Lissajous points for polynomial interpolation on various domains

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Lissajous curves

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$$l_{\boldsymbol{\alpha},\mathbf{u}}^{(\boldsymbol{q})}(t) = \left(u_1 \cos\left(\frac{lcm[\boldsymbol{q}] \cdot t - \alpha_1 \pi}{q_1} \right), u_2 \cos\left(\frac{lcm[\boldsymbol{q}] \cdot t - \alpha_2 \pi}{q_2} \right) \right).$$

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- *degenerate*, if there exists $t' \in \mathbb{R}$ and $\boldsymbol{u'} \in \{-1, 1\}^2$, such that $l_{\alpha,\boldsymbol{u}}^{(\boldsymbol{q})}(\cdot - t') = l_{\mathbf{0},\boldsymbol{u'}}^{(\boldsymbol{q})}$,

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- If the Lissajous curve is $non-degenerate$, we can find, up to finitely many exceptions, only one value of $t \in [0, 2\pi)$ corresponding to a point on the curve.

Degenerate Lissajous curves

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Consider the Lissajous figures of the type:

$$\gamma_{n,p} : [0, \pi] \to [-1, 1]^2$$
$$\gamma_{n,p} = \Big(\cos(nt), \cos((n+p)t)\Big) = l_{\mathbf{0},\mathbf{1}}^{(n+p,n)}(t)$$

with n and p positive integers such that n and $n+p$ are relatively prime.

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with n and p positive integers such that n and $n+p$ are relatively prime.
If we sample the curve along $n(n+p)+1$ equidistant points

$$t_k = \frac{\pi k}{n(n+p)}, \qquad k = 0, ..., n(n+p)$$

in the interval $[0,\pi]$, we get the Lissajous points

$$LD_{n,p} := \{\gamma_{n,p}(t_k) : k = 0, ..., n(n+p)\}.$$

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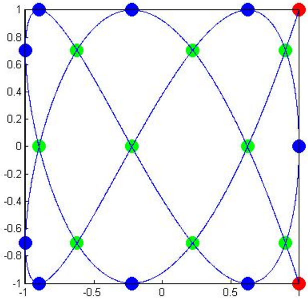

Figure: Lissajous curve $\gamma_{4,3}$ and $LD_{4,3}$.

We can describe the Lissajous curves and points in another way: we consider the algebraic Chebyshev variety

$$\mathcal{C}_{n,p} := \{(x, y) \in [-1, 1]^2 : T_{n+p}(x) = T_n(y)\}$$

where $T_n(x) = cos(\mathrm{n}arcos(x))$ denotes the Chebyshev polynomial of the first kind.

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This algebraic variety corresponds to the degenerate Lissajous curve and the singular points of $\mathcal{C}_{n,p}$ to the Lissajous points.

If we use the notation

$$z_k^n := cos\left(\frac{k\pi}{n}\right), \quad n \in \mathbb{N}, \quad k = 0, ...n$$

to abbreviate the Chebyshev-Lobatto points, we can describe the Lissajous points as

$$LD_{n,p} = \{(z_i^{n+p}, z_j^n) : i = 0, ..., n+p, j = 0, ..., n, i+j \equiv 0 \mod 2\}$$

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Example: for $d = 1$ and $n \in \mathbb{N}$, we obtain $l_{0,1}^n(t) = cos(t)$, $l_{0,1}^n([0, \pi]) = [-1, 1]$ and the points $LD_n = \{z_i^n : i = 0, ..., n\}$ are the univariate Chebyshev-Lobatto points.

The points can be arranged, also, in two rectangular grids:

$$LD^{r}_{n,p} = \{(z_i^{n+p}, zj^{n}) : i = 0, ..., n+p, j = 0, ..., n, i, j even\}$$

$$LD^{b}_{n,p} = \{(z_i^{n+p}, zj^{n}) : i = 0, ..., n+p, j = 0, ..., n, i, j odd\}$$

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Introducing the index sets

$$\Gamma^L_{n,p} = \left\{(i,j) \in \mathbb{N}_0^2 : \frac{i}{n+p} + \frac{j}{n} < 1\right\} \cup \{(0,n)\},$$

the note set $LD_{n,p}$ can be characterized as

$$LD_{n,p} = \{(z^{n+p}_{in+j(n+p)}, z^n_{in+j(n+p)}) : (i,j) \in \Gamma^L_{n,p})\}.$$

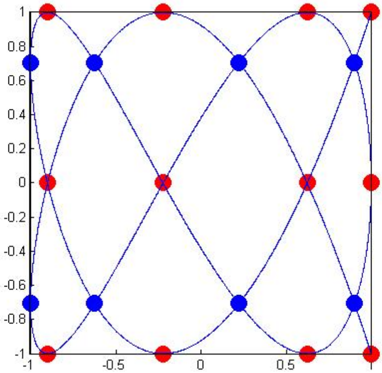

Figure: Lissajous curve $\gamma_{4,3}$ and $LD_{4,3}$.

Non-degenerate Lissajous curves

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We can consider the non-degenerate Lissajous curves of the form

$$\lambda_{n,p} = (sin(nt), sin((n+p)t)) = l^{2(n+p,n)}_{(n+p,n),\mathbf{1}}(t)$$

where n and p are relatively prime. The curve is non-degenerate if and only if p is odd.

