Radial Basis Functions and their Applications to Landmark-based Image Registration

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Radial Basis Functions (RBFs)

- DEFINITION: A function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is called radial if there exsits a univariate function $\phi : [0, +\infty) \rightarrow \mathbb{R}$, such that

$$\Phi(\mathbf{x}) = \phi(r)$$

  here $r = \|\mathbf{x}\|$, and $\|.\|$ is Euclidean norm on $\mathbb{R}^d$.

- PROPERTIES:

  Radially or spherically symmetric about the center.

  Under all euclidean transformations, (i.e, translations, rotations, reflections) radial function interpolants have nice property of being invariant.

- APPLICATIONS: Can be applied in solving scattered data interpolation problem or multivariate approximation problems.

RBFs Multivariate Interpolation Scheme

- THE MULTIVARIATE INTERPOLATION SCHEME is defined as follows: given data $\mathbf{X} = \{x_1, x_2, ..., x_N\},\ x_i \in \mathbb{R}^d$ and the corresponding values is $f_{\mathbf{X}} = \{f_{x_1}, f_{x_2}, ..., f_{x_N}\},\ f_{x_i} \in \mathbb{R}, i = 1, 2, ..., N$. where d is the dimension of the working space and N is the number of the data sites, choosing the interpolate kernel $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that $\Phi(x) = f_x$. Usually, the RBFs interpolant $\Phi(x)$ is a linear combination that is

$$\Phi(x) = \sum_{j=1}^{N} \alpha_j \phi(\|x - x_j\|). \quad (1)$$

then the parameters α can be obtained through solving the linear system $A\alpha = f_x, A(i,j) = \phi(\|x_i - x_j\|)$.

$$\begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \cdots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \cdots & \phi(\|x_N - x_N\|) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_N} \end{pmatrix}$$

Positive Definite Functions

- For obtaining the unique solution of the system $A\alpha = f_x$, the coefficient matrix should be NON-SINGULAR.
- Strictly Positive Definite Functions $\rightarrow$ Non-singular.
- DEFINITION: A real valued continuous function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is called positive definite on $\mathbb{R}^d$ if for all pairwise distinct $x_1, x_2, ...x_N$, and for all $\alpha \in \mathbb{R}^N$

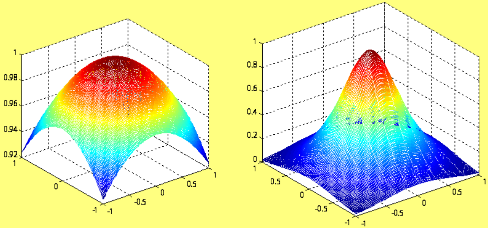
$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k) \geq 0. \tag{2}$$

function Φ is called strictly positive definite (SPD) when the equality of (2) holds iff $\alpha \equiv \mathbf{0}$.

- GLOBALLY SUPPORTED RBFS (GSRBFS)

GSRBFs	$\Phi(r)$	Property
Gaussian	e^{-r^2/c^2}, $c > 0$	SPD $\cap C^\infty(0)$
Matérn $M(r \mid v)$	$\frac{2^{1-v}}{\Gamma(v)}(\frac{r}{c})^v K_v(\frac{r}{c})$, $v > 0$	SPD $\cap C^{2v-1}(0)$

- EXAMPLES OF GSRBFS: images of $M_{5/2}$ with the shape parameter $c = 2$ (left) and $c = 0.2$ (right)



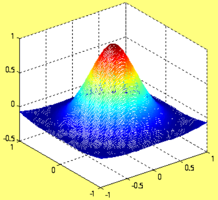
(a) $c = 2$ (b) $c = 0.2$

Illustrations of RBFs-CSRBFs

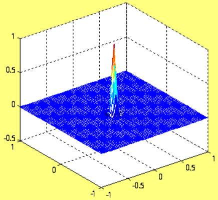
- COMPACTLY SUPPORTED RBFS (CSRBFS)

CSRBFs	$\Phi(\frac{r}{c})$	Property
Wendland $\varphi_{3,1}$	$(1-\frac{r}{c})_+^4(4\frac{r}{c}+1)$, $\frac{r}{c}\leq 1, d\leq 3$	SPD $\cap C^2(0)$
Wu $\psi_{1,2}$	$(1-\frac{r}{c})_+^4(1+4\frac{r}{c}+3\frac{r}{c}^2+\frac{3}{4}\frac{r}{c}^3)$, $\frac{r}{c}\leq 1, d\leq 3$	SPD $\cap C^2(0)$
Gneiting $\tau_{2,l}$	$(1-\frac{r}{c})_+^l(1+l\frac{r}{c}-\frac{(l+1)(l+4)}{2}\frac{r}{c}^2)$, $r\leq 1, l\geq \frac{7}{2}$	SPD $\cap C^2(0)$

- EXAMPLES OF CSRBFS: images of $\tau_{2,5}$ with the shape parameter $c=2$ (left) and $c=0.2$ (right)



(c) $c=2$



(d) $c=0.2$

Conditionally Positive Definite (CPD) Functions

- REASON: Not all popular choices of RBFs that are used fit into multivariate interpolant scheme, such as Thin Plate Spline (TPS).
- DEFINITION: A continuous function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called conditionally positive definite of order m on $\mathbb{R}^d$ if

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k) \geq 0$$

for any N pairwise distinct points $x_1, ..., x_N \in \mathbb{R}^d$, and $\alpha = [\alpha_1, ..., \alpha_N]^T \in \mathbb{R}^N$ satisfying

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0$$

for any polynomial of degree at most $m - 1$. The function Φ is called strictly conditionally positive definite (SCPD) of order m on $\mathbb{R}^d$ if the quadratic form is zero only for $\alpha \equiv 0$.

Advantages of RBFs

- THIN PLATE SPLINE: $\Phi(r) = r \log r$.
- EFFICIENT: RBF is one efficient, frequently used way to solve multivariate approximation problems.
- LITTLE RESTRICTIONS: Its applicability in almost any dimension because there are generally little restrictions on the way the data are prescribed.
- FAST CONVERGENCE: When data become dense, RBFs can produce high accuracy to the approximated target function in many cases.

