New cubature and hyperinterpolation in the cube

Stefano De Marchi[1], Marco Vianello[2] and Yuan Xu[3]

[1]Dept. of Computer Science, University of Verona
http://www.sci.univr.it/~demarchi

[2]Dept. Pure and Applied Mathematics, University of Padua
http://www.math.unipd.it/~marcov

[3]Dept. of Mathematics, University of Oregon
http://www.uoregon.edu/~yuan

Tønsberg, 28 June 2008

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Motivations

- Finding Padua points in the cube (in the square Bos&al JAT06) .... so far, only attempts... no general answers.
- In Caliari&al. JCAM 08, we studied hyperinterpolation in the cube on MPX $\times$ Chebyshev points.
- In Sommariva&al. NA 08, the authors studied a non-tensorial Clenshaw-Curtis cubature in the square, integrating the hyperinterpolant on MPX and Padua points.
- Does exist any "suitable" set of points for near-minimal (Chebyshev and Clenshaw-Curtis formulae) cubature in the cube ?

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

- Finding Padua points in the cube (in the square Bos&al JAT06) .... so far, only attempts... no general answers.
- In Caliari&al. JCAM 08, we studied hyperinterpolation in the cube on MPX $\times$ Chebyshev points.
- In Sommariva&al. NA 08, the authors studied a non-tensorial Clenshaw-Curtis cubature in the square, integrating the hyperinterpolant on MPX and Padua points.
- Does exist any "suitable" set of points for near-minimal (Chebyshev and Clenshaw-Curtis formulae) cubature in the cube ?

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Motivations

- Finding Padua points in the cube (in the square Bos&al JAT06) .... so far, only attempts... no general answers.
- In Caliari&al. JCAM 08, we studied hyperinterpolation in the cube on MPX $\times$ Chebyshev points.
- In Sommariva&al. NA 08, the authors studied a non-tensorial Clenshaw-Curtis cubature in the square, integrating the hyperinterpolant on MPX and Padua points.
- Does exist any "suitable" set of points for near-minimal (Chebyshev and Clenshaw-Curtis formulae) cubature in the cube ?

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Motivations

- Finding Padua points in the cube (in the square Bos&al JAT06) .... so far, only attempts... no general answers.
- In Caliari&al. JCAM 08, we studied hyperinterpolation in the cube on MPX $\times$ Chebyshev points.
- In Sommariva&al. NA 08, the authors studied a non-tensorial Clenshaw-Curtis cubature in the square, integrating the hyperinterpolant on MPX and Padua points.
- Does exist any "suitable" set of points for near-minimal (Chebyshev and Clenshaw-Curtis formulae) cubature in the cube ?

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Cubature formulae (1)

A cubature formula of degree of exactness $2n+1$ with N nodes w.r.t. a measure $d\mu$ supported on a set Ω takes the form

$$\int_\Omega p(x)\, d\mu = \sum_{\xi \in X_n} w_\xi\, p(\xi) \qquad \text{forall} \quad p \in \Pi^d_{2n+1}(\Omega)\ , \tag{1}$$

where the weights $\{w_\xi\}$, are (positive) numbers; the nodes

$$\xi := (\xi_1, \xi_2, \ldots, \xi_d) \in X_n \subset \Omega \tag{2}$$

with $\mathrm{card}(X_n) = N$, and Π^d_m denotes the subspace of d-variate polynomials of total degree $\leq m$ restricted to Ω.

For a cubature formula of degree $2n+1$ to exist, it is necessary that

$$N := \mathrm{card}(X_n) \geq \dim(\Pi^d_n(\Omega)) = \binom{n+d}{d} = \frac{n^d}{d!}(1+o(1)). \tag{3}$$

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \qquad (4)$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \tag{4}$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \tag{4}$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \tag{4}$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \tag{4}$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Cubature formulae (2)

A challenging problem is to construct cubature formulae with fewer nodes, that is, with N close to the lower bound (3).

In this work we consider the case that the measure is given by the product Chebyshev weight function

$$d\mu = W_d(x)\,dx, \qquad W_d(x) := \frac{1}{\pi^d}\prod_{i=1}^{d}\frac{1}{\sqrt{1-x_i^2}} \tag{4}$$

on the cube $\Omega := [-1,1]^d$. Our main result is a new family of cubature formulae that uses $N \approx n^d/2^{d-1}$ many nodes.

- $d = 1$: Gauss formulae are the minimal, $N = n + 1$.
- $d = 2$: these formulae are known to have minimal number of nodes.
- $d \geq 3$: they are still far from the lower bound, but they appear to be the best ones known at the moment.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Hyperinterpolation (1) (cf. Sloan JAT (1995))

Hyperinterpolation is an approximation method constructed by applying the cubature formula to the Fourier coefficients of the orthogonal projection operator.

For every function $f \in C(\Omega)$ the μ-orthogonal projection of f on $\Pi_n^d(\Omega)$ is

$$\mathcal{S}_n f(x) = \sum_{|\alpha| \le n} a_\alpha \, p_\alpha(x), \qquad a_\alpha := \int_\Omega f(x) \, p_\alpha(x) \, d\mu \,, \quad (5)$$

where x is a d-dimensional point, α is a d-index of length $|\alpha|$ and the polynomials $\{p_\alpha \,, \, 0 \le |\alpha| \le n\}$ is any μ-orthonormal basis of $\Pi_n^d(\Omega)$ with p_α of total degree $|\alpha|$.

Clearly, $\mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Hyperinterpolation (1) (cf. Sloan JAT (1995))

Hyperinterpolation is an approximation method constructed by applying the cubature formula to the Fourier coefficients of the orthogonal projection operator.

For every function $f \in C(\Omega)$ the μ-orthogonal projection of f on $\Pi_n^d(\Omega)$ is

$$\mathcal{S}_n f(x) = \sum_{|\alpha| \le n} a_\alpha \, p_\alpha(x), \qquad a_\alpha := \int_\Omega f(x) \, p_\alpha(x) \, d\mu \,, \quad (5)$$

where x is a d-dimensional point, α is a d-index of length $|\alpha|$ and the polynomials $\{p_\alpha \,, \, 0 \le |\alpha| \le n\}$ is any μ-orthonormal basis of $\Pi_n^d(\Omega)$ with p_α of total degree $|\alpha|$.