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where n and p are relatively prime. The curve is non-degenerate if and only if p is odd.
In this case, $\lambda_{n,p} : [0, 2\pi) \to \mathbb{R}^2$ is sampled along the $4n(n+p)$ equidistant points

$$t_k := \frac{2\pi k}{4n(n+p)}, \quad k = 1, ..., 4n(n+p).$$

In this way we get the following set of Lissajous node points:

$$LND_{n,p} := \{\lambda_{n,p}(t_k) : k = 1, ..., 4n(n+p)\}.$$

- The non-degenerate Lissajous points are the self-intersections and boundary contacts of the Lissajous curve in the square $[-1,1]^2$;
- $|LND_{n,p}| = 2n(n+p) + 2n + p$;
- We can describe the points using the Chebyschev-Lobatto points;
- The algebraic Chebyshev variety in this case is

$$C_{n,p} = \{(x,y) \in [-1,1]^2 : (-1)^{n+p} T_{2n+2p}(x) = (-1)^n T_{2n}(y)\}.$$

- The points can be arranged in two rectangular grids.

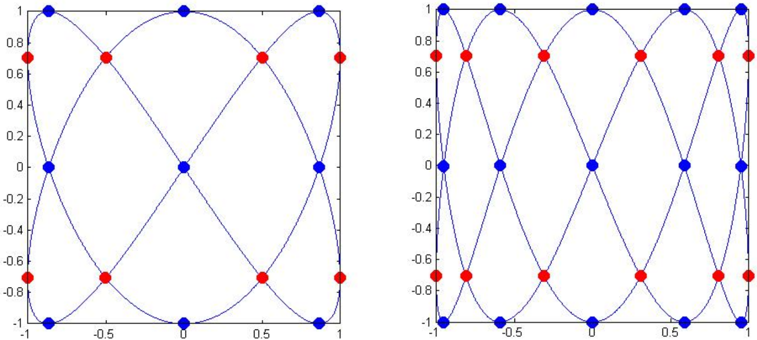


Figure: Lissajous curves $\lambda_{2,1}$, $|LND_{2,1}| = 17$ (left) and $\lambda_{2,3}$, $|LND_{2,3}| = 27$ (right).

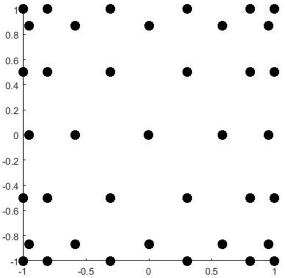
Particular set of Lissajous points $LC_{\boldsymbol{k}}^{(2\boldsymbol{n})}$

Now we consider this set of Lissajous points:

$$LC_{\mathbf{0}}^{(10,6)} = LC_{\mathbf{0},0}^{(10,6)} \cup LC_{\mathbf{0},1}^{(10,6)}$$

$$LC_{\mathbf{0},\tau}^{2\boldsymbol{n}} = \{(z_i^{(2n_1)}, z_j^{(2n_2)}) : (i,j) \in \mathbb{I}_{\mathbf{0},\tau}^{(2\boldsymbol{n})}\}$$

$$\mathbb{I}_{\mathbf{0},\tau}^{(2\boldsymbol{n})} = \{(i,j) \in \mathbb{N}_0 : 0 \le i \le 2n_1 \quad i \equiv \tau mod2, \\ 0 \le j \le 2n_2 \quad j \equiv \tau mod2\}$$



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Now, we return to the inititial general notation:

$$l_{\boldsymbol{k},\mathbf{u}}^{(2\boldsymbol{n})}(t) = (u_1 \cos(2n_1 t - k_1), u_2 \cos(2n_2 t - k_2)).$$

The affine Chebyshev variety can be written as:

$$C_{\mathbf{k}}^{(2\mathbf{n})} = \bigcup_{\mathbf{u} \in \{-\mathbf{1},\mathbf{1}\}^{\mathbf{2}}} l_{\boldsymbol{k},\mathbf{u}}^{(2\boldsymbol{n})}([0,\pi)).$$

Similar as for the degenerate curves, a point is a singular point of $C_{\mathbf{k}}^{(2\mathbf{n})}$ if and only if it is a Lissajous point.

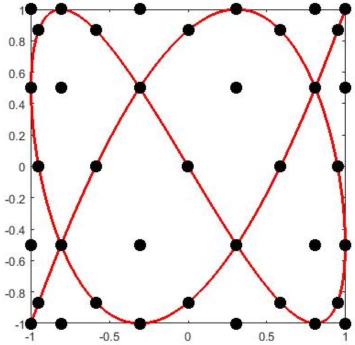

Figure: Lissajous curve $l^{(10,6)}_{\mathbf{0},(1,1)}([0,\pi])$ and $LC^{(10,6)}_{\mathbf{0}}$.

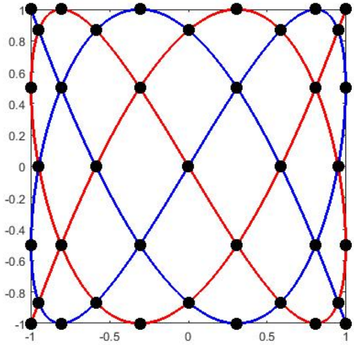

Figure: Lissajous curves $l^{(10,6)}_{\mathbf{0},(1,1)}([0,\pi]) \cup l^{(10,6)}_{\mathbf{0},(-1,1)}([0,\pi])$ and $LC^{(10,6)}_{\mathbf{0}}$.

