Master degree course on Approximation Theory and Applications,

Lab exercises on

Generalized Hermite interpolation and solution of elliptic PDEs

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December 13, 2016

1. With examples we try to illustrate the symmetric approach of Hermite interpolation on a problem that interpolates values and derivatives.

   Consider
   $$f(x,y) = \frac{\tanh(9(y-x))+1}{\tanh(9)+1}\,, \qquad (1)$$
   whose first order partial derivatives can easily be computed.

   By using multiquadrics (MQ) with $\epsilon = 6$, compare these 4 problems

   (a) Lagrange interpolation at N equispaced points on $[0,1]^2$;

   (b) Lagrange interpolation at $3N$ *clustered points* with *separation distance* $q = h/10$ (where $h = 1/(\sqrt{N}-1)$ is the *fill distance* of a set of equispaced points in the square);

   (c) as point (b) with $q = h/100$;

   (d) Hermite interpolation of the values of the function and its first order derivatives on N equispaced points, as chosen in (a).

   The files to be used are

   - for (a), `RBFInterpolation2D.m` by a suitable substitution of line 1 (that is the definition of the RBF) and lines 2-6 (test function) with the expression of the function (1) ;
   - for (b), again using `RBFInterpolation2D.m`, changing lines 8 and 9 with following ones (remember to compute or load the appropriate `dsites`)

   ```
   q=0.1/(sqrt(N)-1);
   grid=linspace(0,1,sqrt(N));
   shifted=linspace(q,1+q,sqrt(N)); shifted(end)=1-q;
   [xc1,yc1]=meshgrid(shifted,grid);
   [xc2,yc2]=meshgrid(grid,shifted);
   dsites=[dsites;xc1(:) yc1(:);  xc2(:) yc2(:)];
   ```

   - for (c), as in (b) with the appropriate changes
   - for (d) use the script `RBFHermite_2D.m` (which makes use of the function `DifferenceMatrix.m`).

   Produce two tables, showing for each problem the RMSE changes and the condition number of the interpolation matrices

	Lagrange	clustered $q = h/10$
mesh	RMSE | Condition number	RMSE| Condition number

	clustered $q = h/100$	Hermite
mesh	RMSE | Condition number	RMSE | Condition number

Consider the meshes with 3×3, 5×5, 9×9, 17×17 and 33×33 points

We shall see that the Hermite interpolation approach w.r.t. *clustered points* by using Lagrange, improves the results especially with the mesh 33×33, where we can also appreciate an (small) improvement of the condition number and of the error.

2. Consider the Poisson equation with Dirichlet boundary conditions

$$\begin{aligned} \nabla^2 u(x,y) &= -\frac{5}{4}\pi^2 \sin(\pi x)\cos\left(\frac{\pi y}{2}\right), \quad (x,y) \in \Omega = [0,1]^2 \\ u(x,y) &= \sin(\pi x) \quad (x,y) \in \Gamma_1 \\ u(x,y) &= 0 \qquad (x,y) \in \Gamma_2 \end{aligned} \tag{2}$$

where $\Gamma_1 = \{(x,y):\ 0 \le x \le 1,\ y = 0\}$ e $\Gamma_2 = \partial\Omega \backslash \Gamma_1$.

The exact solution is $u(x,y) = \sin(\pi x)\cos\left(\frac{\pi y}{2}\right)$.

- Use the script `KansaLaplace_2D.m` for solving (2) with the non-symmetric Kansa method with IMQ and Gaussian, with $\epsilon = 3$ (note: the script is based on IMQ!).

  For using the script with the Gaussian we have to substitute the function and the laplacian as follows

```
rbf = @(e,r) exp(-(e*r).^2); ep=3;
Lrbf = @(e,r) 4*e^2 exp( -(e*r).^2).*((e*r).^2-1);
```

  In both cases take $N = 3^2, 5^2, 9^2, 17^2$ inner centers and $M = 2^2, 4^2, 6^2, 8^2$ additional along the boundary (along the sides of the square or outside the border, as in the provided script).

  Take the internal points as Halton or equispaced points.

  Again produce a table with RMSE and condition number of the collocation matrix.

  Note. Since it is necessary the laplacian and the bi-laplacian of the Gaussian, the formulas are (with $\varphi(r) = \varphi(\|x\|)$)

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(\|x\|) &= \frac{d^2}{dr^2}\varphi(r) + \frac{1}{r}\varphi(r) \quad \text{laplacian} \\ \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\varphi(\|x\|) &= \frac{d^4}{dr^4}\varphi(r) + \frac{2}{r}\frac{d^3}{dr^3}\varphi(r) - \frac{1}{r}\frac{d^2}{dr^2}\varphi(r) + \frac{1}{r^3}\frac{d}{dr}\varphi(r) \quad \text{bi}-\text{lapl.} \end{aligned}$$

- Finally use `HermiteLaplace_2D.m` for solving (2) with the symmetric method of Fasshauer based on Hermite collocation approach. Perform the same experiments with IMQ and Gaussian.

For the scripts the link is
`http://www.math.unipd.it/~demarchi/TAA2010`