Clearly, $\mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$

Hyperinterpolation (2)

Given a cubature formula (1) of degree $\leq 2n$ we can construct an hyperinterpolant as follows.

- From (5), the polynomial approximation of degree n can be obtained by the *discretized Fourier coefficients* $\{c_\alpha\}$, i.e.

$$f(x) \approx \mathcal{L}_n f(x) := \sum_{|\alpha| \leq n} c_\alpha \, p_\alpha(x) \, , \quad c_\alpha := \sum_{\xi \in X_n} w_\xi \, f(\xi) \, p_\alpha(\xi) \tag{6}$$

  where $c_\alpha = a_\alpha$ and thus $\mathcal{L}_n p = \mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$.

- Moreover, for every $f \in C(\Omega)$, the basic estimate holds:

$$\|f - \mathcal{L}_n f\|_{L^2_{d\mu}(\Omega)} \leq 2\sqrt{\mu(\Omega)} \, E_n(f) \to 0 \, , \quad n \to \infty \, , \tag{7}$$

  where $E_n(f) := \inf\{\|f - p\|_\infty \, , \; p \in \Pi_n^d(\Omega)\}$, so that it converges in mean.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Hyperinterpolation (2)

Given a cubature formula (1) of degree $\leq 2n$ we can construct an hyperinterpolant as follows.

- From (5), the polynomial approximation of degree n can be obtained by the *discretized Fourier coefficients* $\{c_\alpha\}$, i.e.

$$f(x) \approx \mathcal{L}_n f(x) := \sum_{|\alpha| \leq n} c_\alpha \, p_\alpha(x) \; , \quad c_\alpha := \sum_{\xi \in X_n} w_\xi \, f(\xi) \, p_\alpha(\xi) \; , \tag{6}$$

  where $c_\alpha = a_\alpha$ and thus $\mathcal{L}_n p = \mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$.

- Moreover, for every $f \in C(\Omega)$, the basic estimate holds:

$$\|f - \mathcal{L}_n f\|_{L^2_{d\mu}(\Omega)} \leq 2\sqrt{\mu(\Omega)}\, E_n(f) \to 0 \; , \quad n \to \infty \; , \tag{7}$$

  where $E_n(f) := \inf \{\|f - p\|_\infty \; , \; p \in \Pi_n^d(\Omega)\}$, so that it converges in mean.

Hyperinterpolation (2)

Given a cubature formula (1) of degree $\leq 2n$ we can construct an hyperinterpolant as follows.

- From (5), the polynomial approximation of degree n can be obtained by the *discretized Fourier coefficients* $\{c_\alpha\}$, i.e.

$$f(x) \approx \mathcal{L}_n f(x) := \sum_{|\alpha| \leq n} c_\alpha \, p_\alpha(x) \; , \quad c_\alpha := \sum_{\xi \in X_n} w_\xi \, f(\xi) \, p_\alpha(\xi) \tag{6}$$

  where $c_\alpha = a_\alpha$ and thus $\mathcal{L}_n p = \mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$.

- Moreover, for every $f \in C(\Omega)$, the basic estimate holds:

$$\|f - \mathcal{L}_n f\|_{L^2_{d\mu}(\Omega)} \leq 2\sqrt{\mu(\Omega)}\, E_n(f) \to 0 \; , \quad n \to \infty \; , \tag{7}$$

  where $E_n(f) := \inf\{\|f - p\|_\infty \, , \; p \in \Pi_n^d(\Omega)\}$, so that it converges in mean.

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Minimal cubature formulae: from the square to the cube

- Bojanov&Petrova (JCAM 1997) derived minimal cubature formulae in the square for the product Chebsyhev measure, splitting the Gauss-Lobatto quadrature into two sums, over even indices and odd indices, respectively.
- This factorization method was also used for $d > 2$ and yields a cubature formula of degree $2n-1$ for W_d with roughly $n^d/2^{d/2}$ many nodes.
- A close inspection to the Bojanov&Petrova's technique allowed us to derive cubature formulae of degree $2n-1$ for W_d with roughly $2\left(\frac{n}{2}^d\right) = \frac{n^d}{2^{d-1}}$ many nodes.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Recalls from 1d Gauss-Lobatto

- The 1d Gauss-Lobatto formula for $w(= W_1)$ on $[-1, 1]$ is

$$\int_{-1}^{1} f(x)w(x)dx = \frac{1}{n}\left(\frac{1}{2}f(-1) + \sum_{j=1}^{n-1} f\left(\cos\frac{j\pi}{n}\right) + \frac{1}{2}f(1)\right) := I_n f \; , \tag{8}$$

- We factor this rule depending on n even or odd.

$$n = 2m: \quad \begin{aligned} I_n^E f &:= \frac{1}{n}\left(\frac{1}{2}f(-1) + \sum_{j=1}^{m-1} f\left(\cos\frac{2j\pi}{n}\right) + \frac{1}{2}f(1)\right) \\ I_n^O f &:= \frac{1}{n}\sum_{j=1}^{m} f\left(\cos\frac{(2j-1)\pi}{n}\right) \end{aligned} \tag{9}$$

... similarly for odds, $n = 2m - 1$.

- Then, the quadrature (8) becomes

$$\int_{-1}^{1} f(x)w(x)dx = I_n^E f + I_n^O f \; , \qquad \forall f \in \Pi_{2n-1} \; ,$$

A key lemma

Lemma

Let T_k be the Chebyshev pol. of deg. k. For $n \geq 0$ and $k \in \mathbb{Z}$,

$$I_n^E T_k = \begin{cases} 0, & k \neq 0 \mod n \\ \frac{1}{2}, & k = 0 \mod n \end{cases} \quad \textit{and} \quad I_n^O T_k = \begin{cases} 0, & k \neq 0 \mod n \\ \frac{1}{2}, & k = 0, 2n, 4n, \ldots \\ -\frac{1}{2}, & k = 0, n, 3n, \ldots \end{cases}$$

Proof. ... from elementary trig. identities. For instance, $n = 2m$, $I_n^O T_k = \frac{\sin k\pi}{2n \sin \frac{k\pi}{n}}$ from which $I_n^O T_k = 0$ for $k \neq 0 \mod n$.

