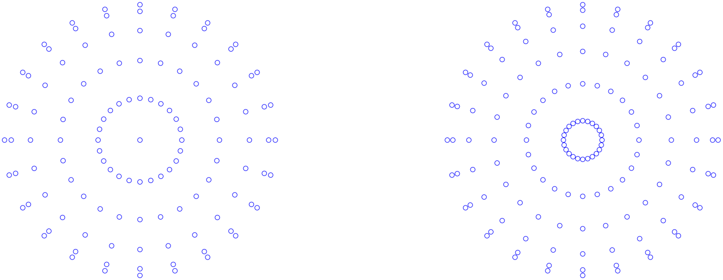

Figure: Symmetric polar WAM for the disk for degree $n = 10$ (left) and $n = 11$ (right).

WAMs for quadrant and the triangle

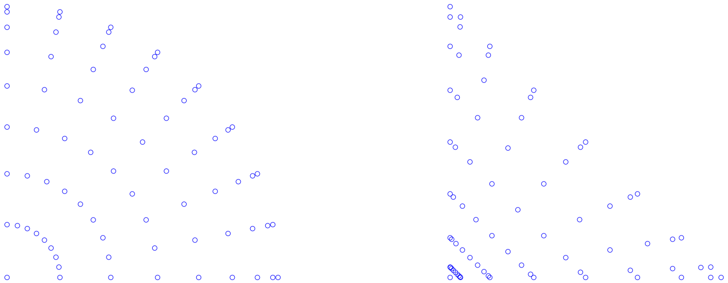

Figure: A WAM of the first quadrant for polynomial degree $n = 16$ (left) and the corresponding WAM of the simplex for $n = 8$ (right).

Fekete points

Consider

- $K \subset \mathbb{R}^d (or\ \mathbb{C}^d)$, $S_N = \text{span}(p_j)_{1 \le j \le N}$, $\{\xi_1, \ldots, \xi_N\} \subset K$
- Vandermonde-like matrix $V(\xi; \mathbf{p}) = [p_j(\xi_i)]$, $1 \le i, j \le N$. Assume unisolvency.
- $$\ell_j(x) = \frac{\det V(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_N; \mathbf{p})}{\det V(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N; \mathbf{p})}, \quad j = 1, \ldots, N, \quad (2)$$
- $\ell = (\ell_1, \ldots, \ell_N)^t$ is obtained from the basis $\mathbf{p} = (p_1, \ldots, p_N)^t$ as the solution of $\ell = L\mathbf{p}$, $L := (V(\xi; \mathbf{p}))^{-t}$.

Definition

The points that maximize the (absolute value of the) Vandermonde-like determinant in K^N are the Fekete points.

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Fekete points

Then it follows:

- $\|\ell_j\|_\infty \le 1$ for every j.
- the norm of the operator $L_{S_N} : C(K) \to S_N$ is bounded as follows

$$\|L_{S_N}\| = \max_{x \in K} \sum_{j=1}^{N} |\ell_j(x)| = \max_{x \in K} \|L\mathbf{p}(x)\|_1 \le N \,. \qquad (3)$$

- $\Lambda_n := \|L_{S_N}\|$ is the Lebesgue constant at the point set ξ.
- Fekete pts and Lebesgue constant are preserved by affine maps.
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Known open problems with Fekete points

- Analytically known only in the interval (Gauss-Lobatto pts), the complex circle (equispaced pts) and the cube (tensor product of Gauss-Lobatto pts);
- Their asymptotic spacing (i.e. equidistributed w.r.t. the pluripotential equilibrium measure of K) is known only in few cases (cf. BLW In. Math. 08).
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Approximated Fekete Pts and Discrete Leja Pts

An approximate solution can be given by one of the following **2 greedy algorithms**, **A1, A2**, given in Matlab-like notation, which compute what we call **Discrete Extremal Sets**(cf. BCLSV09, BDMSV09).

A1: Approximate Fekete Points (AFP):

- $V = V(\mathbf{a}, \mathbf{p})$; $ind = [\,]$;
- `for` $k = 1 : N$ "select i_k: vol $V([ind, i_k]\,,\,1 : N)$ is maximum";

$ind = [ind, i_k]$; `end`

- $\xi = \mathbf{a}(i_1, \ldots, i_N)$

A2: Discrete Leja Points (DLP):

- $V = V(\mathbf{a}, \mathbf{p})$; $ind = [\,]$;
- `for` $k = 1 : N$ "select i_k: $|\det V([ind, i_k]\,,\,1 : k)|$ is maximum";

$ind = [ind, i_k]$; `end`

- $\xi = \mathbf{a}(i_1, \ldots, i_N)$

Remarks

- **A2** depends on the ordering of the polynomial basis. In 1-d it produces the **Leja points**.
- **A1** does not depend on the ordering. It is based on the notion of volume generated by the rows of a rectangular matrix.
- DLP form a sequence. Once we have computed the points for degree n, we have automatically at hand (nested) interpolation sets for all lower degrees.