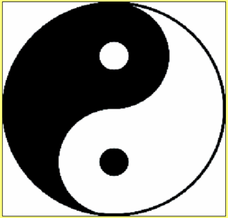
IMAGE REGISTRATION

Image Registration (IR)

- IMAGE REGISTRATION is the process of overlaying two or more images of the same scene at different times, from different viewpoints, or obtained by different sensors.
- APPLICATIONS OF IR:
  1. Image fusion: Combining relevant information from two or more images into a single image.
  2. Remote sensing: Acquisition information of an object or phenomenon without making physical contact with the object–earth science, intelligence, military application.
  3. Medicine: Surgical planning, monitoring of diseases, constructing patient Atlas, computer-aided surgeries.

Simplified Registration Example

(e) Template image



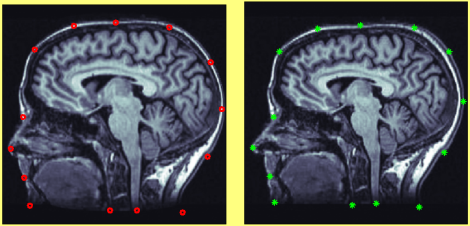
(f) Reference image

(g) Transformed template image

Figure 1: Simple example of IR using TPS transformation.

Given two images, which are named by REFERENCE OR FIXING image (right) and TEMPLATE OR MOVING (left) image. The aim is to determine a CORRESPONDENCE (a transformation function) which connects the points of two images, so that the transformed template image is similar to reference image.

Landmark-based Image Registration (LIR)



(a) Template image (b) Reference image

Figure 2: MRI brain images of an anonymous patient taken at different times.

THE PROBLEM: Find the transformation

$$\mathbf{F}: \mathbb{R}^d \to \mathbb{R}^d,$$

where $d = 2,3$, such that each landmark of the template image is mapped to corresponding landmarks of the reference image, that is $\mathbf{F}(\mathbf{x}_i^T) = \mathbf{x}_i^R$, or $\mathbf{F}_k(\mathbf{x}_i^T) = \mathbf{x}_{k,i}^R$, $i = 1,2,...,n$, $k = 1,2,...,d$.

- This problem can be formulated in the context of multidimensional interpolation on scattered data, and solved using the radial basis function(RBF) method.

Two Linear Systems of LIR

- In figure 2, we have 14 template landmarks marked by $\circ$ and the corresponding reference landmarks marked by $*$.
- We denote the template landmarks as $\mathbf{x}_i^T = (x_i^T, y_i^T)$ and the reference landmarks denoted by $\mathbf{x}_i^R = (x_i^R, y_i^R)$, $i = 1, 2, ..., 14$.
- In this talk, $d = 2$, therefore, we get two linear systems:
  $\Phi_1(x) = \sum_{j=1}^{N} \alpha_{1,j}\phi(\|x - x_j\|) = x^R$,
  $\Phi_2(x) = \sum_{j=1}^{N} \alpha_{2,j}\phi(\|x - x_j\|) = y^R$.

Two Linear Systems of LIR

$$\begin{pmatrix} \phi(\|\mathbf{x}_1^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_1^T - \mathbf{x}_{14}^T\|) \\ \phi(\|\mathbf{x}_2^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_2^T - \mathbf{x}_{14}^T\|) \\ \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_{14}^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_{14}^T - \mathbf{x}_{14}^T\|) \end{pmatrix} \begin{pmatrix} \alpha_{1,1} \\ \alpha_{1,2} \\ \vdots \\ \alpha_{1,14} \end{pmatrix} = \begin{pmatrix} x_1^R \\ x_2^R \\ \vdots \\ x_{14}^R \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \phi(\|\mathbf{x}_1^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_1^T - \mathbf{x}_{14}^T\|) \\ \phi(\|\mathbf{x}_2^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_2^T - \mathbf{x}_{14}^T\|) \\ \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_{14}^T - \mathbf{x}_1^T\|) & \cdots & \phi(\|\mathbf{x}_{14}^T - \mathbf{x}_{14}^T\|) \end{pmatrix} \begin{pmatrix} \alpha_{2,1} \\ \alpha_{2,2} \\ \vdots \\ \alpha_{2,14} \end{pmatrix} = \begin{pmatrix} y_1^R \\ y_2^R \\ \vdots \\ y_{14}^R \end{pmatrix} \tag{4}$$

Radial Basis Function Transformations

Generally, applying a radial basis function approach, the general coordinate of the transformation $F_k(x)$, k = 1,2,...,d, (for simplicity, we write $F(x)$ instead of it, the same as following) is assumed to have the form

$$F(\mathbf{x}) = \varphi(\mathbf{x}) + p(\mathbf{x}),$$

$$\Downarrow$$

$$F(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i \Phi(\|\mathbf{x} - \mathbf{x}_i^T\|) + \sum_{j=1}^{M} a_j \pi_j(\mathbf{x}),$$

where $\|\mathbf{x} - \mathbf{x}_i^T\|$ is the Euclidean distance from $\mathbf{x}$ to $\mathbf{x}_i^T$, and α_i and a_j are coefficients.

Specific Functions of Gneiting and Matérn Families

In 2002, Gneiting obtained a family of compactly supported functions, which started with Wendland's functions, for example

$$\varphi_{s+2,1} = (1-r)_+^{l+1}[(l+1)r+1]$$

and using the turning bands operator

$$\tau_s(r) = \varphi_{s+2}(r) + \frac{r\varphi_{s+2}'(r)}{s}$$

Two Special Gneiting Functions

Now we set $s = 2$, the formula of Gneiting function is

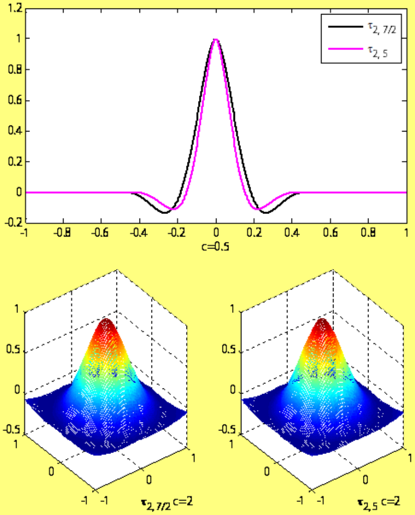
$$\tau_{2,l}(r) = (1-r)_+^l[1 + lr - \frac{(l+1)(l+4)}{2} r^2]$$

Both of them are in $C^2(\mathbb{R})$ and SPD when $l \geq 7/2$.