... to dimension d

Let $\sigma \in \{E, O\}^d$, so that $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_i = E$ or $\sigma_i = O$. We then define the sum

$$I_n^{\sigma_1} \cdots I_n^{\sigma_d} f$$

as a d-fold multiple sum in which $I_n^{\sigma_k}$ is applied to the k-th variable of f. Let us define

$$\tilde{\sigma}_i = \begin{cases} E & \sigma_i = O \\ O & \sigma_i = E \end{cases} \tag{10}$$

For each $\sigma \in \{E, O\}^d$, we then define

$$I_{n,d}^{\sigma} f := I_n^{\sigma_1} \ldots I_n^{\sigma_d} f + I_n^{\widetilde{\sigma}_1} \ldots I_n^{\widetilde{\sigma}_d} f \, . \tag{11}$$

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Main theorem

Theorem

For $d \geq 1$ and each $\sigma \in \{E, O\}^d$, the cubature formula

$$\int_{[-1,1]^d} f(x) W_d(x) dx = 2^{d-1} I_{n,d}^{\sigma} f \qquad (12)$$

is exact for $f \in \Pi_{2n-1}^d$ and its number of nodes, N, satisfies

$$N = 2 \left(\left\lfloor \frac{n}{2} \right\rfloor \right)^d (1 + o(n^{-1})) \, .$$

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

What the theorem says ...

- For $d = 2$, we have 2 distinct cubature formulae of degree $2n-1$ (cf. Morrow&Patterson (SIAM J. Numer. Anal. 1978); Y. Xu (JAT 1996), and Bojanov&Petrova (JCAM 1997)).:
  1. $\sigma = (E, E)$ and $N = \dim(\Pi^2_{n-1}) + \lfloor \frac{n}{2} \rfloor$ many nodes;
  2. $\sigma = (E, O)$ and $N + 1$ nodes.
- For $d = 3$, there are 4 distinct formulae for $\sigma = (E, E, E), (E, E, O), (E, O, E), (O, E, E)$, respectively. For $n = 2m$, the number of nodes is

$$N = \frac{(n+1)^3 + (n+1)}{4}$$

for $\sigma = (E, E, E)$ and

$$N = \frac{(n+1)^3 - (n+1)}{4}$$

for $\sigma = (E, E, O), (E, O, E), (O, E, E)$, respectively. Asymptotically $n^3/4$.

What the theorem says ...

- For $d = 2$, we have 2 distinct cubature formulae of degree $2n - 1$ (cf. Morrow&Patterson (SIAM J. Numer. Anal. 1978); Y. Xu (JAT 1996), and Bojanov&Petrova (JCAM 1997)).:
  1. $\sigma = (E, E)$ and $N = \dim(\Pi^2_{n-1}) + \lfloor \frac{n}{2} \rfloor$ many nodes;
  2. $\sigma = (E, O)$ and $N + 1$ nodes.
- For $d = 3$, there are 4 distinct formulae for $\sigma = (E, E, E), (E, E, O), (E, O, E), (O, E, E)$, respectively. For $n = 2m$, the number of nodes is

  $$N = \frac{(n+1)^3 + (n+1)}{4}$$

  for $\sigma = (E, E, E)$ and

  $$N = \frac{(n+1)^3 - (n+1)}{4}$$

  for $\sigma = (E, E, O), (E, O, E), (O, E, E)$, respectively. Asymptotically $n^3/4$.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

A Matlab code for generating the points

```
n= input('Give me the degree = ');
fam=input('Which family = ');
zn = -cos([0:n]*pi/n);
E=zn(1:2:length(zn)); O=zn(2:2:length(zn));
switch(fam)
case 1 [X1,Y1,Z1]=meshgrid(E,E,E);
[X2,Y2,Z2]=meshgrid(O,O,O);
case 2 [X1,Y1,Z1]=meshgrid(E,E,O);
[X2,Y2,Z2]=meshgrid(O,O,E);
case 3 [X1,Y1,Z1]=meshgrid(E,O,O);
[X2,Y2,Z2]=meshgrid(O,E,E);
case 4 [X1,Y1,Z1]=meshgrid(O,O,O);
[X2,Y2,Z2]=meshgrid(E,E,E);
end
---> Plot of the points <---
```

http://profs.sci.univr.it/∼demarchi/software/Hyper3

A Matlab code for generating the points

```
n= input('Give me the degree = ');
fam=input('Which family = ');
zn = -cos([0:n]*pi/n);
E=zn(1:2:length(zn)); O=zn(2:2:length(zn));
switch(fam)
case 1 [X1,Y1,Z1]=meshgrid(E,E,E);
[X2,Y2,Z2]=meshgrid(O,O,O);
case 2 [X1,Y1,Z1]=meshgrid(E,E,O);
[X2,Y2,Z2]=meshgrid(O,O,E);
case 3 [X1,Y1,Z1]=meshgrid(E,O,O);
[X2,Y2,Z2]=meshgrid(O,E,E);
case 4 [X1,Y1,Z1]=meshgrid(O,O,O);
[X2,Y2,Z2]=meshgrid(E,E,E);
end
---> Plot of the points <---
```

http://profs.sci.univr.it/∼demarchi/software/Hyper3

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

The points

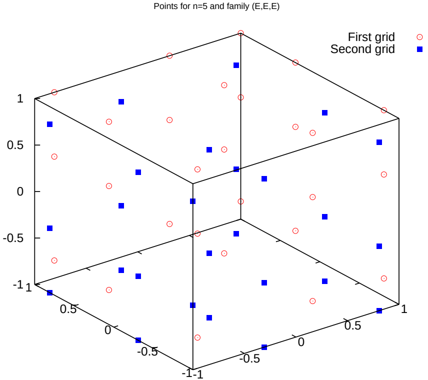


Figure: These are the cubature points for $n = 5$ in the 3-cube $[-1, 1]^3$. They are 54 i.e the union of two grids, both formed by 3^3 points.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Example: new cubature versus tensor-product ones

