# On Leja sequences: some results and applications

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**Abstract**

In this paper we collect and review some results concerning Leja sequences on one and two dimensional domains. Then, we present two applications and discuss some open problems on the asymptotic distribution of Leja points on two dimensional domains.

**Keywords:** Leja sequences, interpolation, discretized integral operators, radial basis functions.

## 1 Introduction

Working on approximation of data and functions, often require the availability of sets of points on which the interpolation error is as small as possible. The well-known *Chebyshev points* are maybe the first and most used in many applications especially in the one dimensional setting for their important property to minimize the error functional. In more than one dimension many of the results and properties of Chebyshev points are not easily extensible, except maybe domains for which tensor product rules can be applied, like hypercubes. In the last century there were an increasing interest on points with properties similar to Chebyshev points but whose definition is *independent* of the space dimension. Fekete was one of the first that proposed such a set of *extremal* points (cf. [1]). Fekete points are the ones that *maximize the Vandermonde determinant* associated to the interpolation problem, among all possible point sets. Obviously these points are not unique. Following the work of Fekete, Fejér (cf. [2] proved that the *Fekete points can be obtained as a solution of a global optimization problem* involving the fundamental Lagrange polynomials (see Section 3, below). Only some years later Leja studied a sequence of points that is easily to be computed and that can be extracted also from a discretization of a (compact) set of $\mathbb{R}^2$ (cf. [3]). Even if the original work by Leja was done in $\mathbb{R}^2$, Leja sequences can be defined in any compact set of $\mathbb{R}^d$, $d \geq 1$.

The aim of this paper is to give an overview that allow to understand what are Leja sequences, their properties and some of their applications.

The paper is organized as follows. In section 2 we define Leja sequences and we describe how to compute then in some special cases. In section 3, we compare Fekete and Leja points showing that there are some interesting connections among these two set of points. In section 4 we discuss the computational cost of Leja sequences showing that there exist a fast way to compute them. In the successive section 5, we preset Leja sequences on two dimensional domains showing different distributions depending on the choice of a *weight* function obtained the so called *weighted Leja points*. Moreover, we present a new approach to determine Leja points without resort to any weight function. Since the idea is due to Len Bos, we call this *Leja-Bos points*. We also conjectured that Leja-Bos sequences asymptotically distribute uniformly.

In section 6, we discuss two applications. One is connected to the compression of discretized integral operators on an interval of the real line. The second concerns with the use of different Leja sequences for interpolation on the unit square by polynomials and three different radial functions (gaussian with scale 1, multiquadrics and inverse multiquadrics also with coefficient 1). We consider interpolation by radial basis functions since they provide an up-to-date and well-established method for reconstructing multivariate functions from scattered data. We show that different weighted-Leja points give rise to different interpolants, as expected. As for the goodness of the interpolation, we also estimated the *Lebesgue constant* on those different Leja sequences, showing that the Leja-Bos points provide in general smaller Lebesgue constants. Finally we conclude with some *open problems* that are essentially connected with the asymptotic distribution of Leja points on two dimensional domains.

## 2 Leja sequences

Let $E \subset \mathbb{C}$ be compact. F. Leja (cf. [3]) defined a sequence of extremal points as follows. Choose arbitrarily $\lambda_1 \in E$. The next point $\lambda_2$ is such that $|\lambda_2 - \lambda_1| = \max_{z \in E} |z - \lambda_1|$. Then $\lambda_3$ such that

$$|(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)| = \max_{z \in E} |z - \lambda_1||z - \lambda_2|$$

and in general

$$|(\lambda_{n+1} - \lambda_n) \cdots (\lambda_{n+1} - \lambda_1)| = \max_{z \in E} \prod_{k=1}^{n} |z - \lambda_k| \ .$$

Since the function $\Phi(z) = \prod_{k=1}^{n}(z - \lambda_k)$ is analytic, then from the *maximum principle of analytic functions* all points $\{\lambda_p\}$ (except $\lambda_1$ at most) belong to the boundary of $E$.

The points $\lambda_p$ are in general *not unique*, as shown in the next examples.

EXAMPLE 1 $E \subset \mathbb{C} \approx \mathbb{R}^2$. $E = \{z : |z| \leq 1\}$. *We take* $\lambda_1 = 1$ *so that* $\lambda_2 = -1$. *Now for* $\lambda_3$ *we have* `two` *choices: either* $i$ *or* $-i$. *Then* $\lambda_4 = -\lambda_3$.

EXAMPLE 2 $E = [0, 1]$ *and let* $\lambda_1 = 1$ *so that* $\lambda_2 = 0$. *Then* $\lambda_3 = 1/2$ *while for* $\lambda_4$ *we have* `two` *choices: that is one of the points where the function* $|z - \lambda_1||z - \lambda_2||z - \lambda_3|$ *assumes its maxima. That is* $\lambda_4$ *will between* $[\lambda_2, \lambda_3]$ *or between* $[\lambda_3, \lambda_1]$. *Again two candidates.*

It is worth noticing, that Leja sequences was already discovered by A. Edrei (cf. [4, p. 78-79]) but only after the paper by Leja the sequence so defined were called *Leja sequences.*

## 2.1 Some properties of Leja sequences

These properties are conveniently described in terms of the *Green function* of the *complement* $E_c$ of the set $E$.

