Master degree course on Approximation Theory and Applications,

Lab exercises on

Least squares and MLS

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November 29, 2016

1. Take a function f and the data sites $X = \{x_1, ..., x_N\}$, while the basis functions centers at the points $\Xi = \{\xi_1, \ldots, \xi_M\}$ con $M \le N$. Let Q_f be the approximant

$$Q_f(x) = \sum_{i=1}^{M} c_j \Phi(x, \xi_j), \quad x \in \mathbb{R}^s. \tag{1}$$

with the coefficients c_j determined as the least-squares solution of $A\mathbf{c} = \mathbf{f}$ (A is $N \times M$ with components $A_{j,k} = \Phi(x_j, \xi_k)$, $\mathbf{c}, \mathbf{f}$, are vectors of length M and N, respectively), that is by minimizing the quadratic functional $E(\mathbf{c}) = \|Q_f - \mathbf{f}\|_2^2$.

Consider the script `RBFApproximation2D.m`, which uses two sets of points for the data sites X and centers Ξ, for approximating the function `sinc` in $[0,1]^2$ by the Gaussian RBF (with $\epsilon = 1$).

   (a) increase and decrease ϵ in order to see how the error and the rank of the matrix A changes;

   (b) repeat the experiment with different set of node and centers: equispaced, Halton, Chebyshev.

2. This exercise is a 1d approximate Moving Least-Square example, showing the effect of the *scaling* on the convergence behavior, for Gaussian-Laguerre generating functions.

Take, as test function, the 1d *mollified* Franke function

$$f(x) = \left(15\mathrm{e}^{-\frac{1}{1-(2x-1)^2}}\right) \left[\frac{3}{4}\left(\mathrm{e}^{-(9x-2)^2/4} + \mathrm{e}^{-(9x+1)^2/49}\right) + \frac{1}{2}\mathrm{e}^{-(9x-7)^2/4} - \frac{1}{5}\mathrm{e}^{-(9x-4)^2}\right],$$

with mollifier $g(x) = 15\mathrm{e}^{-\frac{1}{1-(2x-1)^2}}$.

Let D be the scaling parameter with values $D \in \{0.4, 0.8, 1.2, 1.6, 2.0, 4.0, 8.0\}$. Then take a grid consisting of $N = 2^k+1$, $k = 1, \ldots, 14$ equispaced points of $[0,1]$ on which we sample the test function.

The approximant is then

$$P_f(x) = \frac{1}{\pi D} \sum_{k=1}^{N} f(x_i) \mathrm{e}^{\frac{(x-x_i)^2}{Dh^2}}, \quad x \in [0,1]$$

and $h = 1/(N-1)$.

This corresponds to take the usual shape parameter

$$\epsilon = \frac{1}{\sqrt{D}h} = \frac{N-1}{\sqrt{D}} = \frac{2^k}{\sqrt{D}}$$

which corresponds to a *stationary* approximation.

Use the script `ApproxMLSApprox1D.m` which allows to plot the approximation of the mollified Franke function by using Gaussian-Laguerre RBF at various values of D

We will notice that, for a fixed Gaussian-Laguerre RBF, when $D \geq 2$ we reach the approximation order $\mathcal{O}(h^2)$ or grater, while for $D < 2$ we will see a stalling of the error.

We recall that in the 1d case the Gaussian-Laguerre RBFs are $\Phi(x) = e^{-x^2} L_n^{p/2}(x^2)$ where $L_n^{p/2}$ is the Laguerre polynomial of degree n and order $p/2$

$$L_n^{p/2}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+p/2}{n-k} x^k \,.$$

Remark. The function `ApproxMLSApprox1D.m` uses Gaussian-Laguerre of order $p = 0, 1, 2$, values of $D = 2, 4, 6$ and convergence order h^2, h^4, h^6 respectively.

The Matlab functions can be dowloaded at the link
http://www.math.unipd.it/∼demarchi/TAA2010