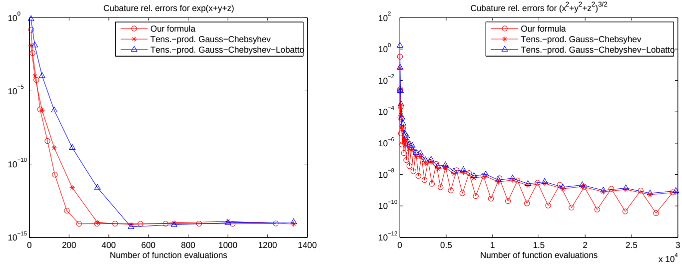


Figure: Relative cubature errors versus the number of function evaluations for the exponential (left) and a C^2 function (right). Here, $\sigma = (E, E, E)$.

Note: the superiority for less smooth functions arises for even n.

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Hyperinterpolation in the cube (1)

Having the cubature formula (12) of Theor. 1, we can construct the hyperinterpolant.

- $$p_\alpha(x) = \hat{T}_{\alpha_1}(x_1)\hat{T}_{\alpha_2}(x_2)\hat{T}_{\alpha_3}(x_3)\,, \tag{13}$$

  where $\hat{T}_k(\cdot) = \sqrt{2}\cos(k\arccos(\cdot)),\ k > 0$ and $\hat{T}_0(\cdot) = 1$.

- $$C_n = \left\{\cos\frac{k\pi}{n},\ k = 0,\ldots,n\right\}$$

  be the set of $n+1$ Chebyshev-Lobatto points, and C_n^E, C_n^O its restriction to even and odd indices, respectively. Then,

  $$X_n = \left(C_{n+1}^{\sigma_1} \times C_{n+1}^{\sigma_2} \times C_{n+1}^{\sigma_3}\right) \cup \left(C_{n+1}^{\tilde{\sigma}_1} \times C_{n+1}^{\tilde{\sigma}_2} \times C_{n+1}^{\tilde{\sigma}_3}\right)\,, \tag{14}$$

  with $(\sigma_1,\sigma_2,\sigma_3) \in \{E,O\}^3$, see (10).

- The weights of the cubature formula (12) for $\xi \in X_n$, are

  $$w_\xi = \frac{4}{(n+1)^3}\cdot\begin{cases} 1 & \text{if } \xi \text{ is an interior point} \\ 1/2 & \text{if } \xi \text{ is a face point} \\ 1/4 & \text{if } \xi \text{ is an edge point} \\ 1/8 & \text{if } \xi \text{ is a vertex point}\end{cases} \tag{15}$$

  Note: since

  $$\dim(\Pi_n^3(\Omega)) = (n+1)(n+2)(n+3)/6 < N = \mathrm{card}(X_n) \approx n^3/4\,,$$

  the polynomial $\mathcal{L}_n f$ in (6) is not interpolant.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Hyperinterpolation in the cube (1)

Having the cubature formula (12) of Theor. 1, we can construct the hyperinterpolant.

- $$p_\alpha(x) = \hat{T}_{\alpha_1}(x_1)\hat{T}_{\alpha_2}(x_2)\hat{T}_{\alpha_3}(x_3)\ , \tag{13}$$

  where $\hat{T}_k(\cdot) = \sqrt{2}\cos(k\arccos(\cdot)),\ k > 0$ and $\hat{T}_0(\cdot) = 1$.

- $$C_n = \left\{\cos\frac{k\pi}{n},\ k = 0, \ldots, n\right\}$$

  be the set of $n+1$ Chebyshev-Lobatto points, and C_n^E, C_n^O its restriction to even and odd indices, respectively. Then,

  $$X_n = \left(C_{n+1}^{\sigma_1} \times C_{n+1}^{\sigma_2} \times C_{n+1}^{\sigma_3}\right) \cup \left(C_{n+1}^{\tilde{\sigma}_1} \times C_{n+1}^{\tilde{\sigma}_2} \times C_{n+1}^{\tilde{\sigma}_3}\right)\ , \tag{14}$$

  with $(\sigma_1, \sigma_2, \sigma_3) \in \{E, O\}^3$, see (10).

- The weights of the cubature formula (12) for $\xi \in X_n$, are

  $$w_\xi = \frac{4}{(n+1)^3} \cdot \begin{cases} 1 & \text{if } \xi \text{ is an interior point} \\ 1/2 & \text{if } \xi \text{ is a face point} \\ 1/4 & \text{if } \xi \text{ is an edge point} \\ 1/8 & \text{if } \xi \text{ is a vertex point} \end{cases} \tag{15}$$

  Note: since

  $$\dim(\Pi_n^3(\Omega)) = (n+1)(n+2)(n+3)/6 < N = \mathrm{card}(X_n) \approx n^3/4\ ,$$

  the polynomial $\mathcal{L}_n f$ in (6) is not interpolant.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Hyperinterpolation in the cube (1)

Having the cubature formula (12) of Theor. 1, we can construct the hyperinterpolant.

- $$p_\alpha(x) = \hat{T}_{\alpha_1}(x_1)\hat{T}_{\alpha_2}(x_2)\hat{T}_{\alpha_3}(x_3)\ , \qquad (13)$$

  where $\hat{T}_k(\cdot) = \sqrt{2}\cos(k \arccos(\cdot)),\ k > 0$ and $\hat{T}_0(\cdot) = 1$.

- $$C_n = \left\{\cos\frac{k\pi}{n},\ k = 0, \ldots, n\right\}$$

  be the set of $n+1$ Chebyshev-Lobatto points, and C_n^E, C_n^O its restriction to even and odd indices, respectively. Then,

  $$X_n = \left(C_{n+1}^{\sigma_1} \times C_{n+1}^{\sigma_2} \times C_{n+1}^{\sigma_3}\right) \cup \left(C_{n+1}^{\tilde{\sigma}_1} \times C_{n+1}^{\tilde{\sigma}_2} \times C_{n+1}^{\tilde{\sigma}_3}\right)\ , \qquad (14)$$

  with $(\sigma_1, \sigma_2, \sigma_3) \in \{E, O\}^3$, see (10).