Again, we identify the complex plane with $\mathbb{R}^2$: then for $x, y \in \mathbb{R}$, $z \in \mathbb{C}$ such that $z = x + iy$ we may consider $z$ and $(x, y)$ as denoting the same point.

Let $E \subset \mathbb{C}$ compact. Its complement, $E_c = (\mathbb{C} \cup \{\infty\}) \backslash E$ is connected and possesses a Green function with logarithmic singularity at the point $\infty$. Moreover $\partial E = \partial E_c$.

The Green function, $G(x, y)$, is uniquely determined by these three properties:

(i) $\Delta G(x, y) = 0$, in $E_c \backslash \{\infty\}$;

(ii) $G(x, y) = 0$ on $\partial E_c$;

(iii) $\frac{1}{2\pi} \int_{\partial E_c} \frac{\partial}{\partial n} G(x, y) ds = 1$, where $\frac{\partial}{\partial n}$ is the normal derivative directed into $E_c$ and $ds$ is the element of the arc-length.

Moreover, there exists a positive constant $c$ such that

$$c = \lim_{|z| \to \infty} |z| e^{-G(x,y)} .$$

The constant $c$ is known as the *capacity* of the set $E$.

Leja showed that the above extremal sequence for $E$, is *uniformly distributed* with respect to the density $\frac{1}{2\pi} \frac{\partial}{\partial n} G(x, y)$ on the boundary of $E$. That is, each sub-arc $\gamma$ of the boundary $\partial E$ contains a fraction

$$\frac{1}{2\pi} \int_{\gamma} \frac{\partial}{\partial n} G(x, y) ds$$

of Leja points $\{\lambda_j\}_{j=1}^{n}$ for $E$ as $n \to \infty$. We notice that this property is also verified by Fejer and Fekete points.

EXAMPLE 3 Let $E = \{z : |z| \le r\}$, $r > 0$. For this set the $G(x,y) = \log\left|\frac{z}{r}\right|$, $z = x + iy$. $E$ has capacity $c = r$. $\frac{\partial}{\partial n}G(x,y) = \frac{1}{r}$ on $\partial E$. Then, $\frac{1}{2\pi}\int_\gamma \frac{\partial}{\partial n}G(x,y)ds = \frac{1}{2\pi r}|\gamma|$, where $|\gamma|$ is the length of the arc $\gamma$. Thus, Leja points are uniformly distributed.

EXAMPLE 4 $E = [-a,a] \subset \mathbb{C}$. Then, $G(x,y) = \log|(z/a) + \sqrt{(z/a)^2-1}|$ (and $|(z/a)+\sqrt{(z/a)^2-1}| \ge 1$ for $z \in E_c$ ). Then, $c = a/2$. Therefore if $E = [a,b]$ then $c = (b-a)/4$.

In the next example, we present a method to generate Leja points on the circle without the help of any underlying discretisation.

EXAMPLE 5 Let $E$ be the unit circle. Consider the binary representation of nonnegative integers

$$j = \sum_{k=0}^{\infty} j_k 2^k, \; j_k \in \{0,1\}.$$

Then we define the points

$$z_j = \exp\left( i\pi \sum_{k=0}^{\infty} j_k 2^{-k} \right) \;, \quad j = 1, 2, \ldots \tag{1}$$

It can be proved that the points $\{z_j\}$ defined in (1) are Leja points all lying on the circumference (cf. [5]). For instance, some of these points are: $z_0 = 1$, $z_1 = -1$, $z_2 = \exp(i\frac{\pi}{2})$, $z_3 = \exp(i\frac{3\pi}{2})$, $z_4 = \exp(i\frac{\pi}{4})$ and so on. Moreover, the points in every set of $2^k$ Leja points $\{z_j\}_{j=0}^{2^k-1}$, $k = 1, 2, \ldots$ are *equidistributed* on the circumference as shown in Fig. 1 for $k = 4$. In the Fig. 2 we display the points that fall in the upper semi-circumference. Their projections on the interval $[-1, 1]$ seem to distribute like a Leja sequence. If this can be proved to be true even asymptotycally, the formula (1) can be considered as an explicit formula for Leja points on the interval.

Now we can extend the original definition of Leja sequences in the following way.