A. $\tau_{2,7/2}(\frac{r}{c}) \doteq (1-\frac{r}{c})_+^{7/2}(1+\frac{7}{2}\frac{r}{c}-\frac{135}{8}(\frac{r}{c})^2)$

B. $\tau_{2,5}(\frac{r}{c}) \doteq (1-\frac{r}{c})_+^{5}(1+5\frac{r}{c}-27(\frac{r}{c})^2)$

Graphs of two Gneiting functions



Matérn Family

Matérn functions have received a great deal of attention recently and they have the following form

$$M(r \mid v, c) = \frac{2^{1-v}}{\Gamma(v)} \left(\frac{r}{c}\right)^{v} K_v\left(\frac{r}{c}\right),$$

Here K_v is *Modified Bessel Functions of the second kind of order* v.

$$K_n(x) = \left(\frac{\pi}{2x}\right)^{(1/2)} e^{(-x)} \left[1 + \frac{(4n^2-1^2)}{1(8x)} \left(1 + \frac{(4n^2-3^2)}{2(8x)} \left(1 + \frac{(4n^2-5^2)}{3(8x)} (\dots)\right)\right)\right],$$

and c is the coefficient to determine the width or the support of functions. The specific three kinds of Matérn functions are listed:

C. $M_{1/2} \doteq e^{-r/c}$

D. $M_{3/2} \doteq (1 + \frac{r}{c})e^{-r/c}$

E. $M_{5/2} \doteq (1 + \frac{r}{c} + \frac{1}{3}\frac{r^2}{c^2})e^{-r/c}$

Topology Preservation

Criteria to Evaluate Topology Preservation Performances

- The optimal shape parameter c^*, which means the minimum value to preserve the topology. In different cases, the calculations are various.

  1. when support size (or shape parameter) c is larger than c^* $\Rightarrow$ topology preservation is ensured,

     but the locality deformation is extended.

  2. when c is less than c^* $\Rightarrow$ topology violation occurs in the deformation.

- The Jacobian matrix of transformation. $\Rightarrow$ if the determinant of Jacobian matrix is closer 1, then the topology is better preserved

- One-landmark, Two-landmarks and Four-landmarks matching.

One-landmark Matching

- In this case, necessary conditions to have topology preservation are:

  1. Continuity of the function F
  2. Positivity of the Jacobian determinant at each point

$$\Downarrow$$

The coordinates of transformation are

$$F_1(\mathbf{x}) = x + \Delta_x \Phi(||\mathbf{x} - \mathbf{p}||),$$
$$F_2(\mathbf{x}) = y + \Delta_y \Phi(||\mathbf{x} - \mathbf{p}||),$$

$$\Downarrow$$

Requiring the determinant of the Jacobian is positive, we obtain

$$\det(J(x, y)) = 1 + \Delta_x \frac{\partial \Phi}{\partial x} + \Delta_y \frac{\partial \Phi}{\partial y} > 0,$$

$$\Downarrow$$

$$\Delta \frac{\partial \Phi}{\partial r} > -\frac{1}{\sqrt{2}}.$$

here $\Delta = \max\{\Delta_x,\ \Delta_y\}$. c^* can be estimated based on $(\frac{\partial \Phi}{\partial r})_{\min},\ r = \sqrt{x^2 + y^2}$.

Evaluation of c^*

The advantage of having small C is that the influence area of each landmark turns out to be small. This allows us to have a greater local control.

$\varphi_{3,1}$	$\psi_{1,2}$	$\tau_{2,7/2}$	$\tau_{2,5}$
$c > 2.98\Delta$	$c > 2.80\Delta$	$c > 5.09\Delta$	$c > 6.26\Delta$

Table 1: Minimum support size for various CSRBFs, where $d = 2$.

Gaussian	$M_{1/2}$	$M_{3/2}$	$M_{5/2}$
$c > 2.42\Delta$	$c > 1.10\Delta$	$c > 0.52\Delta$	$c > 0.3960\Delta$

Table 2: Minimum support size for GSRBFs, where $d = 2$.

Numerical Results: One-landmark

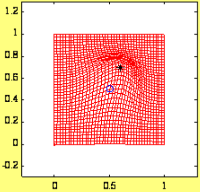
- The considering grid is 40×40 in $[0, 1]^2$. Mapping the template landmark $(0.5, 0.5)$ into the corresponding reference landmark $(0.6, 0.7)$ to transform the grid.
- Here we show two examples:

  i topology preservation results with c^*

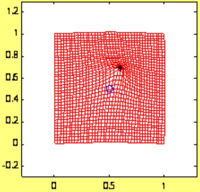
  ii results with a value c which leads topology condition violation

In both examples, $\Delta = 0.2$.

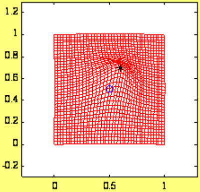
Numerical Results: Topology Preservation for GSRBFs



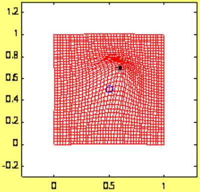
(c) Gaussian $c = 0.5$



(d) Matérn, $M_{1/2}$ $c = 0.22$

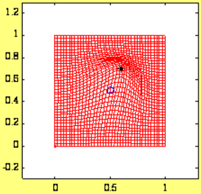


(e) Matérn, $M_{3/2}$ $c = 0.105$

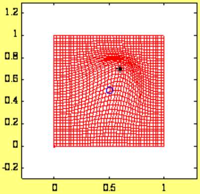


(f) Matérn, $M_{5/2}$ $c = 0.08$

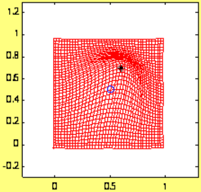
Numerical Results: Topology Preservation for CSRBFs



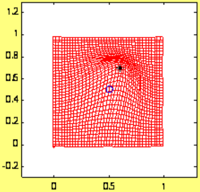
(g) Wendland $\varphi_{3,1}$, $c = 0.6$



(h) Wu $\psi_{1,2}$, $c = 0.58$

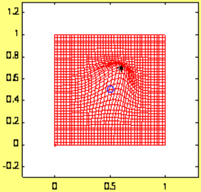


(i) Gneiting $\tau_{2,7/2}$, $c = 1.02$

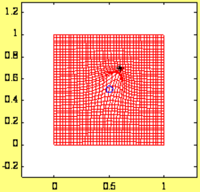


(j) Gneiting $\tau_{2,5}$, $c = 1.26$

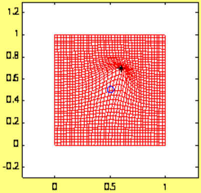
Numerical Results: Topology Violation for GSRBFs