- The weights of the cubature formula (12) for $\xi \in X_n$, are

  $$w_\xi = \frac{4}{(n+1)^3} \cdot \begin{cases} 1 & \text{if } \xi \text{ is an interior point} \\ 1/2 & \text{if } \xi \text{ is a face point} \\ 1/4 & \text{if } \xi \text{ is an edge point} \\ 1/8 & \text{if } \xi \text{ is a vertex point} \end{cases} \qquad (15)$$

  Note: since

  $$\dim(\Pi_n^3(\Omega)) = (n+1)(n+2)(n+3)/6 < N = \mathrm{card}(X_n) \approx n^3/4\ ,$$

  the polynomial $\mathcal{L}_n f$ in (6) is not interpolant.

Hyperinterpolation in the cube (1)

Having the cubature formula (12) of Theor. 1, we can construct the hyperinterpolant.

- $$p_\alpha(x) = \hat{T}_{\alpha_1}(x_1)\hat{T}_{\alpha_2}(x_2)\hat{T}_{\alpha_3}(x_3)\ , \tag{13}$$

  where $\hat{T}_k(\cdot) = \sqrt{2}\cos(k\arccos(\cdot)),\ k > 0$ and $\hat{T}_0(\cdot) = 1$.

- $$C_n = \left\{\cos\frac{k\pi}{n},\ k = 0, \ldots, n\right\}$$

  be the set of $n+1$ Chebyshev-Lobatto points, and C_n^E, C_n^O its restriction to even and odd indices, respectively. Then,

  $$X_n = \left(C_{n+1}^{\sigma_1} \times C_{n+1}^{\sigma_2} \times C_{n+1}^{\sigma_3}\right) \cup \left(C_{n+1}^{\tilde{\sigma}_1} \times C_{n+1}^{\tilde{\sigma}_2} \times C_{n+1}^{\tilde{\sigma}_3}\right)\ , \tag{14}$$

  with $(\sigma_1, \sigma_2, \sigma_3) \in \{E, O\}^3$, see (10).

- The weights of the cubature formula (12) for $\xi \in X_n$, are

  $$w_\xi = \frac{4}{(n+1)^3} \cdot \begin{cases} 1 & \text{if } \xi \text{ is an interior point} \\ 1/2 & \text{if } \xi \text{ is a face point} \\ 1/4 & \text{if } \xi \text{ is an edge point} \\ 1/8 & \text{if } \xi \text{ is a vertex point} \end{cases} \tag{15}$$

Note: since

$$\dim(\Pi_n^3(\Omega)) = (n+1)(n+2)(n+3)/6 < N = \mathrm{card}(X_n) \approx n^3/4\ ,$$

the polynomial $\mathcal{L}_n f$ in (6) **is not interpolant**.

Hyperinterpolation in the cube (2)

- $$F(\xi) = F(\xi_1, \xi_2, \xi_3) = \begin{cases} w_\xi f(\xi) & \xi \in X_n \\ 0 & \xi \in (C_{n+1} \times C_{n+1} \times C_{n+1}) \setminus X_n \end{cases} \qquad (16)$$

- Then, we can write

$$\begin{aligned} c_\alpha &= \sum_{\xi \in X_n} w_\xi f(\xi) p_\alpha(\xi) \\ &= \left(\prod_{s=1}^{3} \beta_{\alpha_s} \right) \sum_{i=0}^{n+1} \left(\sum_{j=0}^{n+1} \left(\sum_{k=0}^{n+1} F_{ijk} \cos \frac{k\alpha_1 \pi}{n+1} \right) \cos \frac{j\alpha_2 \pi}{n+1} \right) \cos \frac{i\alpha_3 \pi}{n+1} \ , \end{aligned}$$

where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \{0, 1, \ldots, n\}^3 \ , \quad \beta_{\alpha_s} = \begin{cases} \sqrt{2} & \alpha_s > 0 \\ 1 & \alpha_s = 0 \end{cases} \ , \quad s = 1, 2, 3 \ .$$

- Hence, $\{c_\alpha\}$ is a scaled Discrete Cosine Tranform of the array

$$F_{ijk} = F\left(\cos \frac{i\pi}{n+1}, \cos \frac{j\pi}{n+1}, \cos \frac{k\pi}{n+1} \right) \ , \quad 0 \leq i, j, k \leq n \ , \qquad (17)$$

where we eventually pick up only the $(n+1)(n+2)(n+3)/6 \approx n^3/6$ hyperinterpolation coefficients corresponding to $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq n$.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Algorithm: **Fast total degree hyperinterpolation in the 3-cube.**

(*i*) construct the point set X_n as union of the two subgrids in (14);

(*ii*) compute the cubature weights in (15);

(*iii*) compute the array $\{F_{ijk}\}$ at the complete grid $C_{n+1} \times C_{n+1} \times C_{n+1}$ by (16) (notice: f is evaluated only at X_n);

(*iv*) compute the array of coefficients $\{c_\alpha\}$ by three nested applications of the 1-dimensional Real(FFT(·));

(*v*) select the coefficients $\{c_\alpha\}$ corresponding to the triples $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq n$.

Remarks

- the number of hyperinterpolation nodes, or function evaluations, is equal to $\mathrm{card}(X_n) \approx n^3/4$;
- the number of hyperinterpolation coefficients is $\dim(\Pi_n^3) \approx n^3/6$.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

Algorithm: Fast total degree hyperinterpolation in the 3-cube.

(*i*) construct the point set X_n as union of the two subgrids in (14);

(*ii*) compute the cubature weights in (15);

(*iii*) compute the array $\{F_{ijk}\}$ at the complete grid $C_{n+1} \times C_{n+1} \times C_{n+1}$ by (16) (notice: f is evaluated only at X_n);

(*iv*) compute the array of coefficients $\{c_\alpha\}$ by three nested applications of the 1-dimensional Real(FFT(·));

(*v*) select the coefficients $\{c_\alpha\}$ corresponding to the triples $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le n$.

