Master degree course on Approximation Theory and Applications,

Lab exercises

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January 18, 2016

1. Take a function f and the set $X = \{x_1, ..., x_N\}$ while the basis functions are centered at $\Xi = \{\xi_1, \dots, \xi_M\}$ con $M \le N$. The approximant is then

$$Q_f(x) = \sum_{i=1}^{M} c_j \Phi(x, \xi_j), \quad x \in \mathbb{R}^s. \tag{1}$$

with the coefficients c_j determined as the least-squares solution of $A\mathbf{c} = \mathbf{f}$ (with A, $N \times M$ with components $A_{j,k} = \Phi(x_j, \xi_k)$, $\mathbf{c},\ \mathbf{f}$,vectors of length M and N, respectively) , that is by minimizing the quadratic functional $E(\mathbf{c}) = \|Q_f - f\|_2^2$.

Consider the script `RBFApproximation2D.m`, which uses two sets of points for the "data sites" and center, for approximating the function `sinc` in $[0,1]^2$ with a gaussian with $\epsilon = 1$.

The exercise consists in increasing and decreasing ϵ in order to see how the error and the rank of the matrix A changes. Repeat the experiment with different set of node and centers: equispaced, Halton, Chebyshev.

2. This exercise is a 1d approximate Moving Least-Square example, showing the effect of scaling on the convergence behavior for Gaussian-Laguerre generating functions.

Take, as test function, the 1d *mollified* Franke function

$$f(x) = \left(15\mathrm{e}^{-\frac{1}{1-(2x-1)^2}}\right)\left[\frac{3}{4}\left(\mathrm{e}^{-(9x-2)^2/4} + \mathrm{e}^{-(9x+1)^2/49}\right) + \frac{1}{2}\mathrm{e}^{-(9x-7)^2/4} - \frac{1}{5}\mathrm{e}^{-(9x-4)^2}\right],$$

with mollifier $g(x) = 15\mathrm{e}^{-\frac{1}{1-(2x-1)^2}}$.

Let D be the scaling parameter with values $D \in \{0.4,\ 0.8,\ 1.2,\ 1.6,\ 2.0,\ 4.0,\ 8.0\}$. Then take a grid consisting of $N = 2^k + 1,\ k = 1, \dots, 14$ equispaced points of $[0,1]$ on which we sample the test function.

The approximant is then

$$P_f(x) = \frac{1}{\pi D} \sum_{k=1}^{N} f(x_i) \mathrm{e}^{\frac{(x-x_i)^2}{Dh^2}}, \ \ x \in [0,1]$$

and $h = 1/(N-1)$.

This corresponds to take the usual shape parameter

$$\epsilon = \frac{1}{\sqrt{D}h} = \frac{N-1}{\sqrt{D}} = \frac{2^k}{\sqrt{D}}$$

which corresponds to a *stationary* approximation.

Use the script `ApproxMLSApprox1D.m` which allows to plot the approximation of the mollified Franke function by using Gaussian-Laguerre RBF at various values of D

We will notice that, for a fixed Gaussian-Laguerre RBF, when $D \geq 2$ we reach the approximation order $\mathcal{O}(h^2)$ or grater, while for $D < 2$ we will see a stalling of the error.

We recall that in the 1d case these functions are $\Phi(x) = e^{-x^2} L_n^{p/2}(x^2)$ where $L_n^{p/2}$ are Laguerre polynomials of degree n and order $p/2$

$$L_n^{p/2}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+p/2}{n-k} x^k .$$

Remark. The function `ApproxMLSApprox1D.m` uses Gaussian-Laguerre of order $p = 0, 1, 2$, values of $D = 2, 4, 6$ and convergence order h^2, h^4, h^6 respectively.

The Matlab functions can be dowloaded at the link
http://www.math.unipd.it/~demarchi/TAA2010