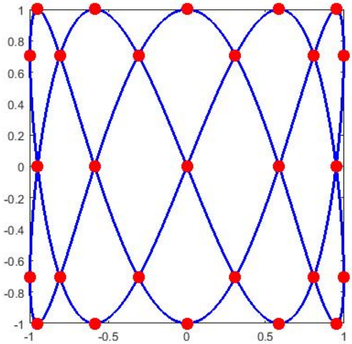

Figure: Lissajous points $LC^{2(5,2)}_{(5,0)}$.

Unified theory for Lissajous points and curves

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We make use of a decomposition of the parameter vector $\boldsymbol{n}$: there exist integer vectors $\boldsymbol{n}^*, \boldsymbol{n}^o \in \mathbb{N}^2$ such that $\forall i = 1, 2$:

- $n_i = n_i^* n_i^o$;
- n_i^* and n_i^o are relatively prime;
- n_1^* and n_2^* are relatively prime;
- $lcm[\boldsymbol{n}] = n_1^* n_2^*$.

We introduce the following sets :

$$R^{(\boldsymbol{n}^o)} = \{0, 1, ..., m_1^o - 1\} \times \{0, 1, ..., m_2^o - 1\}$$

$$\begin{aligned}\mathbb{I}_{\boldsymbol{k},\tau}^{(\boldsymbol{n})} = \{(i, j) \in \mathbb{N}_0 : & 0 \leq i \leq 2n_1 \quad i \equiv \tau mod2, \\ & 0 \leq j \leq 2n_2 \quad j \equiv \tau mod2\}\end{aligned}$$

Using the index sets $\mathbb{I}_{\boldsymbol{k},\tau}^{(\boldsymbol{n})}$, with $\tau = 1, 2$, we obtain the Lissajous points

$$LC_{\boldsymbol{k}}^{(\boldsymbol{n})} = LC_{\boldsymbol{k},0}^{(\boldsymbol{n})} \cup LC_{\boldsymbol{k},1}^{(\boldsymbol{n})}, \quad \text{where}$$

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We consider now the following set of Lissajous curves

$$\mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*,\boldsymbol{n}^o)} = \left\{ l_{(2p_1 m_1^* + k_1, 2p_2 m_2^* + k_2)}^{\boldsymbol{n}} | \boldsymbol{p} \in R^{(\boldsymbol{n}^o)} \right\}.$$

The Chebyshev variety $\mathcal{C}_{\boldsymbol{k}}^{\boldsymbol{n}}$ can be written as:

$$\mathcal{C}_{\boldsymbol{k}}^{\boldsymbol{n}} = \bigcup_{l \in \mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*,\boldsymbol{n}^o)}} l([0, 2\pi])$$

The cardinality of $\mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*,\boldsymbol{n}^o)}$ is $\#\mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*,\boldsymbol{n}^o)} = \frac{1}{2}(n_1^o n_2^o + N_{\deg})$, where the number of degenerate curves $N_{\deg}$ is

$$N_{\deg} = \begin{cases} 1 & \text{if } M_0 = \emptyset, \\ 2^{\#(K_0 \cap M_0)-1} & \text{if } K_0 \cap M_0 \neq \emptyset \text{ and } K_1 \cap M_0 = \emptyset \\ 2^{\#(K_1 \cap M_0)-1} & \text{if } K_0 \cap M_0 = \emptyset \text{ and } K_1 \cap M_0 \neq \emptyset \\ 0 & \text{if } K_0 \cap M_0 \neq \emptyset \text{ and } K_1 \cap M_0 \neq \emptyset \end{cases}$$

where for $\tau \in \{(0,1\}$ we denote

$$M_0 = \{i \in \{1,2\} | n_i \equiv 0 \mod 2\}$$
$$K_\tau = \{i \in \{1,2\} | k_i \equiv \tau \mod 2\}.$$

Example: we consider the bivariate node set $LC^{(10,5)}_{(0,0)}$. We have:

- $\boldsymbol{n} = (10, 5) \quad \boldsymbol{k} = (0, 0)$,
- $\boldsymbol{n}^* = (10, 1) \quad \boldsymbol{n}^o = (1, 5)$,
- $\mathfrak{L}^{(\boldsymbol{n}^*,\boldsymbol{n}^o)}_{\boldsymbol{k}} = \{l^{\boldsymbol{n}}_{(0,2p)} | p \in \{0, 1, 2\}\}$,
- $\mathcal{C}^{\boldsymbol{n}}_{\boldsymbol{k}} = \bigcup_{p\in\{0,2,4\}} l^{(10,5)}_{(0,p)}([0, 2\pi])$,
- $M_0 = \{1\}, \quad K_0 = \{1, 2\} \quad K_1 = \emptyset$,
- $N_{\text{deg}} = 2^{\#(K_0 \cap M_0)-1} = 2^0 = 1$.