DEFINITION 1 *Let $E$ be a compact subset of $\mathbb{C}$ and $w : E \to \mathbb{R}_+$ a real positive function on $E$ that we call* `weight function`. *Let $z_0 \in E$ chosen so that*

$$w(z_0)|z_0| = \max_{z \in E} w(z)|z| \;, \tag{2}$$

*and for $j = 1, 2, \ldots$ $z_j$ are such that*

$$w(z_j)\prod_{k=0}^{j-1}|z_j - z_k| = \max_{z\in E} w(z) \prod_{k=0}^{n-1}|z - z_k| \;, \; z_j \in E\,. \tag{3}$$

*The sequence $\{z_j\}$ not-unique that satisfies (2) and (3) is a sequence of Leja points for $E$.*

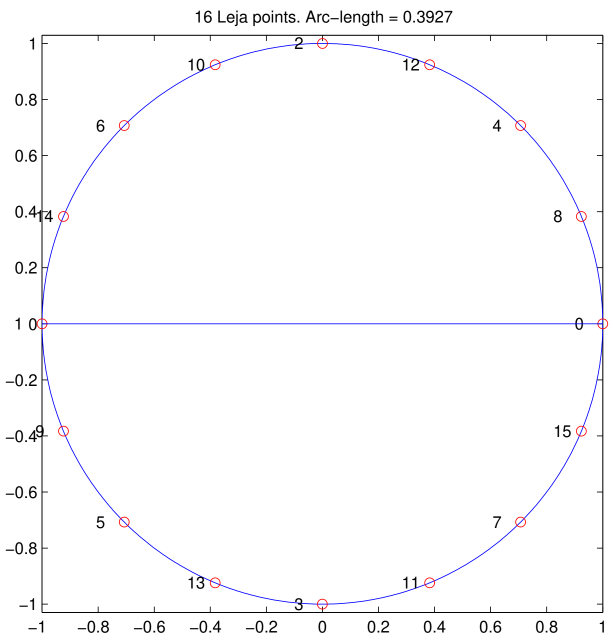


Figure 1: Leja points on the circle computed using formula (1)

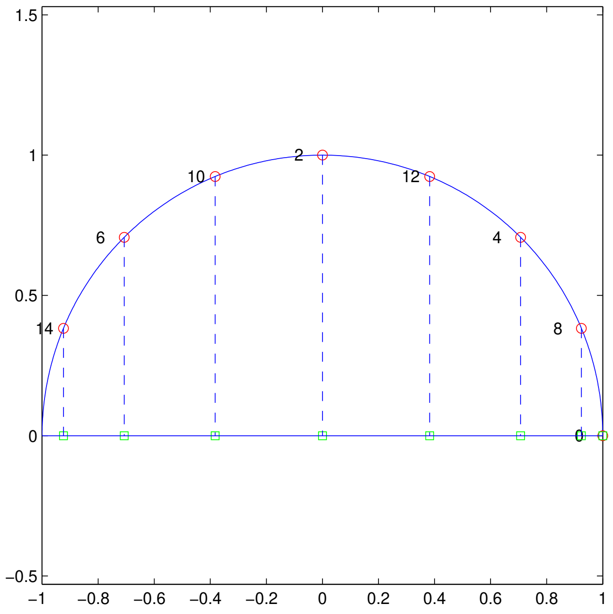

Figure 2: Projections of the Leja points on the upper circumference computed using the formula (1). The leftmost end point is not displayed: it comes from the lower semi circumference.

The classical Leja points (see the previous section) are obtained when $w(z) = 1$. Thus, Definition 1 will be used as the definition of Leja points. When $w(z) \neq 1$, some authors refer to these points as *weighted Leja points.*

**Theorem 1** (cf. [3, 5]) *Let $\{z_j\}_{j=}^{\infty}$ be a sequence of Leja points for $E \subset \mathbb{C}$. Assume that $\beta_1 \leq w(z) \leq \beta_2, \ \ \forall z \in E$ for some constants $0 < \beta_1 < \beta_2 < \infty$. Let $c$ the capacity of $E$. Then*

$$\prod_{j=0}^{n-1} |z_n - z_j|^{1/n} \geq c \left(\frac{\beta_1}{\beta_2}\right)^{1/n}$$

*and*

$$\lim_{n\to\infty} \prod_{j=0}^{n-1} |z_n - z_j|^{1/n} = c \; . \tag{4}$$

The proof of Theorem 1 for $w(z) = 1$ can be found in [3] and in [5] for a general $w(z)$.

The limit (4) is easily to verify numerically and can be used for an accurate estimation of the capacity $c$ of a set $E$ and to evaluate the polynomials $p_n(z) = \prod_{j=0}^{n}(z - z_j)$ for large degree $n$ and $z \in E$ as shown in the next example.

EXAMPLE 6 (cf. also [5]) Let $E = [0, 1](= [a, b])$ and $w(z) = 1$ so that $c = (b-a)/4 = 1/4$. Let $\{z_j\}$ a sequence of Leja points, we define the mapping

$$H(n) = \prod_{j=0}^{n-1} |z_n - z_j|^{1/n}, \ \ n = 1, 2, ...$$

By Theorem 1, $H(n) \geq \frac{1}{4}$ and asymptotically decreases to $\frac{1}{4}$, i.e. $\lim_{n\to\infty} H(n) = \frac{1}{4}$. In Fig. 3 we show the graph of $H(n)$ for $n \leq 200$, by computing the Leja points on a discretization of the interval $E$ (by using 1001 points) and by computing the so-called *Fast Leja Points* (see next Section). As clear from the graphs, both kind of Leja points behave in the same way. The main difference, as it will be clarify later, is the computational cost that is much smaller for *Fast Leja Points.*

The following Theorem can be seen as a consequence of Theorem 1.