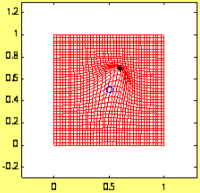
(k) Gaussian $c = 0.34$



(l) Matérn,$M_{1/2}$ $c = 0.09$

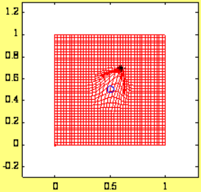


(m) Matérn,$M_{3/2}$ $c = 0.08$

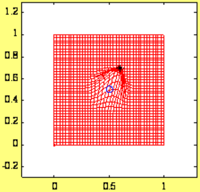


(n) Matérn,$M_{5/2}$ $c = 0.05$

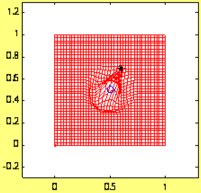
Numerical Results: Topology Violation for CSRBFs



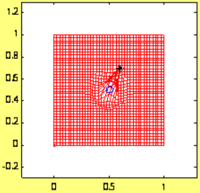
(o) Wendland $\varphi_{3,1}$, $c = 0.25$



(p) Wu $\psi_{1,2}$, $c = 0.25$



(q) Gneiting $\tau_{2,7/2}$, $c = 0.25$



(r) Gneiting $\tau_{2,5}$, $c = 0.25$

Two-landmarks Matching

- In this model, two landmarks are given as $P = \{(0,0),(d,l)\}$ and $Q = \{(0,\Delta),(d,l-\Delta)\}$. In this case, the displacements along x- and y-coordinate are same but with opposite directions. $\Delta < \max\{d,\ l\}$.
    1. the locality parameter c is chosen large enough to ensure the influence regions of the two landmarks intersect each other
    2. small locality parameters result in a non-preserving topology similar to the one-landmark matching case
- Let us now consider components of a generic transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ obtained by a transformation of two points, i.e,
    1. $F_1(\mathbf{x}) = x + \alpha_{1,1}\Phi(||\mathbf{x} - \mathbf{x}_1^T||) + \alpha_{1,2}\Phi(||\mathbf{x} - \mathbf{x}_2^T||)$,
    2. $F_2(\mathbf{x}) = y + \alpha_{2,1}\Phi(||\mathbf{x} - \mathbf{x}_1^T||) + \alpha_{2,2}\Phi(||\mathbf{x} - \mathbf{x}_2^T||)$.

Determinant of Jacobian Matrix

For obtaining $\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{2,1}$ and $\alpha_{2,2}$, we require that

$$F_1((0,0)) = 0, \qquad F_1((d,l)) = d,$$

$$F_2((0,0)) = \Delta, \qquad F_2((d,l)) = l - \Delta.$$

Solving these two systems of two equations in two unknowns, we get

$$\alpha_{1,1} = 0, \;\; \alpha_{1,2} = 0, \;\; \alpha_{2,1} = \frac{\Delta}{1-\Phi\left(\sqrt{d^2+l^2}\right)}, \;\; \alpha_{2,2} = -\alpha_{2,1}.$$

It follows that the determinant of the Jacobian matrix is

$$\det\left(J(x,y)\right) = 1 + \alpha_{2,1}\frac{\partial\Phi\left(\sqrt{x^2+y^2}\right)}{\partial y} + \alpha_{2,2}\frac{\partial\Phi\left(\sqrt{(x-d)^2+(y-l)^2}\right)}{\partial y},$$

Minimum Value of Jacobian Matrix Determinant

The minimum value is occurred at the midpoint between P_1 and P_2, i.e., $\left(\frac{d}{2}, \frac{l}{2}\right)$, when $\Delta > 0$ and the intersection of the influence regions of two landmarks does not turn out to be negligible. We thus obtain the optimal locality parameter when

$$\det\left(J\left(\frac{d}{2}, \frac{l}{2}\right)\right) = 0.$$

Obviously, one can observe that

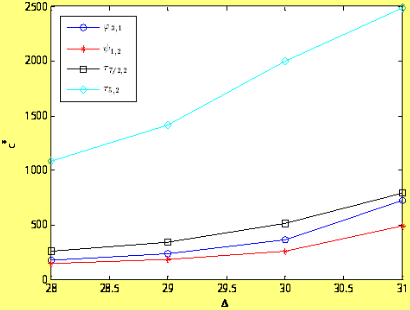
$$\left.\frac{\partial \Phi\left(\sqrt{x^2+y^2}/c\right)}{\partial y}\right|_{x=\frac{d}{2}, y=\frac{l}{2}} = -\left.\frac{\partial \Phi\left(\sqrt{(x-d)^2+(y-l)^2}/c\right)}{\partial y}\right|_{x=\frac{d}{2}, y=\frac{l}{2}},$$

so we get

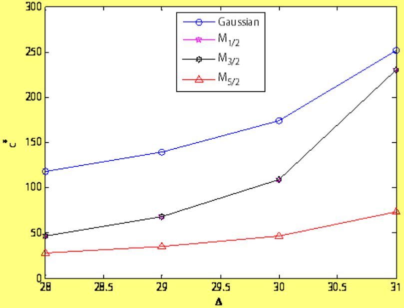
$$\det\left(J\left(\frac{d}{2}, \frac{l}{2}\right)\right) = 1 + 2\alpha_{2,1}\left.\frac{\partial \Phi\left(\sqrt{x^2+y^2}/c\right)}{\partial y}\right|_{x=\frac{d}{2}, y=\frac{l}{2}}.$$