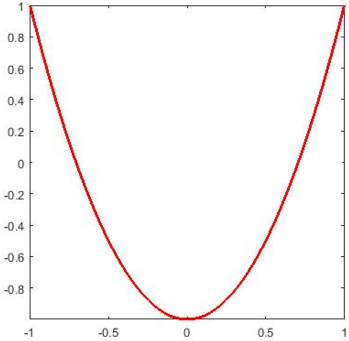

Figure: $I_{(0,0)}^{(10,5)}([0, 2\pi))$

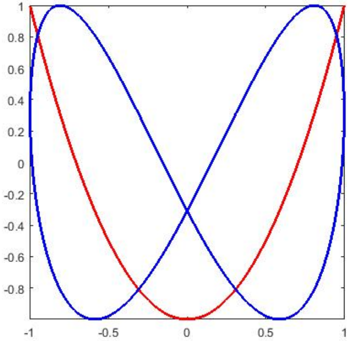

Figure: $I^{(10,5)}_{(0,0)}([0,2\pi)) \cup I^{(10,5)}_{(0,2)}([0,2\pi))$

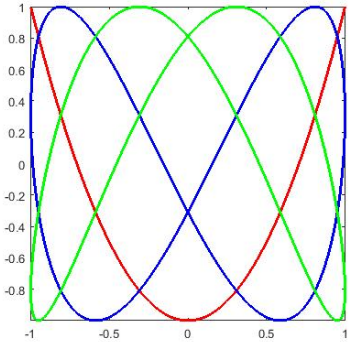

Figure: $l^{(10,5)}_{(0,0)}([0,2\pi)) \cup l^{(10,5)}_{(0,2)}([0,2\pi)) \cup l^{(10,5)}_{(0,4)}([0,2\pi))$

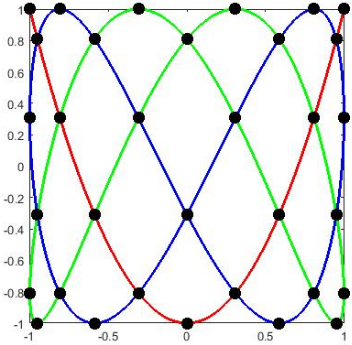

Figure: $I^{(10,5)}_{(0,0)}([0,2\pi)) \cup I^{(10,5)}_{(0,2)}([0,2\pi)) \cup I^{(10,5)}_{(0,4)}([0,2\pi))$ and $LC^{(10,5)}_{(0,0)}$

Interpolation at the degenerate Lissajous points

Given data values $f(\mathcal{A}) \in \mathbb{R}$ at the node points $\mathcal{A} \in LD_{n,p}$, the aim is to find the unique bivariate interpolating polynomial $\mathcal{L}_{n,p}f$ such that

$$\mathcal{L}_{n,p}f(\mathcal{A}) = f(\mathcal{A}), \quad \forall \mathcal{A} \in LD_{n,p}$$

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Which polynomial space has to be chosen for the interpolation problem?
We introduce the space $\Pi^{2,L}_{n,p} = span\{\hat{T}_i(x)\hat{T}_j(y) : (i,j) \in \Gamma^L_{n,p}\}$, where $\hat{T}_i(x)$ is the normalized classical Chebyshev polynomial of the first kind of degree i,

$$\hat{T}_i(x) = \begin{cases} 1, & \text{if } i = 0, \\ \sqrt{2}T_i(x) & \text{if } i \neq 0. \end{cases}$$

$$\Gamma^L_{n,p} = \left\{(i,j) \in \mathbb{N}^2_0 : \frac{i}{n+p} + \frac{j}{n} < 1\right\} \cup \{(0,n)\},$$

$\{\hat{T}_i(x)\hat{T}_j(y) : (i,j) \in \Gamma^L_{n,p}\}$ forms an orthonormal basis for the space $\Pi^{2,L}_{n,p}$ with respect to the inner product

$$\langle f, g\rangle := \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} f(x,y)\overline{g(x,y)}\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}\,dx\,dy.$$

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Since $dim(\Pi^{2,L}_{n,p}) = |\Gamma^L_{n,p}| = |LD_{n,p}|$, our primary choice is the polynomial space $\Pi^{2,L}_{n,p}$.

$\{\hat{T}_i(x)\hat{T}_j(y) : (i,j) \in \Gamma^L_{n,p}\}$ forms an orthonormal basis for the space $\Pi^{2,L}_{n,p}$ with respect to the inner product

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The space $((\Pi^{2,L}_{n,p}, \langle\cdot,\cdot\rangle)$ has the *reproducing kernel*

$$K^L_{n,p}(\mathcal{A},\mathcal{B}) = \sum_{(i,j)\in\Gamma^L_{n,p}} \hat{T}_i(x_\mathcal{A})\hat{T}_i(y_\mathcal{A})\hat{T}_j(x_\mathcal{B})\hat{T}_j(y_\mathcal{B})$$

The fundamental Lagrange polynomials of the Lissajous points are

$$L_{\mathcal{A}}(x,y) := \omega_{\mathcal{A}}[K^{L}_{n,p}((x,y);\mathcal{A}) - \frac{1}{2}\hat{T}_n(y)\hat{T}_n(y_{\mathcal{A}})],$$

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where

$$\omega_{\mathcal{A}} := \frac{1}{n(n+p)}\begin{cases} 1/2, & \text{if } \mathcal{A} \in LD_{n,p} \text{ is a vertex point,} \\ 1, & \text{if } \mathcal{A} \in LD_{n,p} \text{ is an edge point,} \\ 2 & \text{if } \mathcal{A} \in LD_{n,p} \text{ is an interior point,} \end{cases}$$

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Now, the interpolation problem has a unique solution in $\Pi^{2,L}_{n,p}$ given by

$$\mathcal{L}_{n,p}f(x,y) = \sum_{\mathcal{A}\in LD_{n,p}} L_{\mathcal{A}}(x,y)f(\mathcal{A})$$