**Theorem 2** *Let $\{z_j\}_{j=}^{\infty}$ be a sequence of Leja points for $E \subset \mathbb{C}$. Let $z \in E_c$ be given. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |z_n - z_j| = G(x, y) + \log c \; , \tag{5}$$

*where $z = x + iy$, $c$ is the capacity of $E$. The limit is uniform for $z$ belonging to some compact subset of $E_c$.*

**Remark**. The limit (5) can be used to evaluate the convergence of the polynomial interpolation of analytic functions at Leja points.

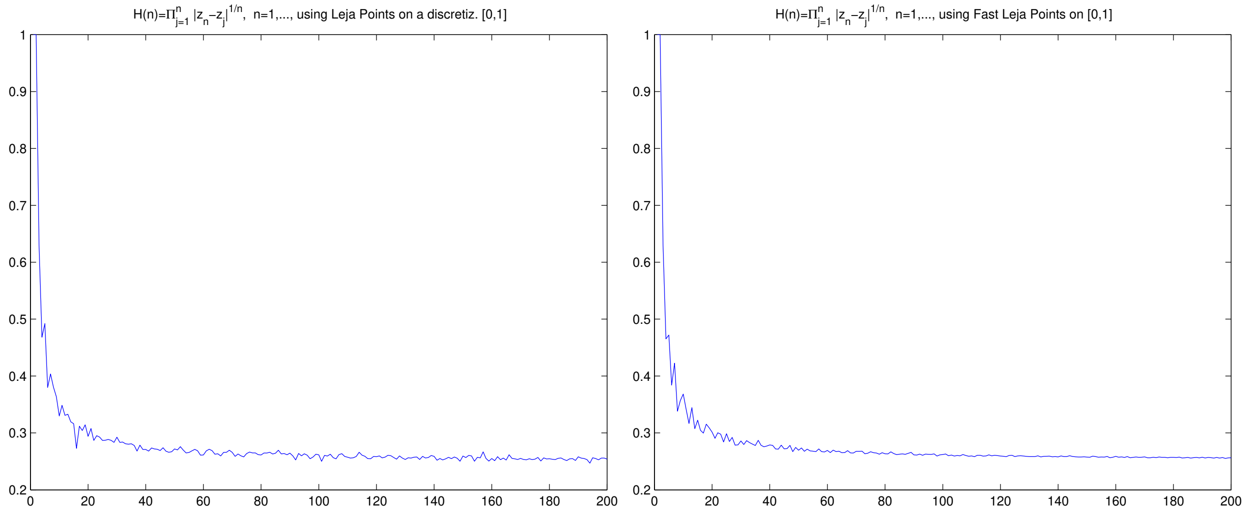


Figure 3: The function $H(n)$ for Leja points on a discretization (left) and using the Fast Leja Points (right).

## 3 Fekete and Leja points

M. Fekete proposed a set of extremal points as a solution of the following prolem (cf. [1]).

Let us consider $n$ unitary masses $n \geq 2$ at the (variable) points $x_1, ..., x_n$ of the interval [-1,1].

Problem 1 *For which positions the expression*

$$\prod_{i \neq j; i,j=1}^{n} (x_j - x_i) \tag{6}$$

*takes its maximum?*

We recognize that this product is nothing else that the *Vandermonde determinant* for the set of points $X_n = \{x_1, ..., x_n\}$, that we denote by $VDM(X_n)$.

**Remark:** Problem 1 is a particular instance of the so-called *electrostatic interpretation of orthogonal polynomials*, for which the function to be maximized is (cf. [6, p.140]).

$$T(x) = \prod_{k=1}^{n} (1 - x_k)^p (1 + x_k)^q \prod_{i \neq j; i,j=1}^{n} |x_j - x_i|$$

with $x = (x_1, ..., x_n)$ and $p, q$ fixed masses in 1 and -1, respectively.

Taking $p = q = 1$ and using $n-2$ instead of $n$, the points that maximize are the one that solve the equation

$$\text{Cost } (1-x^2)P'_{n-1}(x) = 0 \ , \qquad (7)$$

where $P_{n-1}(x)$ is the Legendre polynomial of degree $n-1$ (cf. [6, p. 382], that resorted to a problem posed by Stieltjes in 1885).

Fejér proved that the Fekete points can be obtained as the solution of the following problem [2]:

PROBLEM 2 *Find the set $F$, not necessarily unique, that minimizes the function*

$$\Phi_n(X) = \max_{-1 \le x \le 1} \sum_{k=0}^{n} l_i^2(X;x) \ .$$

where $l_i$ is the $i$-th Lagrange polynomial.

