RBF-based partition of unity method for elliptic PDEs: Adaptivity and stability issues via VSKs

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19 Dicembre 2017

global vs local

Global:

1. Spectral Methods (orthogonal polynomials-based methods)
2. Meshless Methods (RBF-based methods)

Local:

1. Finite Element Method
2. Finite Volume Method
3. Finite Differences Method

Global approach

Global

$$\mathcal{I}(x) = R(x) = \sum_{k=1}^{N} c_k \Phi(x, x_k)$$

$$A\mathbf{c} = \mathbf{f} \quad \textit{with} \quad A_{i,k} = \Phi(x_i, x_k)$$

Partition of Unity

Partition of Unity

$$\bigcup_{j=1}^{d} \Omega_j = \Omega$$

$$W_j \quad s.t. \quad \sum_{j=1}^{d} W_j(x) = 1 \quad \forall x \in \Omega$$

$$\mathcal{I}(x) = \sum_{j=1}^{d} W_j(x) R_j(x) \quad with \quad R_j(x) = \sum_{k=1}^{N_j} c_k^j \Phi(x, x_k^j)$$

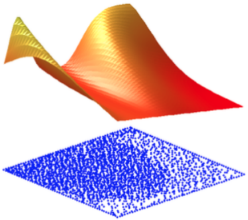
$$A_j \mathbf{c}_j = \mathbf{f}_j$$

Collocation method

Collocation

$$\mathcal{L}(\mathcal{I}(x_i)) = g_1(x_i) \quad x_i \in \Omega$$

$$\mathcal{I}(x_i) = g_2(x_i) \quad x_i \in \partial\Omega$$



Poisson's equation

In our case $\mathcal{L} = -\Delta$.
Local Matrix:

$$L_j = (\bar{W}_j^{\Delta} A_j + 2\bar{W}_j^{\nabla} A_j^{\nabla} + \bar{W}_j A_j^{\Delta}) A_j^{-1}$$

Where:

$$(A_j^{\Delta})_{i,k} = \Delta\Phi(x_i^j, x_k^j) \quad (A_j^{\nabla})_{i,k} = \nabla\Phi(x_i^j, x_k^j)$$

$$(A_j)_{i,k} = \Phi(x_i^j, x_k^j)$$

$$(\bar{W}_j^{\Delta})_{k,k} = \Delta W_j(x_k^j) \quad (\bar{W}_j^{\nabla})_{k,k} = \nabla W_j(x_k^j)$$

$$(\bar{W}_j)_{k,k} = W_j(x_k^j)$$

Poisson's equation

Global Matrix

$$L_{i,k} = \sum_{j=1}^{d} (L_j)_{\zeta_{i,j}\zeta_{k,j}}$$

Finally you solve the global linear system:

$$L\mathbf{I} = \mathbf{f}$$

with

$$\mathbf{I} = (\mathcal{I}(x_1), ..., \mathcal{I}(x_N)) \quad \mathbf{f} = (f_1, ..., f_N)$$

with $f_i = g_1(\mathbf{x}_i)$ for $\mathbf{x}_i \in \dot{\Omega}$ and $f_i = g_2(\mathbf{x}_i)$ for $\mathbf{x}_i \in \partial\Omega$

Stability Issues

1. Tikhonov regularization
2. Variably Scaled Kernel
3. Hybrid Variably Scaled Kernel

Tikhonov regularization

Original LS system

$$\min_{\mathbf{l}}(||L\mathbf{l} - \mathbf{f}||_2^2)$$

Tikhonov regularization

$$\min_{\mathbf{l}}(||L\mathbf{l} - \mathbf{f}||_2^2 + ||\Gamma\mathbf{l}||_2^2)$$

Where usually $\Gamma = \sqrt(\gamma)\mathbb{I}$ and $\gamma = 10^{-10} \sim 10^{-15}$

Separation Distance & fill distance

Separation distance:

$$q_{\mathcal{X}_N} = \frac{1}{2} \min_{i \neq k} ||\mathbf{x}_i - \mathbf{x}_k||_2$$

Fill Distance:

$$h_{\mathcal{X}_N} = \sup_{\mathbf{x} \in \Omega} (\min_{\mathbf{x} \in \mathcal{X}_N} ||\mathbf{x} - \mathbf{x}_k||_2)$$