We can rewrite the interpolating polynomial $\mathcal{L}_{n,p}f(x,y)$ in terms of the orthonormal Chebyshev basis. In this way, we obtain the representation

$$\mathcal{L}_{n,p}f(x,y) = \sum_{i,j \in \Gamma_{n,p}^{L}} c_{i,j} \hat{T}_i(x) \hat{T}_j(y),$$

with the Fourier-Lagrange coefficients $c_{i,j} = \langle \mathcal{L}_{n,p}f, \hat{T}_i(x)\hat{T}_j(y) \rangle$ given by

$$c_{i,j} = \begin{cases} \sum_{\mathcal{A} \in LD_{n,p}} f(\mathcal{A}) \omega_{\mathcal{A}} \hat{T}_i(x_{\mathcal{A}}) \hat{T}_j(y_{\mathcal{A}}), & \text{if } (i,j) \in \Gamma_{n,p}, \\ \frac{1}{2} \sum_{\mathcal{A} \in LD_{n,p}} f(\mathcal{A}) \omega_{\mathcal{A}} \hat{T}_n(y_{\mathcal{A}}), & \text{if (i,j) = (0,n).} \end{cases}$$

The interpolating polynomial can be formulted more compactly using the matrix notation

$$\mathcal{L}_{n,p}f(x,y) = \mathbb{T}_x(x,y)^T \mathbb{C}_{n,p} \mathbb{T}_y(x,y),$$

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- $\mathbb{M}_{n,p} = (m_{i,j}), \qquad m_{i,j} = \begin{cases} 1, & \text{if } (i,j) \in \Gamma_{n,p}, \\ 1/2, & \text{if } (i,j) = (0,n) \\ 0. & (i,j) \notin \Gamma^L_{n,p} \end{cases}$

The Lebesgue constant

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The Lebesgue constant is the operator norm

$$\Lambda_{n,p} = \max_{\substack{f \in \mathcal{C}([-1,1]^2) \\ f \neq 0}} \frac{\|\mathcal{L}_{n,p}(f)\|_\infty}{\|f\|_\infty} = \max_{(x,y)\in[-1,1]^2} \sum_{\mathcal{A}\in LD_{n,p}} |L_{\mathcal{A}}(x,y)|.$$

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We investigate the Lebesgue constant Λ_{n,p_n} for the parameters $p_n \in \{1, n+1\}$. The values are illustrated for $1 \leq n \leq 50$. For a better comparison we plot also the functions $f_1(n)$ and $f_2(n)$, as a lower and an upper benchmark:

$$f_1(n) = \Big(\frac{2}{\pi} log(n+1) + 1\Big)^2$$

$$f_2(n) = \Big(\frac{2}{\pi} log(n+1) + \frac{3}{2}\Big)^2$$

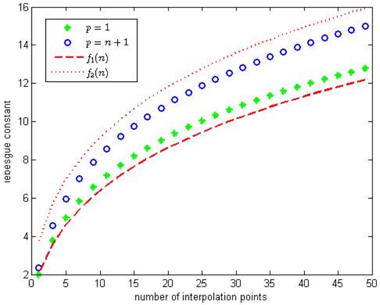
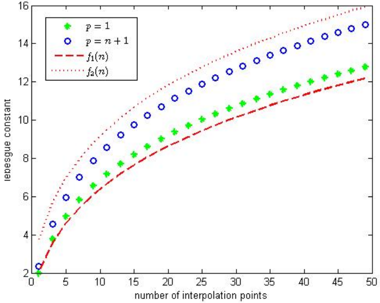


Figure: The Lebesgue constant for the parameters $p_n \in \{1, n+1\}$.



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Theorem

The Lebesgue constant $\Lambda_{n,p}$ is bounded

$$D_\Lambda \ln^2(n) \leq \Lambda_{n,p} \leq C_\Lambda \ln^2(n+p),$$

where the constants D_Λ and C_Λ are independent of n and p.

Numerical examples

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We use the four test functions:

- $f_1(x,y) = 0.75e^{-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}} + 0.75e^{-\frac{(9x+1)^2}{49}-\frac{9y+1}{10}} + 0.5e^{-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}} - 0.2e^{-(9x-4)^2-(9y-7)^2}$

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- $f_2(x,y) = (x^2+y^2)^{\frac{5}{2}}$,
- $f_3(x,y) = e^{-\frac{(5-10x)^2}{2}} + 0.75e^{-\frac{(5-10y)^2}{2}} + 0.75e^{-\frac{(5-10x)^2+(5-10y)^2}{2}}$

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- $f_4(x,y) = (x+y)^{20}$.

$\|\mathcal{L}_{n,p_n}(f) - f\|_\infty$ on the domain $[-1,1]^2$, for three different parameters $p_n \in \{1, n+1, n^2+1\}$ and $2 \le n \le 50$.
The maximal error is computed on a uniform grid of 60×60 points defined in the square $[-1,1]^2$.

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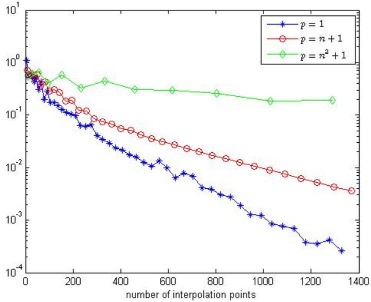

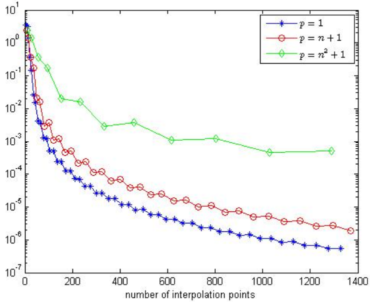

Figure: Absolute errors for f_1 (left) and f_2 (right).