$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)$ of Various RBFs

Radial Basis Functions	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)$
Gneiting $\tau_{2,7/2}$	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)=1-\frac{99}{16c^2}\frac{\Delta l\left(1-\frac{\sqrt{d^2+l^2}}{2c}\right)^{5/2}\left(8-15\frac{\sqrt{d^2+l^2}}{2c}\right)}{1-\left(1-\frac{\sqrt{d^2+l^2}}{c}\right)^{7/2}\left(1+\frac{7}{2}\frac{\sqrt{d^2+l^2}}{c}-\frac{135}{8}\frac{d^2+l^2}{c^2}\right)},$
Gneiting $\tau_{2,5}$	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)=1-\frac{21\Delta l\left(4-\frac{7}{2}\frac{z^{1/2}}{c}-3\frac{z}{c^2}+\frac{19}{4}\frac{z^{3/2}}{c^3}-2\frac{z^2}{c^4}+\frac{9}{2}\frac{z^{5/2}}{c^5}\right)}{42z-175\frac{z^{3/2}}{c}+315\frac{z^2}{c^2}-294\frac{z^{5/2}}{c^3}+140\frac{z^3}{c^4}-27\frac{z^{7/2}}{c^5}},$
Matérn $M_{1/2}$	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)=1-\frac{2l\Delta e^{-\sqrt{d^2+l^2}/2c}}{c\sqrt{d^2+l^2}\left(1-e^{-\sqrt{d^2+l^2}/c}\right)}$
Matérn $M_{3/2}$	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)=1-\frac{l\Delta e^{-\sqrt{d^2+l^2}/2c}}{c^2\left(1-\left(1+\frac{\sqrt{d^2+l^2}}{c}\right)\right)e^{-\sqrt{d^2+l^2}/c}}$
Matérn $M_{5/2}$	$\det\left(J\left(\frac{d}{2},\frac{l}{2}\right)\right)=1-\frac{l\Delta e^{-\sqrt{d^2+l^2}/2c}\left(1+\frac{\sqrt{d^2+l^2}}{2c}\right)}{3c^2\left(1-\left(1+\frac{\sqrt{d^2+l^2}}{c}+\frac{d^2+l^2}{3c^2}\right)\right)e^{-\sqrt{d^2+l^2}/c}}.$

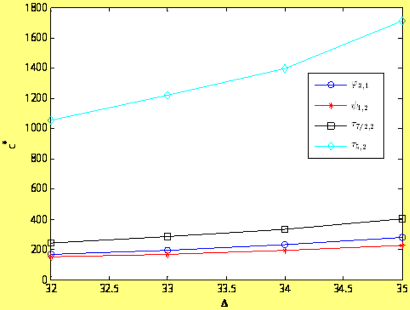


(s) $d = l = 32$

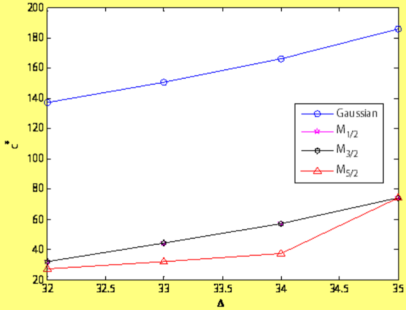


(t) $d = l = 32$

Figure 3: Comparing support size c with fixed d, l and different Δ



(a) $d = 32$, $l = 64$



(b) $d = 32$, $l = 64$

Figure 4: Comparing support size c with fixed d,l and different Δ

Evaluation of the Jacobian Determinant

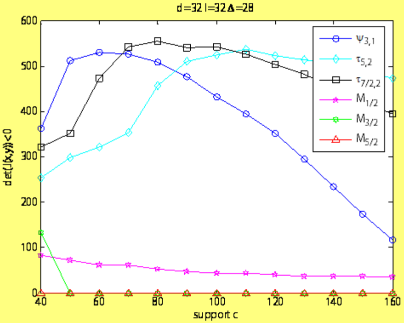
By varying the displacement of such points, we can compare results by two criteria:

1. The number of points where the determinant of the Jacobian is negative

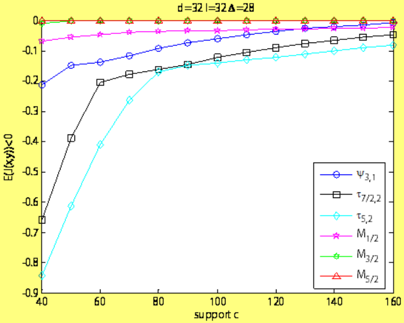
   $\Rightarrow$Such number indicates the size of the region with violated topology preservation.

2. The average of the negative Jacobian determinants

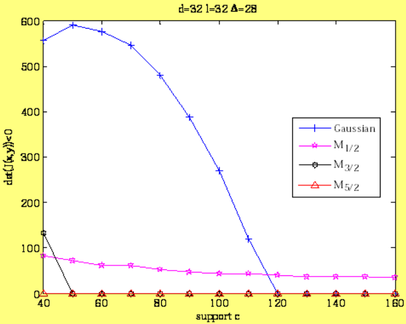
   $\Rightarrow$ This parameter represents the severity of topology violation,The more the value is negative, the more the transformation might be bent or broken compared to the original structure.



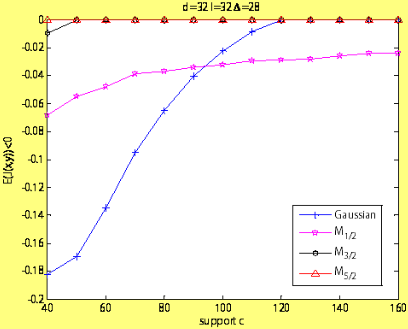
(a) $d = l = 32, \Delta = 28$



(b) $d = l = 32, \Delta = 28$

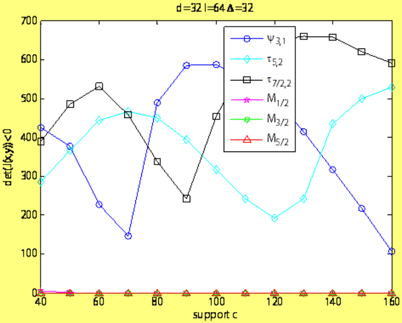


(c) $d = l = 32, \Delta = 28$

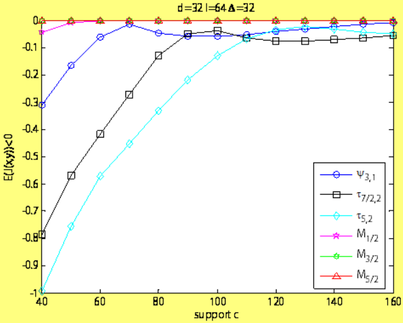


(d) $d = l = 32, \Delta = 28$

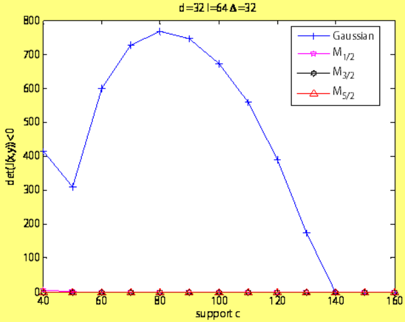
Evaluation of the Jacobian Determinant–Result 2