Convergence Estimate for interpolation

Suppose $\Omega \subseteq \mathbb{R}^M$ is bounded and satisfies an interior cone condition. Suppose that $\Phi \in C^{2k}(\Omega \times \Omega)$ is symmetric and strictly positive definite and let $f \in \mathcal{N}_\Phi(\Omega)$, where $\mathcal{N}_\Phi$ is the native space of Φ. Then, there exist positive constants h_0 and C, independent of $\mathbf{x}$, f, and Φ, such that:

$$|f(\mathbf{x}) - R(\mathbf{x})| \leq C h_{\mathcal{X}_N}^k \sqrt{C_\Phi(\mathbf{x})} ||f||_{\mathcal{N}_\Phi(\Omega)}$$

Provided $h_N \leq h_0$ and $f \in \mathcal{N}_\Phi(\Omega)$, where

$$C_\Phi(\mathbf{x}) = \max_{|\beta|=2k} \big(\max_{\mathbf{x},\mathbf{z} \in \Omega \cap B(w, C_2 h_N} (|D_2^\beta \Phi(\mathbf{w}, \mathbf{z})|)\big)$$

Variably Scaled Kernel

$$\Psi_j : \mathbf{x} \to (\mathbf{x}, \psi_j(\mathbf{x}))$$

Increase of distances:

$$||\Psi_j(\mathbf{x}) - \Psi_j(\mathbf{y})||_2^2 \geq ||\mathbf{x} - \mathbf{y}||_2^2$$

VSK:

$$\mathcal{K}((\mathbf{x}, \psi(\mathbf{x})), (\mathbf{y}, \psi(\mathbf{y}))) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^M$$

Local interpolant:

$$R_{\psi_j}(\mathbf{x}) = \sum_{k=1}^{N_j} c_k^j \mathcal{K}((\mathbf{x}, \psi(\mathbf{x})), (\mathbf{x}_k^j, \psi(\mathbf{x}_k^j))) \quad \mathbf{x} \in \Omega_j$$

Variably Scaled Kernel

Discrete local operator:

$$\bar{L}_{\psi_j} = (\bar{W}_j^{\Delta} A_{\psi_j} + 2\bar{W}_j^{\nabla} A_{\psi_j}^{\nabla} + \bar{W}_j A_{\psi_j}^{\Delta}) A_{\psi_j}^{-1}$$

Where simply $A_{\psi_j}^{\Delta}$, $A_{\psi_j}^{\nabla}$, A_{ψ_j} are little modifications of the previous definitions arising from the fact that:

$$\Phi = \Phi((||\mathbf{x} - \mathbf{x}_k^j||^2 + (\psi_j(\mathbf{x}) - \psi_j(\mathbf{x}_k^j))^2)^{\frac{1}{2}})$$

HVSK

Algorithm 1 HVSK

```
for j = 1 to d do
    Computing A_j
    Checking σ_m
    if σ_m < (1e − 16)/ε^4 then
        VSK
    else
        STANDARD
    end if
end for
```

Residual Adaptive Subsampling Scheme

We first start from a set

$$\mathcal{X}_N = \{\mathbf{x}_i^{(1)}, \quad i = 1, ..., N^{(1)}\}$$

where we compute the solution and a test set of interior points:

$$\mathcal{Y}_{\tilde{N}^{(1)}} = \{\mathbf{y}_i^{(1)}, \quad i = 1, ..., \tilde{N}^{(1)}\}$$

then we compute the residual on the test set:

$$r_i^{(1)} = f(\mathbf{y}_i^{(1)}) - \mathcal{I}_{N^{(1)}}(\mathbf{y}_i^{(1)})$$

Residual Adaptive Subsampling Scheme

then we define the following two sets of points:

$$S_{T_1^{(1)}} = \{\mathbf{y}_i^{(1)} : r_i^{(1)} > \tau_1, \quad i = 1, ..., T_1^{(1)}\}$$

$$S_{T_2^{(1)}} = \{\bar{\mathbf{x}}_i^{(1)} : r_i^{(1)} < \tau_2, \quad i = 1, ..., T_2^{(1)}\}$$