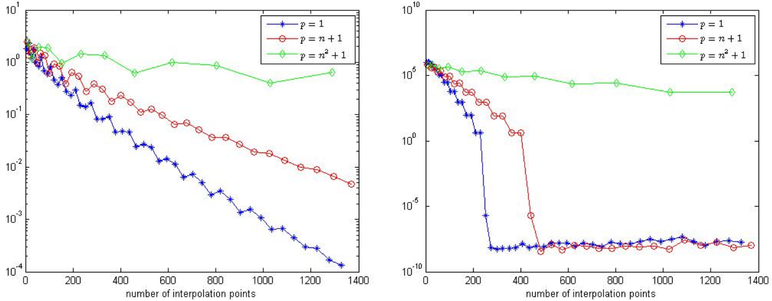

Figure: Absolute errors for f_3 (left) and f_4 (right).

The Lissajous points can be extendend to other domains through a suitable mapping of the square:

$$\sigma : [-1, 1]^2 \to K.$$

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We can construct the interpolation formula on the new domain,

$$\mathcal{L}_{n,p} f(x_1,x_2) := \mathbb{T}(\sigma_1^{-1}(x_1,x_2))^t \mathbb{C}_0(f \circ \sigma)\mathbb{T}(\sigma_2^{-1}(x_1,x_2)), \qquad (1)$$

where $\sigma_i^{-1}(x_1,x_2)$, with $i = 1,2$, denotes the component of the inverse transformation.

Rectangle

Rectangle

$$\sigma(t_1, t_2) = \left(\frac{b-a}{2} t_1 + \frac{b+a}{2}, \frac{d-c}{2} t_2 + \frac{d+c}{2} \right).$$

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The inverse is given by

$$t_1(x_1, x_2) = -1 + 2\frac{x_1 - a}{b - a}, \qquad t_2(x_1, x_2) = \begin{cases} -1 + 2\frac{x_2 - c}{d - c} & \text{if } c \neq d \\ -1 & \text{if } c = d \end{cases}$$

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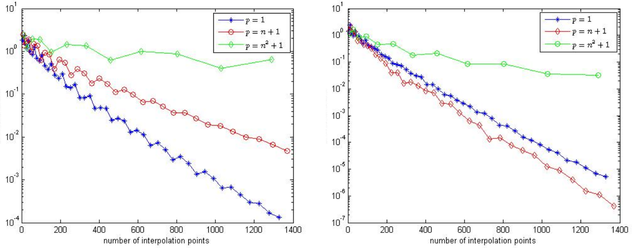

Figure: Absolute errors for f_3 on $[-1, 1]^2$ (left) and on $[0, 2] \times [0, 1]$ (right).

Ellipse

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We consider the ellipse centered in $c = (c_1, c_2)$, with x_1-semiaxis α and x_2-semiaxis β.

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- starlike-polar coordinates

$$\sigma_1(t_1, t_2) = c_1 - \alpha t_2 \sin\left(\frac{\pi}{2} t_1\right), \qquad \sigma_2(t_1, t_2) = c_2 + \beta t_2 \cos\left(\frac{\pi}{2} t_1\right).$$

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- standard polar coordinates

$$\sigma_1(t_1, t_2) = \rho \cos(\theta), \qquad \sigma_2(t_1, t_2) = \rho \sin(\theta)$$

$$\theta(t_1, t_2) = \pi(t_1 + 1), \quad \rho(t_1, t_2) = (t_2 + 1)\frac{r(\theta)}{2}, \quad r(\theta) = \frac{\beta^2/\alpha}{1 - ecos(\theta)}.$$

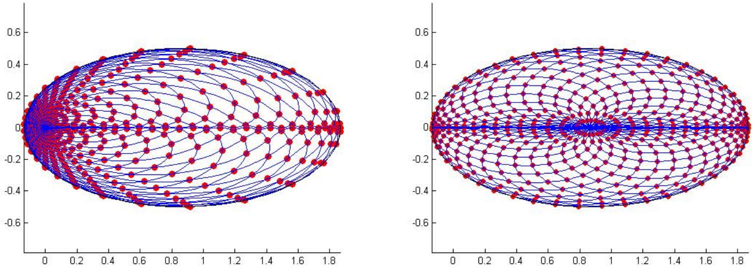

Figure: The distribution of the Padua points and the Lissajous curves in the ellipse with polar (left) and starlike-polar (right) for $n = 33$.

We are interested to understand how $\|\mathcal{L}_{n,1}(f) - f\|_\infty$, for $0 \le n \le 50$, changes in relation with the trasformations which we have chosen. The maximal error is computed on a uniform grid of 100×100 points in the ellipse with semi-major axis 1, semi-minor axis 0.5 and centered in $c = (\sqrt{0.75}, 0)$.

