Master degree course on Approximation Theory and Applications,

Lab exercises

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1. Plot some RBF strictly positive definite centered at the origin

   - *Gaussian - Laguerre* for $n = 1, 2$ and $s = 1, 2$

s	$n = 1$	$n = 2$
1	$(3/2 - x^2)\mathrm{e}^{-x^2}$	$(15/8 - 5/2\,x^2 + 1/2x^4)\mathrm{e}^{-x^2}$
2	$(2 - \Vert\mathbf{x}\Vert^2)\mathrm{e}^{-\Vert\mathbf{x}\Vert^2}$	$(3 - 3\Vert\mathbf{x}\Vert^2 + 1/2\Vert\mathbf{x}\Vert^4)\mathrm{e}^{-\Vert\mathbf{x}\Vert^2}$

     for $s = 1$, $x \in [-1, 1]$ while for $s = 2$, $\mathbf{x} \in [-1, 1]^2$.

   - *Poisson* functions for $s = 2, 3, 4$ in $[-1, 1]^2$ using the shape parameter $\epsilon = 10$ (to be introduced in the below definitions)

$s = 2$	$s = 3$	$s = 4$
$J_0(\Vert\mathbf{x}\Vert)$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\Vert\mathbf{x}\Vert)}{\Vert\mathbf{x}\Vert}$	$\frac{J_1(\Vert\mathbf{x}\Vert)}{\Vert\mathbf{x}\Vert}$

     where J_p is the *Bessel function of first kind and order p* (in Matlab `besselj(p,z)` where `z` is an evaluation vector of points).

   - *Matérn* functions in $[-1, 1]^2$, for three values of β (again with $\epsilon = 10$)

$\beta_1 = \frac{s+1}{2}$	$\beta_2 = \frac{s+3}{2}$	$\beta_3 = \frac{s+5}{2}$
$\mathrm{e}^{-\Vert x\Vert}$	$(1 + \Vert x\Vert)\,\mathrm{e}^{-\Vert x\Vert}$	$\left(3 + 3\Vert x\Vert + \Vert x\Vert^2\right)\mathrm{e}^{-\Vert x\Vert}$

     Notice that *Matérn* for β_1 is not differantiable at the origin. For β_2 is $\mathcal{C}^2(\mathbb{R}^s)$ and for β_3 is $\mathcal{C}^4(\mathbb{R}^s)$.

   - *Generalized inverse multiquadrics* $\Phi(x) = (1 + \Vert x\Vert^2)^{-\beta}$, $s < 2\beta$, in $[-1, 1]^2$ (always using $\epsilon = 5$) for $\beta = 1/2$ (Hardy) and $\beta = 1$ (inverse quadrics).

   - *Truncated powers* $\Phi(x) = (1 - \Vert x\Vert)^l_+$ when $l = 2, 4$ (in $[-1, 1]^2$).

   - *Whittaker's potentials* in $[-1, 1]^2$, for the following α, k e β

α	$k = 2$	$k = 3$
0	$\frac{\beta - \Vert x\Vert + \Vert x\Vert\mathrm{e}^{-\beta/\Vert x\Vert}}{\beta^2}$	$\frac{\beta^2 - 2\beta\Vert x\Vert + 2\Vert x\Vert^2 - 2\Vert x\Vert^2\mathrm{e}^{-\beta/\Vert x\Vert}}{\beta^3}$
1	$\frac{\beta - 2\Vert x\Vert + (\beta + 2\Vert x\Vert)\mathrm{e}^{-\beta/\Vert x\Vert}}{\beta^3}$	$\frac{\beta^2 - 4\beta\Vert x\Vert + 6\Vert x\Vert^2 - (2\beta\Vert x\Vert + 6\Vert x\Vert^2)\mathrm{e}^{-\beta/\Vert x\Vert}}{\beta^4}$

For the plots take $\beta = 1$.

Exercise Construct the RBF interpolant of the Franke functions on a grid 20×20 of Chebyshev points with Poisson and Matérn RBF. Compute also the RMSE.

2. Plot the most important CPD radial basis functions

   - *Generalized multiquadrics*

     $$\Phi(\mathbf{x}) = (1 + \|\mathbf{x}\|^2)^\beta, \;\; \mathbf{x} \in \mathbb{R}^s, \; \beta \in \mathbb{R} \backslash \mathbb{N}_0$$

     with $\beta = 1/2$ (Hardy) we get SCDP of order 1 while with $\beta = 5/2$ we get a SCDP of order 3.

   - *Power functions* (shape parameter free!)

     $$\Phi(\mathbf{x}) = \|\mathbf{x}\|^\beta, \;\; \mathbf{x} \in \mathbb{R}^s, 0 < \beta \notin 2\mathbb{N}$$

     For $\beta = 3$ we get a SCPD of order 2 while for $\beta = 5$ a SCPD of order 3.

   - *thin-plate splines* (shape parameter free!)

     $$\Phi(\mathbf{x}) = \|\mathbf{x}\|^{2\beta} \log \|\mathbf{x}\|, \;\; \mathbf{x} \in \mathbb{R}^s, \beta \in \mathbb{N}$$

     The "classic" one is for $\beta = 1$ (SCPD of order 2), while for $\beta = 2$ we get a function SCPD of order 3. Verify that also these functions are *shape parameter free*.

The Matlab functions can be dowloaded at the link
http://www.math.unipd.it/~demarchi/TAA2010