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Algorithm 1

The core in **A1**

select i_k: vol $V([ind, i_k], 1:N)$ is maximum

can be implemented as

select the largest norm row $row_{i_k}(V)$ and remove from every row of V its orthogonal projection onto row_{i_k}

This process is then equivalent to the **QR factorization with column pivoting**.

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Algorithm 2

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Discrete Extremal Sets

The computation of such sets can be done by basic linear algebra operations

- the QR factorization with column pivoting of the transposed Vandermonde matrix (cf. SV09-1).
- LU factorization with row pivoting of the Vandermonde matrix (cf. SDM09).

This is summarized in the following Matlab-like scripts.

A1-AFP :
• $W = (V(\mathbf{a},\mathbf{p}))^t$; $\mathbf{b} = (1,\ldots,1)^t \in \mathbb{C}^N$; $\mathbf{w} = W\backslash\mathbf{b}$;
$ind = \mathtt{find}(\mathbf{w} \neq \mathbf{0})$; $\xi = \mathbf{a}(ind)$

A2-DLP:
• $V = V(\mathbf{a},\mathbf{p})$; $[L, U, \boldsymbol{\sigma}] = \mathtt{LU}(V, \text{“}\mathtt{vector}\text{”})$; $ind = \boldsymbol{\sigma}(1,\ldots,N)$;
$\xi = \mathbf{a}(ind)$

Theorem

Theorem

(cf. BCLSV Math. Comp. 09)
Suppose that $K \subset C^d$ is compact, non-pluripolar, polynomially convex and regular (in the sense of Pluripotential theory) and that for $n = 1, 2, \ldots, \mathcal{A}_n \subset K$ is a WAM. Let $\{\xi_1, \ldots, \xi_N\}$ be the AFP selected from $\mathcal{A}_n$ by the algorithm **A1-AFP**, *using any polynomial basis $\mathbf{p} = \{p_1, \ldots, p_N\}$ or the DLP selected from $\mathcal{A}_n$ by the algorithm* **A2-DLP** *using any basis of the form* $\mathbf{p}$=$L\mathbf{e}$ *where* $\mathbf{e}=\{e_1, ..., e_N\}$ *is any ordering of the standard monomials $\mathbf{x}^\alpha$ consistent with the degree and $L \in C^N \times C^N$ is lower triangular. Then*

- $\lim_{n\to\infty} |\mathrm{vdm}(\xi_1, \ldots, \xi_N)|^{1/m_n} = \tau(K)$, *the transfinite diameter of K ($m_n = dnN/(d+1)$ is the sum of the degrees of the N monomials of degree $\leq n$);*
- *the discrete probability measures $\mu_n := \frac{1}{N}\sum_{j=1}^N \delta_{\xi_j}$ converge weak-* to the pluripotential-theoretic equilibrium measure $d\mu_K$ of K.*

Example1: AMs for a circular sector

- In the case of a quadrant, an AM has $\mathtt{card}(\mathcal{A}_n) \approx 7n^4$.
- By Properties **P1** and **P4** we get an AM of the 3/4-circular sector as union of 3 meshes of the three quadrants, with cardinality $\mathtt{card}(\mathcal{A}_n) \approx 3 \times 7\,n^4 = 21\,n^4$.

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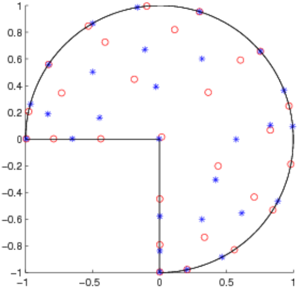

Figure: $N = 28$ AFP (Approximate Fekete Points, circles) and DLP (Discrete Leja Points, asterisks) for degree $n = 6$ extracted from an AM of a circular sector.

We used the Koornwinder orthogonal basis of the unit disk for setting the Vandermonde matrix.

Remarks for this first example

1. The computation of DLP is 3 times faster than that of AFP.
2. The quality of AFP is better than that of DLP.

   For example, in the case $n = 6$ and the quadrant(s), the AFP not only do they appear more evenly distributed than the DLP, but in addition the absolute value of the Vandermonde determinant and the Lebesgue constant (numerically evaluated) are $|vdm| \approx 2 \cdot 10^4$ and $\Lambda_6 \approx 4$ for the AFP, whereas $|vdm| \approx 7 \cdot 10^2$ and $\Lambda_6 \approx 12$ for the DLP. Notice that both the Lebesgue constants are much below the theoretical bound for Fekete points extracted from an AM, namely $\Lambda_n \leq CN$.