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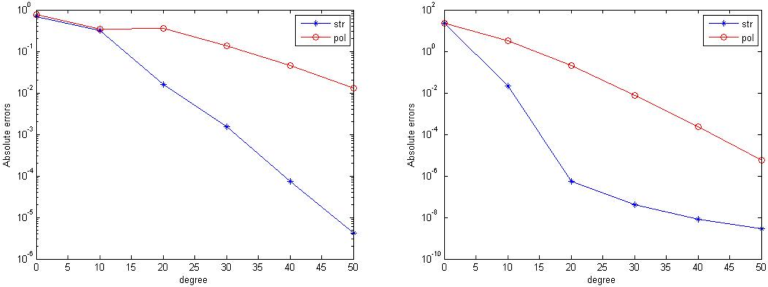

Figure: The absolute errors on the ellipse in polar and starlike-polar coordinates for f_1 (left) and f_2 (right) .

Diamond

Diamond

The maximal error is compute on a WAM with about 15000 points, which is generated by minimal triangulation.

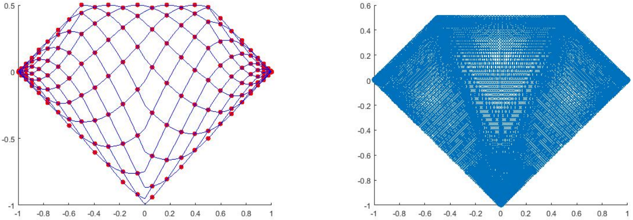

Figure: Lissajous points and curves in the diamond with $n = 13$ and $p_n = n + 1 = 14$ (left), WAM generates by minimal triangulation, 14913 points (right) .

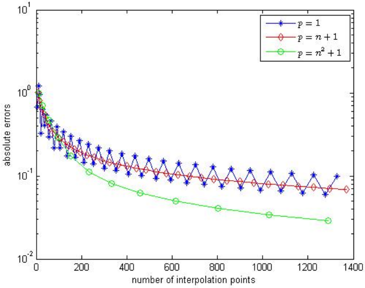

Figure: Absolute errors for f_2 (left) and f_3 (right).

Intersection of disks

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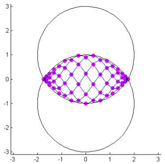
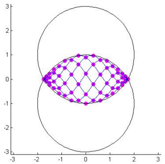


Figure: Lissajous points and curves in the intersection of disks with $n = 7$ and $p = 11$.



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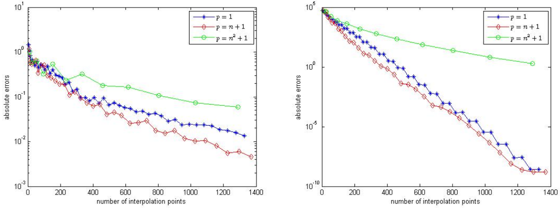

Figure: Absolute errors for f_1 (left) and f_4 (right).

Star

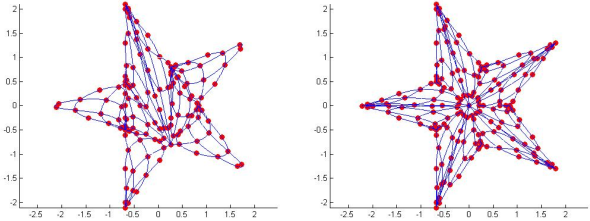

Figure: Distribution of Padua points and curves with minimal and barycentric triangulation. $n = 5$ (168 points left, 210 points right).

Star

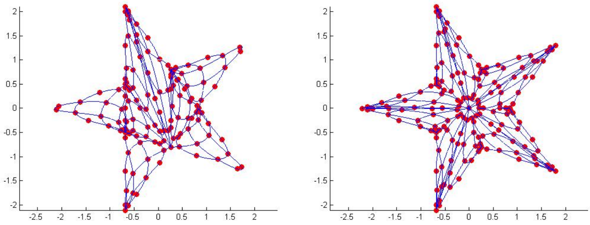

Figure: Distribution of Padua points and curves with minimal and barycentric triangulation. $n = 5$ (168 points left, 210 points right).

Triangulation for f_2	$n = 2$	$n = 10$	$n = 20$
minimal triang.	$7.98e+01$	$3.81e-03$	$9.85e-05$
barycentric triang.	$7.87e+01$	$9.90e-07$	$1.68e-10$

Table: Absolute errors with different triangulations for f_2.

To map, instead, the Lissajous points in the star we use the Schwarz-Christoffel functions. These are conformal maps from the unit disk onto various domains. In the case of our star the mapping is

$$f(z) = \int_0^z \frac{(1 - w^5)^{2/5}}{(1 + w^5)^{4/5}} \, dw$$

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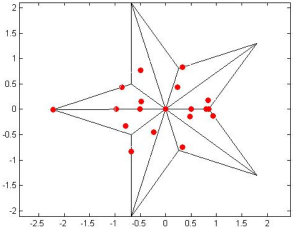

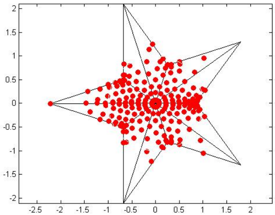

Figure: Padua points in the star with $n = 5$, i.e. with 21 points (left) and $n = 20$ i.e. 231 points (right).

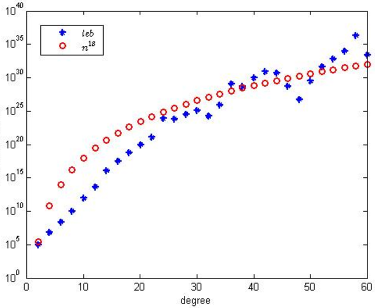

Figure: Lebesgue constant for Padua points.