Remarks

- the number of hyperinterpolation nodes, or function evaluations, is equal to $\mathrm{card}(X_n) \approx n^3/4$;
- the number of hyperinterpolation coefficients is $\dim(\Pi_n^3) \approx n^3/6$.

An example

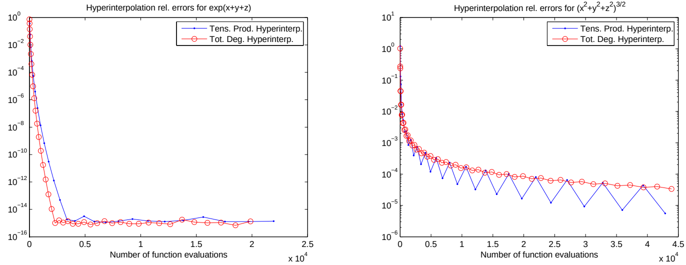
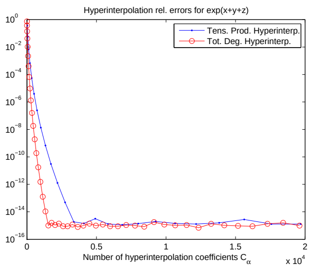


Figure: Hyperinterpolation relative errors (to the max. function deviation from its mean!) versus the number of function evaluations again for the exponential and a C^2 function.

Remarks

- Errors are computed on a uniform grid
- Total degree hyperinterpolation is superior to the tensor-product one for smooth functions.

An example... continue



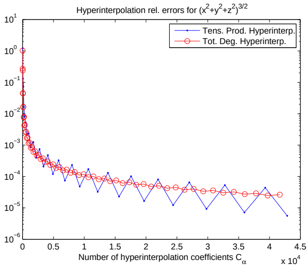


Figure: Hyperinterpolation relative errors versus the number of hyperinterpolation coefficients for the exponential and a C^2 function.

Application to surface compression

Here we report the compression errors, obtained by hyperinterpolating the cubic spline interpolant, $S_3 f(\cdot)$, of our two test functions, sampled on a grid, say C_m, consisting of $m^3 = 20^3 = 8000$ points.

$$E = \|f - \mathcal{L}_n f\|_{\infty, C_m} .$$

Function	$n = 10$	$n = 15$	$n = m = 20$	"true" error
e^{x+y+z}	4.1E-3	4.0E-4	1.7E-5	1.1E-4
$\sqrt{(x^2+y^2+z^2)^3}$	1.1E-3	3.4E-4	6.3E-5	1.2E-4

Table: Compresssion errors compared with the "true" ones

1. The "true" error is the error obtained by the cubic spline on a finer grid (bigger and different of C_m).
2. The compression ratio is given by $6\frac{m^3}{(n+1)(n+2)(n+3)}$. In this example, the ratios are: $36:1$, $12:1$, $5:1$, respectively.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

A "new" Clenshaw-Curtis like cubature formula

- In Sommariva&al. NA2008, has been shown that hyperinterpolation can be used to construct new cubature formulae in the square $[-1,1]^2$.
- Given $h \in L^2_{d\mu}(\Omega)$ and $f \in C(\Omega)$, we can approximate $\int_\Omega h(x)\, f(x)\, d\mu$ as

$$\int_\Omega h(x)\, \mathcal{L}_n f(x)\, d\mu = \sum_{|\alpha| \le n} c_\alpha\, m_\alpha = \sum_{\xi \in X_n} \lambda_\xi\, f(\xi)\ , \qquad (18)$$

where the generalized "orthogonal moments" $\{m_\alpha\}$ and the weights $\{\lambda_\xi\}$ are defined by

$$m_\alpha := \int_\Omega h(x)\, p_\alpha(x)\, d\mu\ ,\ \ \lambda_\xi := w_\xi \sum_{|\alpha| \le n} p_\alpha(\xi)\, m_\alpha\ . \qquad (19)$$

- The cubature formula (18) is exact for every $f \in \Pi^d_n(\Omega)$, and $\{m_\alpha\}$ are just Fourier coefficients of h w.r.t. the μ-orthonormal basis $\{p_\alpha\}$.

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

A "new" Clenshaw-Curtis like cubature formula

- In Sommariva&al. NA2008, has been shown that hyperinterpolation can be used to construct new cubature formulae in the square $[-1,1]^2$.
- Given $h \in L^2_{d\mu}(\Omega)$ and $f \in C(\Omega)$, we can approximate $\int_\Omega h(x)\, f(x)\, d\mu$ as

$$\int_\Omega h(x)\, \mathcal{L}_n f(x)\, d\mu = \sum_{|\alpha|\le n} c_\alpha\, m_\alpha = \sum_{\xi \in X_n} \lambda_\xi\, f(\xi)\ , \qquad (18)$$

where the generalized "orthogonal moments" $\{m_\alpha\}$ and the weights $\{\lambda_\xi\}$ are defined by

$$m_\alpha := \int_\Omega h(x)\, p_\alpha(x)\, d\mu\ , \quad \lambda_\xi := w_\xi \sum_{|\alpha|\le n} p_\alpha(\xi)\, m_\alpha\ . \qquad (19)$$

- The cubature formula (18) is exact for every $f \in \Pi^d_n(\Omega)$, and $\{m_\alpha\}$ are just Fourier coefficients of h w.r.t. the μ-orthonormal basis $\{p_\alpha\}$.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

A "new" Clenshaw-Curtis like cubature formula

- In Sommariva&al. NA2008, has been shown that hyperinterpolation can be used to construct new cubature formulae in the square $[-1,1]^2$.
- Given $h \in L^2_{d\mu}(\Omega)$ and $f \in C(\Omega)$, we can approximate $\int_\Omega h(x)\, f(x)\, d\mu$ as

$$\int_\Omega h(x)\, \mathcal{L}_n f(x)\, d\mu = \sum_{|\alpha| \le n} c_\alpha\, m_\alpha = \sum_{\xi \in X_n} \lambda_\xi\, f(\xi)\ , \qquad (18)$$

where the generalized "orthogonal moments" $\{m_\alpha\}$ and the weights $\{\lambda_\xi\}$ are defined by

$$m_\alpha := \int_\Omega h(x)\, p_\alpha(x)\, d\mu\ , \quad \lambda_\xi := w_\xi \sum_{|\alpha| \le n} p_\alpha(\xi)\, m_\alpha\ . \qquad (19)$$

- The cubature formula (18) is exact for every $f \in \Pi^d_n(\Omega)$, and $\{m_\alpha\}$ are just Fourier coefficients of h w.r.t. the μ-orthonormal basis $\{p_\alpha\}$.