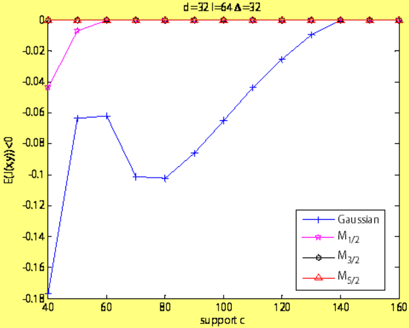
(e) $d = 32, l = 64, \Delta = 28$



(f) $d = 32, l = 64, \Delta = 28$

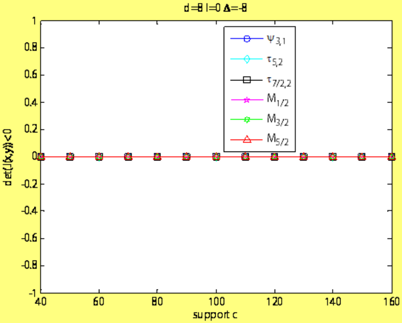


(g) $d = 32, l = 64, \Delta = 28$

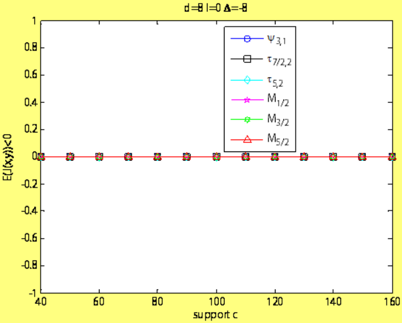


(h) $d = 32, l = 64, \Delta = 28$

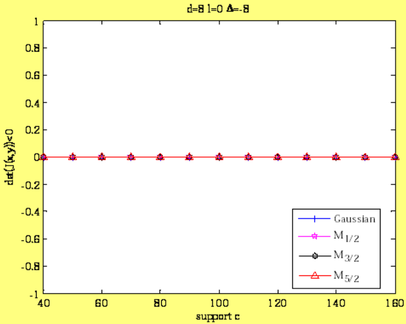
Evaluation of the Jacobian Determinant–Result 3



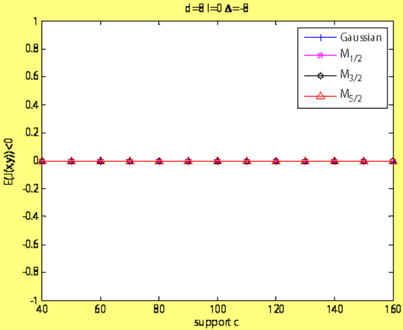
(i) $d = 8, l = 0, \Delta = -8$



(j) $d = 8, l = 0, \Delta = -8$

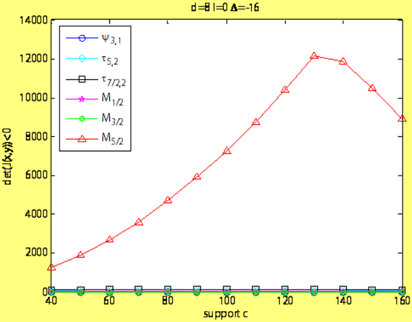


(k) $d = 8, l = 0, \Delta = -8$

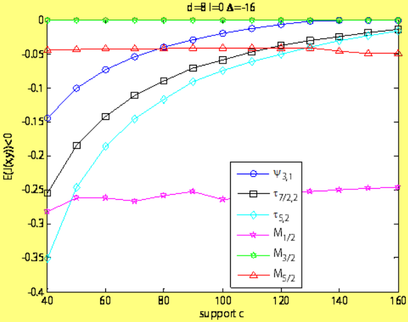


(l) $d = 8, l = 0, \Delta = -8$

Evaluation of the Jacobian Determinant–Result 4

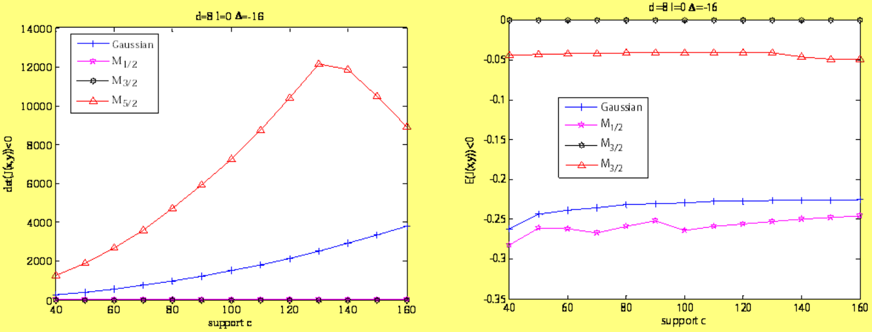


(m) $d = 8, l = 0, \Delta = -16$



(n) $d = 8, l = 0, \Delta = -16$

Figure 5: Comparison of the negative number and the average of Jacobian determinant



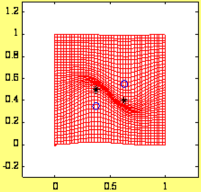
(a) $d = 8, l = 0, \Delta = -16$ (b) $d = 8, l = 0, \Delta = -16$

Figure 6: Comparison of the negative number and the average of Jacobian determinant

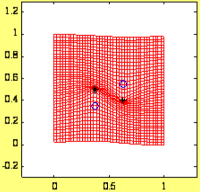
Numerical Results: Two-landmarks

- Considering 40×40 grid in $[0,1]^2$ and then compare results obtained by the grid distortion in the shift case of two landmarks $\{(0.375, 0.350), (0.625, 0.55)\}$ in $\{(0.375, 0.5), (0.625, 0.4)\}$ respectively.

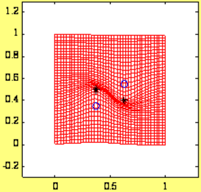
Numerical results: Two-landmarks for GSRBFs



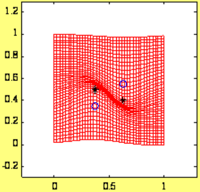
(a) Gaussian, $c = 0.5$



(b) Matérn,$M_{1/2}$, $c = 0.25$

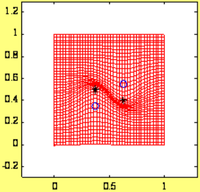


(c) Matérn,$M_{3/2}$, $c = 0.1$

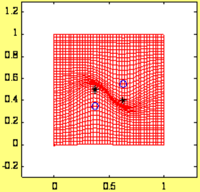


(d) Matérn,$M_{5/2}$, $c = 0.1$

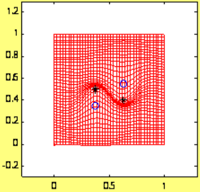
Numerical Results: Two-landmarks for CSRBFs