Moreover, he verified that

$$\Phi_n(F) = 1, \qquad \Lambda_n(F) \le \sqrt{n+1} \ .$$

where $\Lambda_n(F)$ is the *Lebesgue constant.*

The Fekete points satisfy some important properties:

1. In [-1,1] they are zeros of $(x^2-1)P'_n(x)$ with $P_n(x)$ the *Legendre* polynomial of degree $n$.
2. They can be defined on every compact $E \subset \mathbb{R}^d$.
3. They *maximize* the Vandermonde determinant.
4. $\Lambda_n = \mathcal{O}(\log(n)) \Rightarrow \lim_{n\to\infty} \Lambda_n^{\frac{1}{n}} = 1$
5. Let $E$ be a compact set, then

   $$\lim_{n\to\infty} |VDM(f_1, ..., f_n)|^{\frac{2}{n(n-1)}} = \rho(E)$$

   where $\rho(E) = \lim_{n\to\infty}(M_n(E))^{\frac{1}{n}}$ and $M_n(E) = \max_E |T_n(x;\, E)|$, where $T_n$ is the Chebyshev polynomial of degree $n$, is known as *Chebyshev constant* or *transfinite diameter* of E.

The relation that makes the first connection between Fekete and Leja points is the maximization of the Vandermonde determinant $VDM(X_n)$ on a *fixed* set $X_n = \{x_1, ..., x_n\} \in E$. The set $F_n = \{f_1, ..., f_n\}$ of Fekete points is the one that *globally* solve the (multi-dimensional optimization problem) $\max_{X_n \in E} |VDM(X_n)|$, where we use the modulus avoiding to order the points.

Since $|VDM(X_n)| = \prod_{i=1}^{n-1} |x_n - x_i| \cdot |VDM(X_{n-1})|$, to determine the $n$-th point of the set $L_n = \{x_1, ..., x_n\}$ of Leja points, once we have computed the $x_1, ..., x_{n-1}$, we simply *locally* solve (a one-dimensional problem) $\max_{x \in E} \prod_{i=1}^{n-1} |x - x_i|$.

Both set of points $F_n$ or $L_n$, allow to minimize the Lebesgue constant associated, since from their definition we bound the elementary Lagrange polynomials,

$$l_i(x) = \frac{VDM(X_n^{(i)})}{VDM(X_n)} \ ,$$

where $X_n^{(i)}$ is the set $X_n$ in which $x$ is used instead of $x_i$.

# 4 Computational cost of Leja points

Let $E = [a, b]$ and $w(z)$ as in Definition 1, and $E'_m = \{$set of $m$ distinct points in $E\}$, that is a discretization of $E$. We introduce the quantity

$$\epsilon_m = \max_{z \in E} \min_{z' \in E'_m} |z - z'| \tag{8}$$

The number $m$ has to be taken so that it *increases rapidly enough with respect to* $n$, $n$ being the number of Leja points in $E$ to be computed. One possible choice is that

$$\epsilon_m = o\left(\frac{1}{n^2}\right) , \quad n \to \infty \ . \tag{9}$$

As an example, the property (8) is satisfied for

$$E'_m = \left\{ a + (b - a) \left( \frac{j - 1}{m - 1} \right) , \ j = 1, ..., m \right\} \tag{10}$$

and $n = o(\sqrt{m})$.

Saff and Totik [7], showed that if (9) holds, then the Leja points $\{z_j\}_{j=0}^{n-1}$ for $E'_m$ have the same limit distribution as the Leja points for $E$.

EXAMPLE 7 The set of the first $m$ zeros of Chebyshev polynomials of first kind on [a,b], that is

$$E_m = \left\{ \frac{a + b}{2} + \frac{a - b}{2} \cos\left( \frac{2j - 1}{2m} \pi \right) , \ j = 1, ..., m \right\} \tag{11}$$

is a suitable set that allows to compute quickly Leja points.

**Proposition 1** *Let $E = [a,b]$, $w(z) = 1$ and $E_m$ as in Example 7 expression (11). Let $n = n(m)$ be a non decreasing function of $m$ such that for all $m > 0$ we have that $1 \leq n(m) \leq \kappa m$, for some constant $\kappa \in (0,1)$. Then the Leja points $\{z_j\}$ for $E_m$ have the same limit distribution as the Leja points for $E = [a,b]$ as $n \to \infty$.*

**Proof.** See [5] p. 127. □.

The main conclusion from the previous discussion is that instead of $E = [a,b]$ we can use $E_m$ that certainly simplify the computations.

Using $E'_m$ or $E_m$ (as defined by (10) and (11)) instead of $E$, the set of points $\{w_k,\ k = 1,...,m\}$, we then compute the first $n$ Leja points. The *computational cost* requires roughly $2mn$ flops, since we may store the products

$$\prod_{j=0}^{k-1} |w_k - z_j|, \quad 1 \leq k \leq m$$

for all $j = 1, 2, ....$ Of course, when $m$ grows, new products have to be computed increasing the number of products that one has to store.