Stability and convergence

Concerning stability and convergence of such cubature formulae, the following result has been proved in Sommariva&al. NA2008 (in the square and nontensorial Clenshaw-Curtis):

Theorem

Let all the assumptions for the construction of the cubature formula (18) be satisfied, and in particular let $h \in L^2_{d\mu}(\Omega)$. Then the sum of the absolute values of the cubature weights has a finite limit

$$\lim_{n\to\infty} \sum_{\xi \in X_n} |\lambda_\xi| = \int_\Omega |h(x)|\, d\mu \ . \tag{20}$$

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

"new" Clenshaw-Curtis cubature ... continue

- Applying (18)-(19) when

$$d\mu = w(x)\,dx\ ,\quad w \in L^1_{dx}(\Omega)\ ,\quad \text{with}\ \ h = \frac{1}{w} \in L^1_{dx}(\Omega)\ , \tag{21}$$

(since then $h^2 = 1/w^2 \in L^1_{d\mu}(\Omega)$) we obtain, via hyperinterpolation, a cubature formula for the standard Lebesgue measure from an algebraic cubature formula for another measure.

- Specializing this approach to the 1-dimensional Chebyshev measure gives the popular Clenshaw-Curtis quadrature formula.
- An extension to dimension 2 has been studied in Sommariva&al. (NA 2008).
- Here we apply the method in dimension 3, obtaining a **new nontensorial Clenshaw-Curtis-like cubature formula in the 3-cube**.

"new" Clenshaw-Curtis cubature ... continue

- Applying (18)-(19) when

$$d\mu = w(x)\,dx \;, \quad w \in L^1_{dx}(\Omega) \;, \quad \text{with} \quad h = \frac{1}{w} \in L^1_{dx}(\Omega) \;, \tag{21}$$

  (since then $h^2 = 1/w^2 \in L^1_{d\mu}(\Omega)$) we obtain, via hyperinterpolation, a cubature formula for the standard Lebesgue measure from an algebraic cubature formula for another measure.

- Specializing this approach to the 1-dimensional Chebyshev measure gives the popular Clenshaw-Curtis quadrature formula.

- An extension to dimension 2 has been studied in Sommariva&al. (NA 2008).

- Here we apply the method in dimension 3, obtaining a **new nontensorial Clenshaw-Curtis-like cubature formula in the 3-cube**.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

"new" Clenshaw-Curtis cubature ... continue

- Applying (18)-(19) when

$$d\mu = w(x)\,dx\ ,\quad w \in L^1_{dx}(\Omega)\ ,\quad \text{with}\quad h = \frac{1}{w} \in L^1_{dx}(\Omega)\ , \tag{21}$$

  (since then $h^2 = 1/w^2 \in L^1_{d\mu}(\Omega)$) we obtain, via hyperinterpolation, a cubature formula for the standard Lebesgue measure from an algebraic cubature formula for another measure.

- Specializing this approach to the 1-dimensional Chebyshev measure gives the popular Clenshaw-Curtis quadrature formula.

- An extension to dimension 2 has been studied in Sommariva&al. (NA 2008).

- Here we apply the method in dimension 3, obtaining a **new nontensorial Clenshaw-Curtis-like cubature formula in the 3-cube**.

"new" Clenshaw-Curtis cubature ... continue

- Applying (18)-(19) when

$$d\mu = w(x)\,dx\ , \quad w \in L^1_{dx}(\Omega)\ , \quad \text{with} \ \ h = \frac{1}{w} \in L^1_{dx}(\Omega)\ , \tag{21}$$

  (since then $h^2 = 1/w^2 \in L^1_{d\mu}(\Omega)$) we obtain, via hyperinterpolation, a cubature formula for the standard Lebesgue measure from an algebraic cubature formula for another measure.

- Specializing this approach to the 1-dimensional Chebyshev measure gives the popular Clenshaw-Curtis quadrature formula.

- An extension to dimension 2 has been studied in Sommariva&al. (NA 2008).

- Here we apply the method in dimension 3, obtaining a **new nontensorial Clenshaw-Curtis-like cubature formula in the 3-cube**.

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

An example

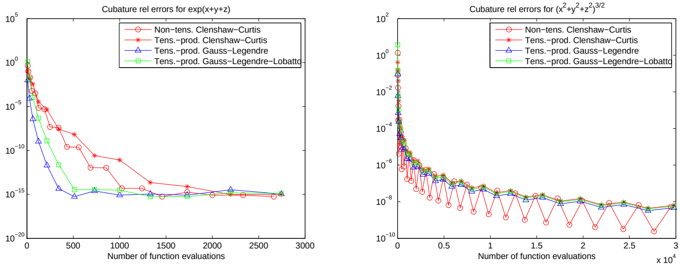


Figure: Relative cubature errors versus the number of cubature points for the exponential and a C^2 function.

Remark

- Nontensorial Clenshaw-Curtis cubature is superior to all the tensor-product ones, on less smooth functions (non-entire) (for $d = 1$ see N. L. Trefethen, Siam Rev. 2008).

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

Work to do

1. Compute numerically the Lebesgue constant (reproducing kernel?)
2. Find bound(s) for the Lebesgue constant growth
3. Make the software more efficient (based on Trefethen's definition: 10 digits, 5 sec. and 1 page!)
4. Understand the very good behavior of the nontensorial Clenshaw-Curtis cubature formulae;
5. Extensions to $d > 3$ (it seems "easy" to pass to $d = 4, 5$.)