Finally at the next step we will consider a new set $\mathcal{X}$:

$$\mathcal{X}_N^{(k+1)} = \mathcal{X}_{N_b}^{(k+1)} \bigcup \mathcal{X}_{N_c}^{(k+1)}$$

where

$$\mathcal{X}_{N_c}^{(k+1)} = (\mathcal{X}_{N_c}^{(k)} \bigcup S_{T_1^{(k)}}) \backslash S_{T_2^{(k)}}$$

Results

RMSE

$$RMSE := \sqrt{\frac{1}{s}\sum_{i=1}^{s} |f(\tilde{\mathbf{x}}_i) - \mathcal{I}(\tilde{\mathbf{x}}_i)|^2}$$

ψ_j Map

$$\psi_j = \sum_{i=1}^{N_j} |p_i^j(\mathbf{x}, \mathbf{x}_i^j)|$$

with

$$p_i^j(\mathbf{x}, \mathbf{x}_i^j) = \frac{1}{\pi} \arctan(h_i^j(x_1 - x_{i1}^j)e^{-5(x_2 - x_{i2}^j)})$$

Map ψ_j

N_c	data set	$h_{\mathcal{X}_N}$	$q_{\mathcal{X}_\mathcal{N}}$
81	original	1.03e-1	1.07e-2
	mapped	2.93e-1	3.68e-2
289	original	5.72e-2	2.07e-3
	mapped	6.34e-2	6.73e-3
1089	original	3.27e-2	1.12e-3
	mapped	5.79e-2	4.34e-3
4225	original	1.68e-2	1.74e-4
	mapped	4.85e-2	6.79e-4

Table: Separation and fill distances of the original data set compared with the ones mapped via VSKs (Halton points)

RMSE vs ε

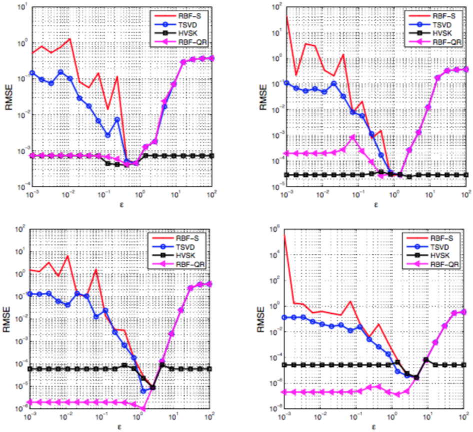


Figure: RMSEs obtained by varying ε for the Gaussian C^{∞} kernel. From left to right, top to bottom, we consider Nc $= 81$, 289, 1089 and 4225 Halton data.

RASS

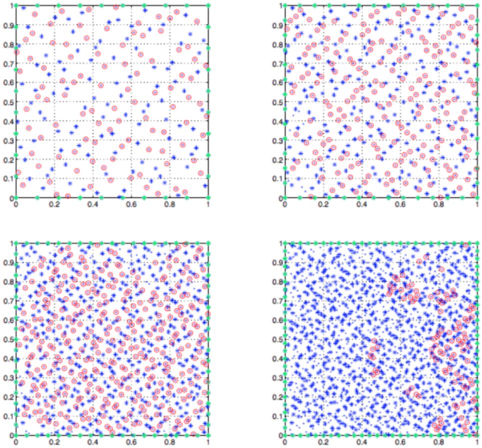


Figure: An illustrative example of RASS procedure

RASS

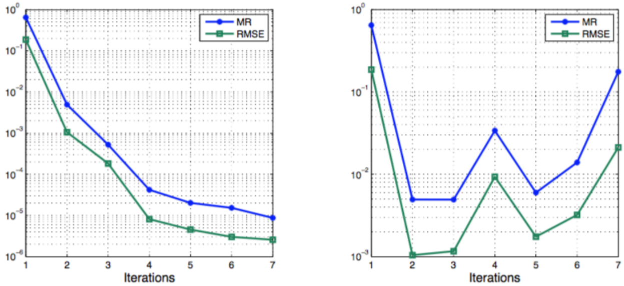


Figure: The iterations versus the MR and RMSE for Halton data. In the left frame we use the HVSK approach, while in the right one the standard bases

Summary

1. Global + Local
    1. Classical PU collocation method $\rightarrow$ stability issues
        1. Tikhonov regularization
        2. VSK $\rightarrow$ HVSK + RASS

Work in progress

1. Different PDEs (parabolic)
2. Parallel computing implementation