Morrow-Patterson points

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- For n a positive *even* integer, the Morrow-Patterson points are the self-intersection points in the interior square $[-1, 1]^2$ of the Lissajous curves

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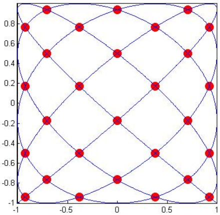

Figure: The curve $\gamma_{6,1}(t)$ and associated MP_6.

Padua points

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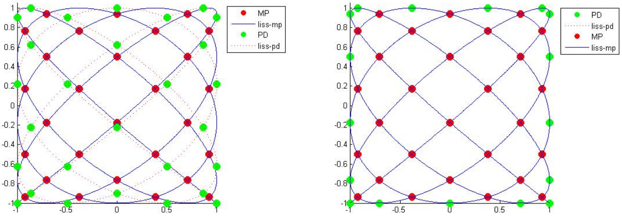

Figure: MP_6 and PD_6 with respective Lissajous curves (left), MP_6 and PD_8 with respective Lissajous curves (right).

Xu points

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- if n is even, i.e. $n = 2m$, there are $\frac{n(n+2)}{2}$ points, $LC^{(2m,2m)}_{(0,1)} =$

$$\begin{aligned} &(z_{2i}, z_{2j+1}), \quad 0 \le i \le m, 0 \le j \le m-1, \\ &(z_{2i+1}, z_{2j}), \quad 0 \le i \le m-1, 0 \le j \le m. \end{aligned}$$

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- if n is odd, i.e. $n = 2m+1$, there are $\frac{(n+1)^2}{2}$ points, $LC_{(0,0)}^{(2m+1,2m+1)} =$

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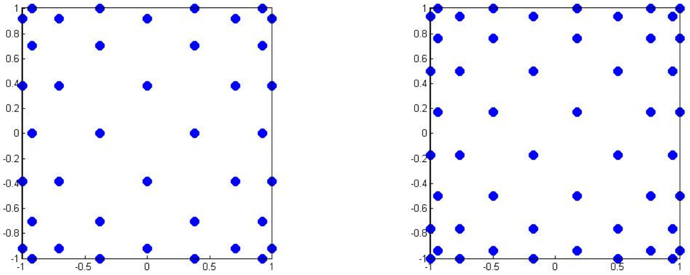


Figure: Xu points for $n = 8$ (left) and $n = 9$ (right).

- $\boldsymbol{n} = (n, n), \boldsymbol{n}^* = (n, 1), \boldsymbol{n}^o = (1, n)$,
- the curves in $\mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*, \boldsymbol{n}^o)}$ are ellipses in $[-1, 1]^2$,
- $\#\mathfrak{L}_{\boldsymbol{k}}^{(\boldsymbol{n}^*, \boldsymbol{n}^o)} = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$
- $N_{deg} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

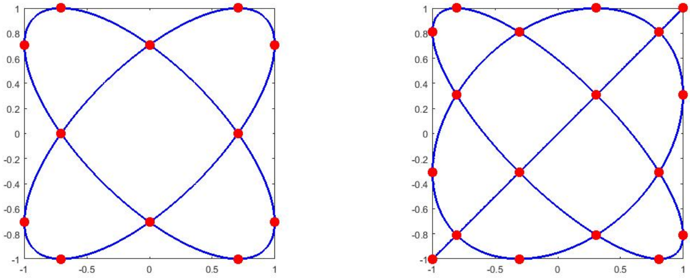


Figure: Xu points and curves for $n = 4$ (left) and $n = 5$ (right).

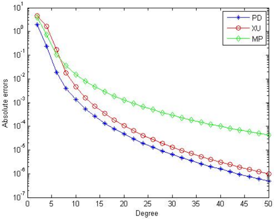
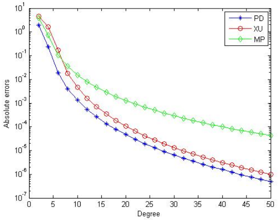


Figure: Interpolation errors for the function f_2 .



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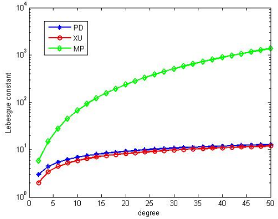

Figure: The behaviour of the Lebesgue constant for Padua, Xu and Morrow-Patterson points.

We compare the distribution of Padua, Xu and degenerate Lissajous points.

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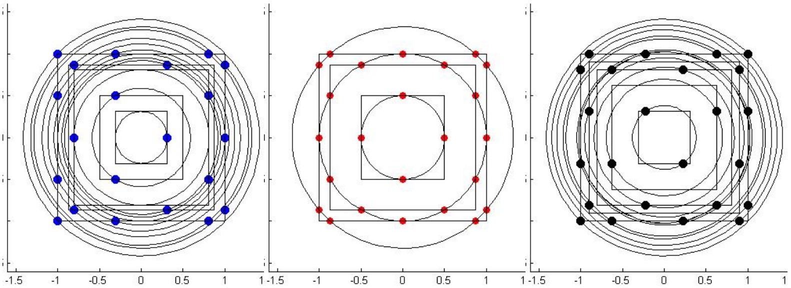

Figure: PD_5 (left), XU_6 (middle) and $LD_{5,2}$ (right) .

Thank you