## 4.1 Fast Leja Points

An alternative set of Leja points, that can be computed in a faster way, were introduced by Baglama, Calvetti and Reichel in [5].

FLP are obtained by maximization over *adaptive* discretization of the *boundary* of $E$. The discretization is finer where more Leja points lye while is coarse where few Leja points are. Then the discretization is updated as new (fast) Leja points are computed.

ALGORITHM 1 We assume that $\partial E$ is a Jordan curve or a Jordan arc and $z(t),\ 0 \leq t \leq T$ a parametrization of $\partial E$ such that $z(t) \neq z(s),\ \ 0 \leq t < s < T$.

FLP can be defined *recursively* as follows. Let $\{z(t_j)\}_{j=0}^{k-1}$ be a set of FLP on $\partial E$ (notice that if $\partial E$ is a Jordan arc than we require that $t_0 = 0,\ \ t_1 = T$).

1. Let $\{z(s_j)\}_{j=0}^{l-1}$ be a set of *candidates* that interlace the set of FLP $\{z(t_j)\}_{j=0}^{k-1}$. Then between *each pair* of adjacent FLP there exists a candidate point so that $l = k-1$ when $\partial E$ is a Jordan arc or $l = k$ when $\partial E$ is a Jordan curve.
2. The next FLP, $z(t_k)$ is chosen among the candidates, that is

$$t_k \in \{s_0, ..., s_{l-1}\}$$

and

$$w(z(t_k))\prod_{l=0}^{k-1}|z(t_k)-z(t_l)| = \max_{0\le j\le l} w(z(s_j))\prod_{l=0}^{k-1}|z(s_j)-z(t_l)| \ .$$

3. Once we have computed $z(t_k)$, we *remove* it from the candidates and a new (in general 2) will be introduced between $z(t_k)$ and the closest FLP, so that the new set of FLP $\{z(t_j)\}_{j=0}^{k}$ and the new set of candidates will interlace.

4. Compute $z(t_{k+1})$ in the same way (steps 1-3, above).

Thus, in the computation of fast Leja points, the optimization over the interval is replaced by an optimization over a set of finitely many grid points. The number of grid points is proportional to the number of fast Leja points already allocated. Therefore the *complexity* of Algorithm 1 is roughly $\frac{1}{2}n^2$ flops to compute $n$ Fast Leja Points for $E$ compared with $2nm$ flops necessary when $E_m$ is used: this says that the Algorithm is much faster also because $m \approx n^2$.

A MATLAB code that implements Algorithm 1 for the case $E = [a, b]$ and $w(z) = 1$ is provided in the Appendix of [5]. Moreover an *Iterative Java code for FLP* for this case is available at ETNA's Web Site `http://etna.mcs.kent.edu`. In the next Example we show how this code works.

EXAMPLE 8 Let $E = [a, b]$ with parametrization $z(t) = t,\ a \le t \le b$ and $w(z) = 1$. We allocate candidate points $s_i$ in a equidistant way between adjacent FLP. Thus, we define $t_0 = b,\ t_1 = a,\ s_0 = (a + b)/2$ which corresponds to the FLP $z(t_0) = b,\ z(t_1) = a$ and $z(s_0) = (a + b)/2$.

The next FLP is *obviously* such that $z(t_2) = z(s_0)$.

Now we re-define $s_0 = (3a + b)/2,\ s_1 = (a + 3b)/2$ so that $t_3$ can be either $s_0$ or $s_1$. Then $z(t_3)$ is defined. And so forth.

## 5 Leja points on 2-dimensional domains

We start this Section by the definition that we shall use for Leja points on 2-dimensional domains.

DEFINITION 2 *Let $E$ be a compact subset of $\mathbb{C} \approx \mathbb{R}^2$ and $w : E \to \mathbb{R}_+$ a real positive function on $E$ that we call* `weight function`. *Let $z_0 \in E$ such that*