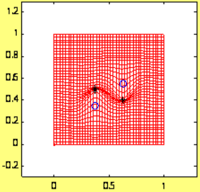
(e) Wendland $\varphi_{3,1}$, $c = 0.5$



(f) Wu $\psi_{1,2}$, $c = 0.5$

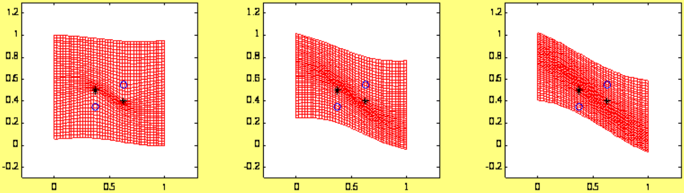


(g) Gneiting $\tau_{2,7/2}$, $c = 0.5$



(h) Gneiting $\tau_{2,5}$, $c = 0.5$

Bad Results: Two-landmarks for the Whole Deformation



(i) $M_{1/2}, c = 0.5$ (j) $M_{3/2}, c = 0.5$ (k) $M_{5/2}, c = 0.5$

Figure 7: Bad results of Matérn transformations

Four-landmarks Matching

1. In the following we compare topology preservation properties for globally supported transformations.
2. Considering four inner landmarks in a grid, located so as to form a rhombus at the center of the figure, and we suppose that only the lower vertex is downward shifted of Δ
3. The landmarks of template and reference images are $P = \{(0,1), (-1,0), (0,-1), (1,0)\}$ and $Q = \{(0,1), (-1,0), (0,-1-\Delta), (1,0)\}$, respectively, with $\Delta > 0$.
4. Let us now consider components of a generic transformation $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ obtained by a transformation of four points P_1, P_2, P_3 and P_4, namely

$$F_1(\mathbf{x}) = x + \sum_{i=1}^{4} \alpha_{1,i} \Phi(||\mathbf{x} - P_i||),$$

$$F_2(\mathbf{x}) = y + \sum_{i=1}^{4} \alpha_{2,i} \Phi(||\mathbf{x} - P_i||).$$

Optimal Locality Parameter c^* in Four-landmarks

- The coefficients $\alpha_{1,i}$ and $\alpha_{2,i}$ are obtained so that the transformation sends P_i to Q_i, with $i = 1, \ldots, 4$. To do that, we need to solve two systems of four equations in four unknowns, whose solutions are

$$\alpha_{1,1} = 0, \quad \alpha_{1,2} = 0, \quad \alpha_{1,3} = 0, \quad \alpha_{1,4} = 0,$$

and

$$\begin{aligned} \alpha_{2,1} &= \tfrac{\beta^2+\beta-2\alpha^2}{(1-\beta)[(1+\beta)^2-4\alpha^2]}\Delta, \quad & \alpha_{2,2} &= \frac{\alpha}{(1+\beta)^2-4\alpha^2}\Delta, \\ \alpha_{2,3} &= -\tfrac{1+\beta-2\alpha^2}{(1-\beta)[(1+\beta)^2-4\alpha^2]}\Delta, \quad & \alpha_{2,4} &= \alpha_{2,2}, \end{aligned}$$

where $\alpha = \Phi\left(\frac{\sqrt{2}}{c}\right)$ and $\beta = \Phi\left(\frac{2}{c}\right)$. For simplicity, we denote $\Phi_1 = \Phi(||(x,y) - P_1||/c)$, $\Phi_2 = \Phi(||(x,y) - P_2||/c)$, $\Phi_3 = \Phi(||(x,y) - P_3||/c)$ and $\Phi_4 = \Phi(||(x,y) - P_4||/c)$.

Four-landmarks: the Determinant of Jacobian matrix

The determinant of the Jacobian matrix is

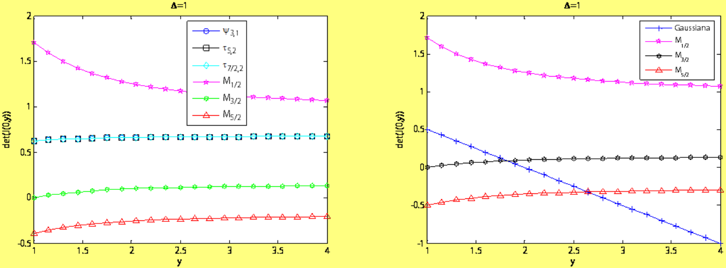
$$\det\left(J(x,y)\right) = 1 + \sum_{i=1}^{4} \alpha_{2,i} \frac{\partial \Phi_i}{\partial y}.$$

The minimum Jacobian determinant is obtained at position $(0, y)$, with $y > 1$. In the following, we analyse the value of the Jacobian determinant at $(0, y)$, with $y > 1$, for different RBFs. Since the support c is very large, in order to have a global transformation, we consider $|| \cdot ||/c$ to be infinitesimal and omit terms of higher order.

$\det(J(0,y))$ of Various RBFs

Radial Basis Functions	$\det(J(0,y))$
Wendland $\varphi_{3,1}$	$\det(J(0,y)) \approx 1 - 0.6402\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$,
Wu $\psi_{1,2}$	$\det(J(0,y)) \approx 1 - 0.6402\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$
Gaussian	$\det(J(0,y)) \approx 1 - \frac{y}{2}\Delta$,
Gneiting $\tau_{2,7/2}$	$\det(J(0,y)) \approx 1 - 0.6402\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$,
Gneiting $\tau_{2,5}$	$\det(J(0,y)) \approx 1 - 0.6402\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$,
Matérn $M_{1/2}$	$\det(J(0,y)) \approx 1 - 2.4142\Delta\left(-1+\frac{y}{\sqrt{y^2+1}}\right)$.
Matérn $M_{3/2}$	$\det(J(0,y)) \approx 1 - 1.7071\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$.
Matérn $M_{5/2}$	$\det(J(0,y)) \approx 1 - 2.5607\Delta\left(y^2+1-y\sqrt{y^2+1}\right)$.

Evaluation of det $(J(0, y))$



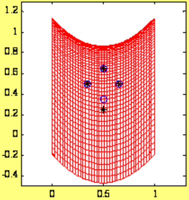
(a) CSRBFs (b) RBFs

Figure 8: Value of $\det(J(0, y))$, with $y > 1$, by varying RBFs.

