Master degree course on Approximation Theory and Applications,

Lab exercises

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Plot some strictly positive definite RBF centered at the origin

- *Gaussian - Laguerre* for $n = 1, 2$ and $s = 1, 2$

s	$n = 1$	$n = 2$
1	$(3/2 - x^2)\mathrm{e}^{-x^2}$	$(15/8 - 5/2\, x^2 + 1/2x^4)\mathrm{e}^{-x^2}$
2	$(2 - \|\mathbf{x}\|^2)\mathrm{e}^{-\|\mathbf{x}\|^2}$	$(3 - 3\|\mathbf{x}\|^2 + 1/2\|\mathbf{x}\|^4)\mathrm{e}^{-\|\mathbf{x}\|^2}$

  for $s = 1$, $x \in [-1, 1]$ while for $s = 2$, $\mathbf{x} \in [-1, 1]^2$.

- *Poisson* functions for $s = 2, 3, 4$ in $[-1, 1]^2$ using the shape parameter $\epsilon = 10$ (to be introduced in the below definitions)

$s = 2$	$s = 3$	$s = 4$
$J_0(\|\mathbf{x}\|)$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}$	$\frac{J_1(\|\mathbf{x}\|)}{\|\mathbf{x}\|}$

  where J_p is the *Bessel function of first kind and order* p (in Matlab `besselj(p,z)` where `z` is an evaluation vector of points).

- *Matérn* functions in $[-1, 1]^2$, for three values of β (again with $\epsilon = 10$)

$\beta_1 = \frac{s+1}{2}$	$\beta_2 = \frac{s+3}{2}$	$\beta_3 = \frac{s+5}{2}$
$\mathrm{e}^{-\|x\|}$	$(1 + \|x\|)\, \mathrm{e}^{-\|x\|}$	$\left(3 + 3\|x\| + \|x\|^2\right) \mathrm{e}^{-\|x\|}$

  Notice that *Matérn* for β_1 is not differantiable at the origin. For β_2 is $\mathcal{C}^2(\mathbb{R}^s)$ and for β_3 is $\mathcal{C}^4(\mathbb{R}^s)$.

- *Generalized inverse multiquadrics* $\Phi(x) = (1 + \|x\|^2)^{-\beta}$, $s < 2\beta$, in $[-1, 1]^2$ (always using $\epsilon = 5$) for $\beta = 1/2$ (Hardy) and $\beta = 1$ (inverse quadrics).

- *Truncated powers* $\Phi(x) = (1 - \|x\|)_+^l$ when $l = 2, 4$ (in $[-1, 1]^2$).

- *Whittaker's potentials* in $[-1, 1]^2$, for the following α, k e β

α	$k=2$	$k=3$
0	$\dfrac{\beta - \|x\| + \|x\| \mathsf{e}^{-\beta/\|x\|}}{\beta^2}$	$\dfrac{\beta^2 - 2\beta\|x\| + 2\|x\|^2 - 2\|x\|^2 \mathsf{e}^{-\beta/\|x\|}}{\beta^3}$
1	$\dfrac{\beta - 2\|x\| + (\beta + 2\|x\|)\mathsf{e}^{-\beta/\|x\|}}{\beta^3}$	$\dfrac{\beta^2 - 4\beta\|x\| + 6\|x\|^2 - (2\beta\|x\| + 6\|x\|^2)\mathsf{e}^{-\beta/\|x\|}}{\beta^4}$

For the plots take $\beta = 1$.

Exercise. Construct the RBF interpolant of the Franke function on a grid 20×20 of Chebyshev points with Poisson and Matérn RBF. Compute also the RMSE.

The Matlab functions can be dowloaded at the usual link
http://www.math.unipd.it/∼demarchi/TAA2010