$$w(z_0)\|z_0\| = \max_{z\in E} w(z)\|z\| \ , \tag{12}$$

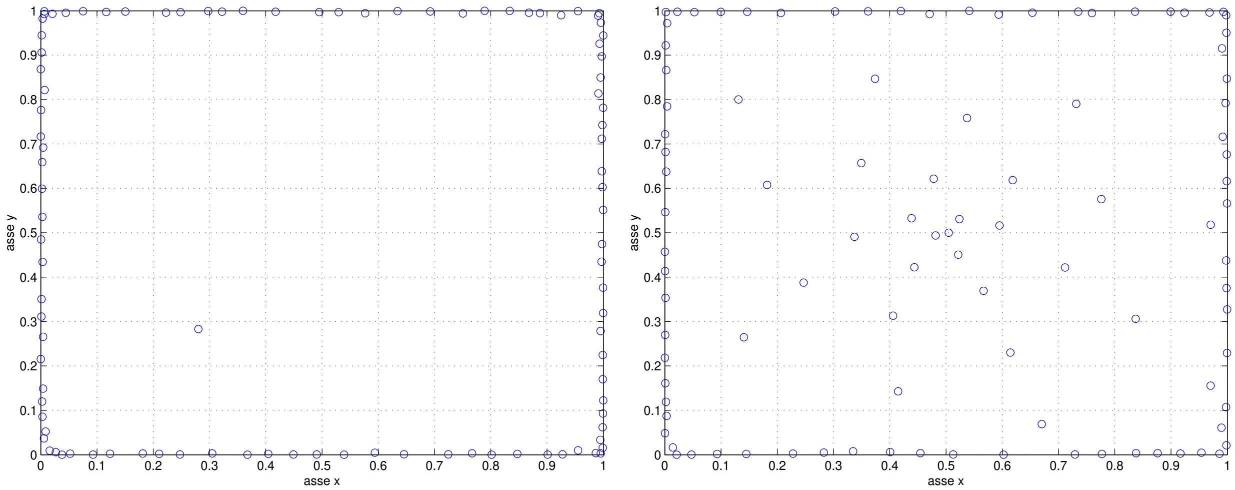


Figure 4: Leja points on the square [0,1]× [0,1] for Example 9 with weight function $w_{1,1}(x,y)$ (left) and $w_{1,50}$ (right)

*and*

$$w(z_j)\prod_{k=0}^{j-1}\|z_j-z_k\| = \max_{z\in E} w(z)\prod_{k=0}^{n-1}\|z-z_k\|\ ,\ z_j\in E\,. \tag{13}$$

*where* $\|\cdot\|$ *is any norm of* $\mathbb{R}^2$ *and* $z=x+iy\,,\ z=(x,y)$ *and* $z_j=(x_j,y_j),\ j=1,2,\ldots$ *The sequence* $\{z_j\}$ *not-unique that satisfies (12) and (13) is a sequence of Leja points for* $E$*.*

We observe that the distribution of the Leja points depend strongly on the choice of the weight function $w$. We provide some examples produced using the software available in [8] that show this dependence.

Example 9 Let $E=[0,1]\times[0,1]$ and $w_{1,k}(x,y)=\exp(-k\sqrt{x^2+y^2}),\ k\in\mathbb{R}_+$. We discretized $E$ with a set of 11000 random points and we computed 100 Leja points. In Figures 4 and 5 we present the distribution of these Leja points obtained using Definition 2 for k=1,50,150. In Figures 6 and 7 we present the distribution of 100 Leja points on $E$ with $w_{2,k}(x,y)=\exp(-k\,||x|-|y||)$

Example 10 Let $E=\{z\ :\ |z|\le r\}$ and we take again $w_{1,k}(x,y)=\exp(-k\sqrt{x^2+y^2})$, $k\in\mathbb{R}_+$. We discretized $E$ by a set of random points and we computed 200 Leja points. In Figure 8 we present the distribution of these points for k=1,50.

**Remark.** The Leja points distribute along the level curves of the weight function for $k\gg 1$ while mostly along the border for $k\le 1$.

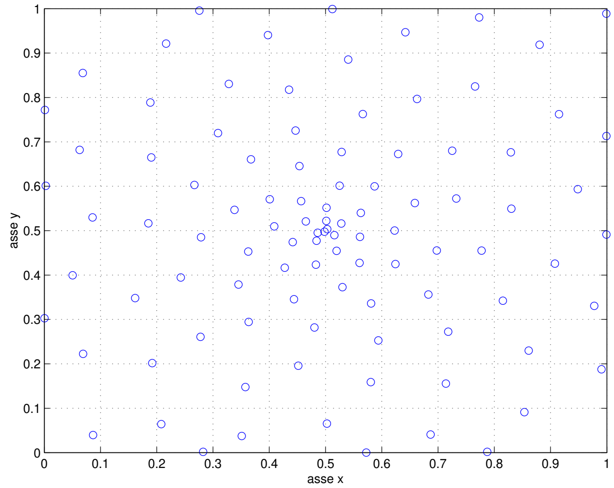


Figure 5: Leja points on the square [0,1]$\times$ [0,1] for Example 9 with weight function $w_{1,150}(x,y)$