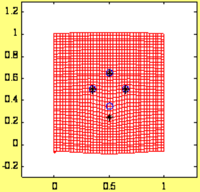
Numerical Results: Four-landmarks

- Considering the same 40×40 grid in $[0, 1]^2$ and compare results obtained by its distortion, which is created by the shift of one of the four landmarks distributed in rhomboidal position.
- The template landmarks are
  $\{(0.5, 0.65), (0.35, 0.5), (0.65, 0.5), (0.5, 0.35)\}$
  and are respectively transformed in the following reference landmarks
  $\{(0.5, 0.65), (0.35, 0.5), (0.65, 0.5), (0.5, 0.25)\}$

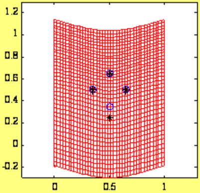
Numerical Results: Four-landmarks for GSRBFs



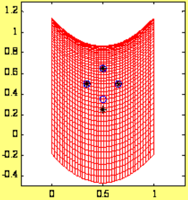
(a) Gaussian, $c = 100$



(b) Matérn,$M_{1/2}$, $c = 100$

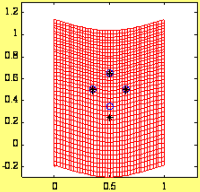


(c) Matérn,$M_{3/2}$, $c = 100$

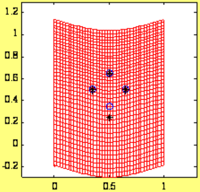


(d) Matérn,$M_{5/2}$, $c = 100$

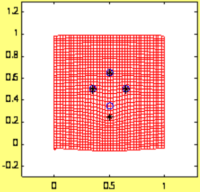
Numerical Results: Four-landmarks for CSRBFs



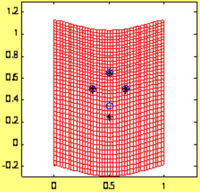
(e) Wendland $\varphi_{3,1}$, $c = 100$



(f) Wu $\psi_{1,2}$, $c = 100$

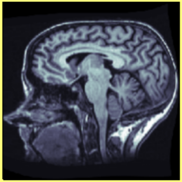


(g) Gneiting $\tau_{2,7/2}$, $c = 100$

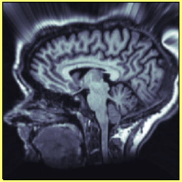


(h) Gneiting $\tau_{2,5}$, $c = 100$

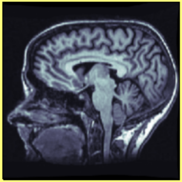
Numerical Results of Medical Brain Images



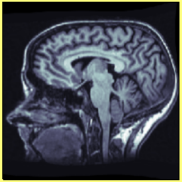
(i) Gaussian, $c = 4$



(j) Matérn,$M_{1/2}$, $c = 4$

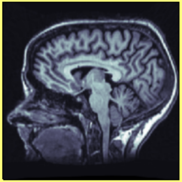


(k) Matérn,$M_{3/2}$, $c = 4$

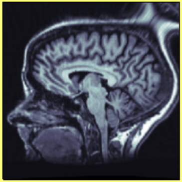


(l) Matérn,$M_{5/2}$, $c = 4$

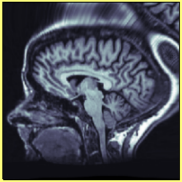
Numerical Results of Medical Brain Images



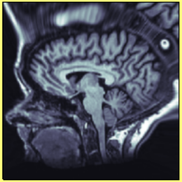
(m) Wendland $\varphi_{3,1}$, $c = 4$



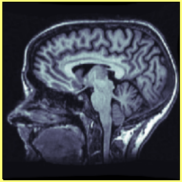
(n) Gneiting $\tau_{2,7/2}$, $c = 4$



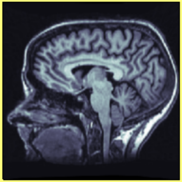
(o) Gneiting $\tau_{2,5}$, $c = 4$



(p) Wu $\psi_{1,2}$, $c = 4$



(q) Gneiting $\tau_{2,7/2}$, $c = 10$



(r) Gneiting $\tau_{2,5}$, $c = 10$

REAL IMAGE SOFTWARE

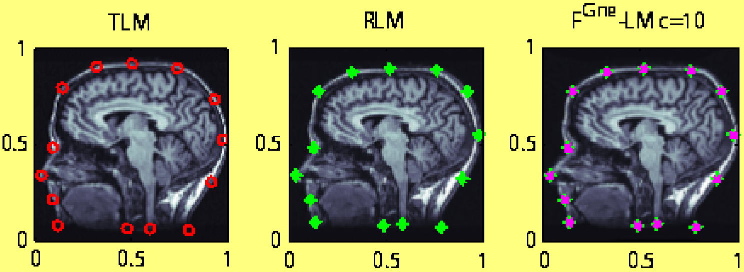
FAIR-Software to Implement LIR

- The software based on FAIR is used to implement LIR, which was created by J. Modersitzki in 2009.

- The process for satisfying LIR is described as the following steps:

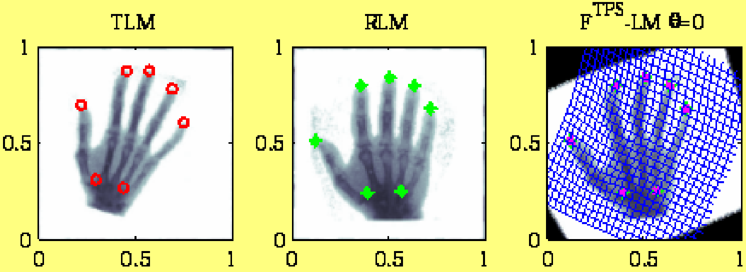
  1. Read into images–setupIMMAGINI.m
  2. Choose landmarks manually–getLandmarks.m
  3. Calcuate the RBF interpolant–getRBFcoefficients.m
  4. Register the template image and view the results–evalRBF.m, viewImage2D.m

How to Implement the Software on MATLAB



(s) E5-2D-Gneiting

The Deformed Grid and the Intuitive Result: OPPOSITE



(t) E5-2D-TPS

Comparison of CPU Time for Various RBFs in Hand Case

RBFs	Gaussian	$M_{1/2}$	$M_{3/2}$	$M_{5/2}$
CPU Time/S	0.3559	0.3593	0.3287	0.3509
RBFs	$\psi_{1,2}$	$\varphi_{3,1}$	$\tau_{2,7/2}$	$\tau_{2,5}$
CPU Time/S	1.9173	1.8389	1.8708	1.9067

Table 3: The CPU time is the average value of 5 experiments. The hand image has 310×310 pixels.

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Thank
you