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Example 2: WAMs for a nonregular convex hexagon

Here the WAMs are generated by 2 different triangulations, barycentric by the "ear-clipping alg.", and minimal with $6-2=4$ triangles), and the Chebyshev product basis of the minimal including rectangle.

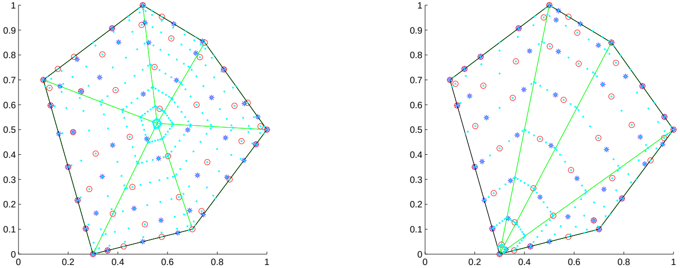

Figure: $N=45$ AFP (circles) and DLP (asterisks) for degree $n=8$ extracted from 2 WAMs of an hexagon (dots). The first WAM has $6n^2-2$ points and the second one has $4n^2+n-1$ points.

Example 2: Lebesgue constants and interpolation errors

Table: Lebesgue constants for AFP and DLP extracted from two WAMs of a nonregular convex hexagon (WAM1: barycentric triangulation, WAM2: minimal triangulation; see Fig. 4).

mesh	points	$n=5$	$n=10$	$n=15$	$n=20$	$n=25$	$n=30$
WAM1	AFP	6.5	18.9	20.4	40.8	73.3	73.0
	DLP	7.1	19.6	49.8	58.3	108.0	167.0
WAM2	AFP	6.8	12.3	34.2	52.3	49.0	80.4
	DLP	10.7	48.4	62.0	91.6	86.6	203.0

Table: Max-norm of the interpolation errors with AFP and DLP extracted from WAM2 for two test functions: $f_1 = \cos(x_1 + x_2)$; $f_2 = ((x_1 - 0.5)^2 + (x_2 - 0.5)^2)^{3/2}$.

function	points	$n=5$	$n=10$	$n=15$	$n=20$	$n=25$	$n=30$
f_1	AFP	6E-06	5E-13	3E-15	3E-15	3E-15	4E-15
	DLP	8E-06	2E-12	2E-15	4E-15	3E-15	4E-15
f_2	AFP	3E-03	2E-04	1E-04	4E-05	2E-05	1E-05
	DLP	3E-03	3E-04	1E-04	3E-05	2E-05	5E-06

Example 3: Sommariva's left hand

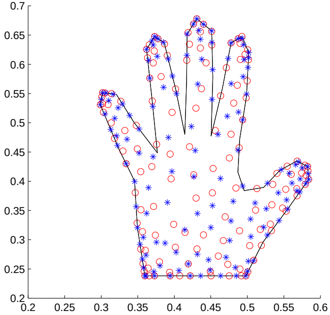

Figure: The hand has 39 sides. By the "ear-clipping algorithm" we computed 37 triangles. The WAM has $37n^2$ points for degree n. For degree $n = 15$ we obtained $N = 136$ AFP (circles) and DLP (asterisks).

Some applications

1. Numerical cubature. If in **A1-AFP** we take as RHS $\mathbf{b} = \mathbf{m} = \int_K \mathbf{p}(x)\, d\mu$ (the *moments*), the vector $\mathbf{w}(ind)$ gives directly the weights of an algebraic cubature formula at the corresponding AFP. The same holds for **A2-DLP**.
2. Weighted polynomial interpolation (cf. SV09-3 with AFP). One considers a basis $w\mathbf{p}$ for the Vandermonde matrix, w being a suitable weight function. Examples: approximation with weighted norm for the construction of digital filters.
3. Solution of PDEs by spectral and high-order methods. Here one needs to locate good points for polynomial approximation on polygonal regions/elements.

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To Borislav Bojanov

On the top of Piz Boè, 3152 m., September 6th, 2006.

This is my simple tribute to Borislav.

Third DOLOMITES RESEARCH WEEK ON APPROXIMATION
Alba di Canazei, Trento (Italy), September 6-9, 2010
http://www.math.unipd.it/∼demarchi/dwcaa10/

THANK YOU
FOR YOUR KIND ATTENTION