Work to do

1. Compute numerically the Lebesgue constant (reproducing kernel?)
2. Find bound(s) for the Lebesgue constant growth
3. Make the software more efficient (based on Trefethen's definition: 10 digits, 5 sec. and 1 page!)
4. Understand the very good behavior of the nontensorial Clenshaw-Curtis cubature formulae;
5. Extensions to $d > 3$ (it seems "easy" to pass to $d = 4, 5$.)

Work to do

1. Compute numerically the Lebesgue constant (reproducing kernel?)
2. Find bound(s) for the Lebesgue constant growth
3. Make the software more efficient (based on Trefethen's definition: 10 digits, 5 sec. and 1 page!)
4. Understand the very good behavior of the nontensorial Clenshaw-Curtis cubature formulae;
5. Extensions to $d > 3$ (it seems "easy" to pass to $d = 4, 5$.)

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Work to do

1. Compute numerically the Lebesgue constant (reproducing kernel?)
2. Find bound(s) for the Lebesgue constant growth
3. Make the software more efficient (based on Trefethen's definition: 10 digits, 5 sec. and 1 page!)
4. Understand the very good behavior of the nontensorial Clenshaw-Curtis cubature formulae;
5. Extensions to $d > 3$ (it seems "easy" to pass to $d = 4, 5$.)

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Work to do

1. Compute numerically the Lebesgue constant (reproducing kernel?)
2. Find bound(s) for the Lebesgue constant growth
3. Make the software more efficient (based on Trefethen's definition: 10 digits, 5 sec. and 1 page!)
4. Understand the very good behavior of the nontensorial Clenshaw-Curtis cubature formulae;
5. Extensions to $d > 3$ (it seems "easy" to pass to $d = 4, 5$.)

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA

References

B. Bojanov and G. Petrova, *On minimal cubature formulae for product weight functions*, J. Comp. Appl. Math. 85 (1997), 113–121.

L. Bos, M. Caliari, S. De Marchi, M. Vianello and Y. Xu, *Bivariate Lagrange interpolation at the Padua points: The generating curve approach*, J. Approx. Theory 143 (2006), 15–25.

C.W. Clenshaw and A.R. Curtis,*A method for numerical integration on an automatic computer*, Numer. Math. 2 (1960), 197–205.

S. De Marchi and M. Vianello, `Hyper3`: a Matlab code for fast polynomial hyperinterpolation in the 3-cube (preliminary version available at: http://profs.sci.univr.it/∼demarchi/software.html).

C.R. Morrow and T.N.L. Patterson, *Construction of algebraic cubature rules using polynomial ideal theory*, SIAM J. Numer. Anal. 15 (1978), 953–976.

I.H. Sloan, *Interpolation and Hyperinterpolation over General Regions*, J. Approx. Theory 83 (1995), 238–254.

A. Sommariva, M. Vianello and R. Zanovello, *Nontensorial Clenshaw-Curtis cubature*, Numer. Algorithms, published on line 27/5/08.

L.N. Trefethen, *Is Gauss quadrature better than Clenshaw-Curtis?*, SIAM Rev. 50 (2008), 67–87.

Y. Xu, *Lagrange interpolation on Chebyshev points of two variables*, J. Approx. Theory 87 (1996), 220–238.

Outline

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

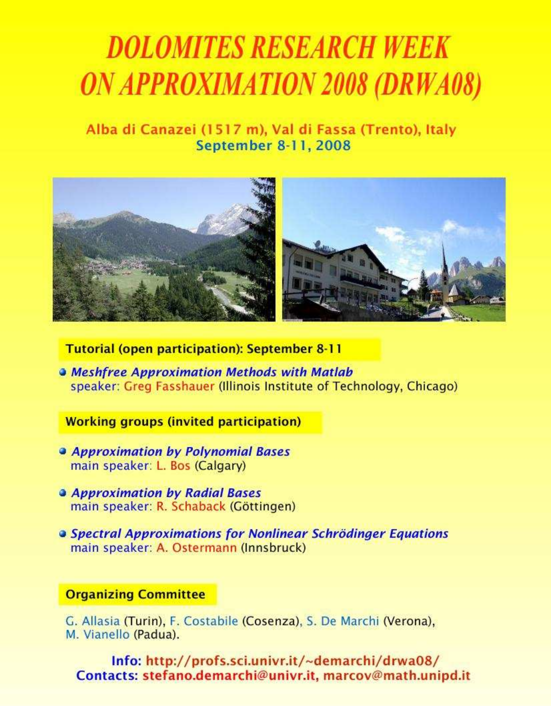
Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA09/DRNA



Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0

First announcement

2nd Dolomites Workshop on Constructive Approximation and Applications

Alba di Canazei, 3-9 Sept. 2009 (?)

Dolomites Research Notes on Approximation (DRNA)

Dolomites Research Notes on Approximation publishes, in open source access, papers and/or slides of the talks presented at the annual Dolomites Research Weeks and Workshops organized regularly (since 2006) by the Padua-Verona research group on Constructive Approximation and its Applications (CAA), at the summer courses site of the University of Verona in Alba di Canazei (Trento, Italy). The journal also publishes, on invitation, survey papers and summaries of Ph.D. theses on approximation theory, algorithms and applications.

- Editor in chief: Stefano De Marchi and Marco Vianello.
- Editorial Board: Borislav Bojanov, Leonard Peter Bos, Martin Buhmann, Armin Iske, Robert Schaback, Shayne Waldron, Holger Wendland and Yuan Xu
- web site (provisional): http://meneghetti.univr.it/ojs/index.php/DRNA/

Summary
Motivations
Generality on cubature and hyperinterpolation
Cubature and hyperinterpolation
Cubature for d-dim. Chebyshev measure
Main result
Hyperinterpolation in the cube
A Clenshaw-Curtis-like formula in the cube
Future work
References
DRWA08/DWCAA0

Thank you for your attention!

Summary

Motivations

Generality on cubature and hyperinterpolation

Cubature and hyperinterpolation

Cubature for d-dim. Chebyshev measure

Main result

Hyperinterpolation in the cube

A Clenshaw-Curtis-like formula in the cube

Future work

References

DRWA08/DWCAA0