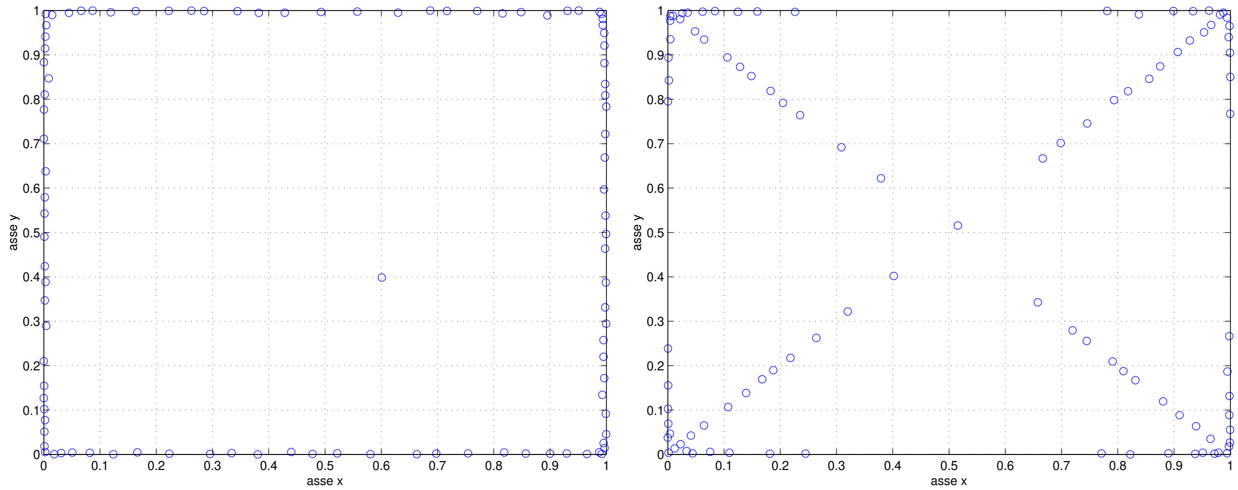


Figure 6: Leja points on the square [0,1]$\times$ [0,1] for Example 9 with weight function $w_{2,1}(x,y)$ (left) and $w_{2,50}(x,y)$ (right)

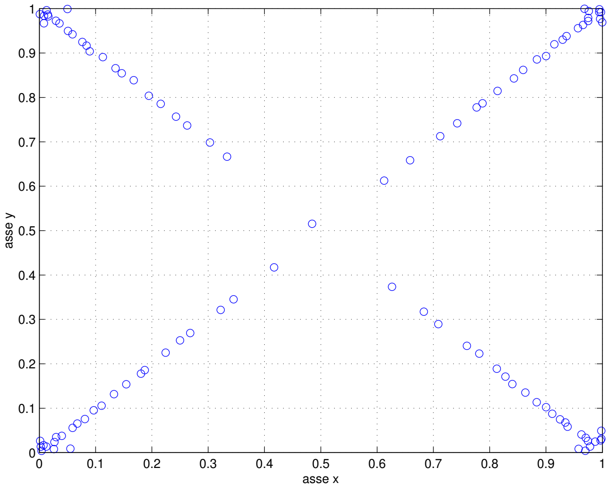


Figure 7: Leja points on the square [0,1]× [0,1] for Example 9 with weight function $w_{2,150}(x, y)$

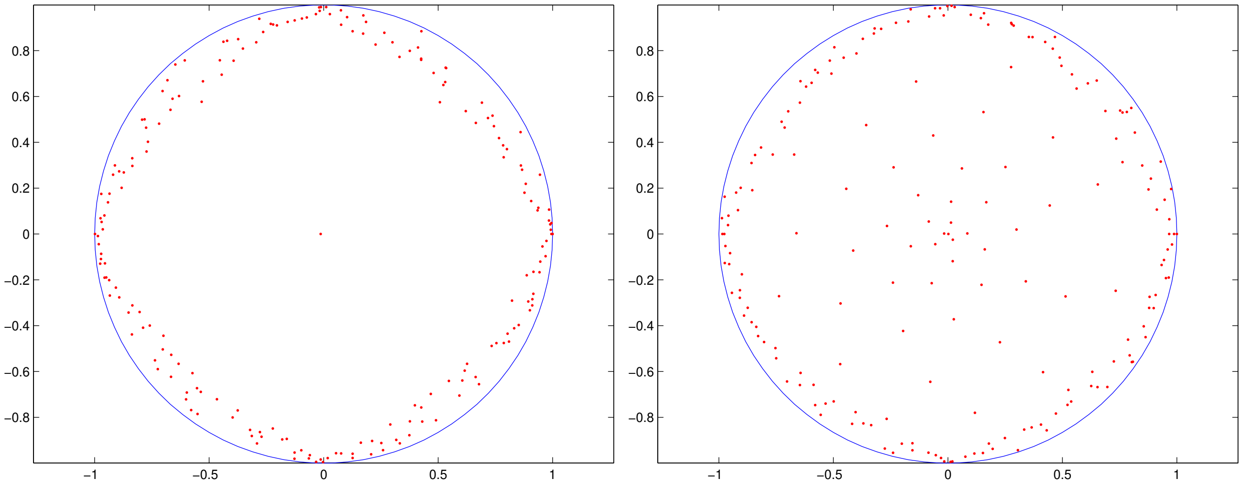

Figure 8: Leja points on the unit circle for Example 10 with weight function $w_{1,1}(x, y)$ (left) and $w_{1,50}(x, y)$ (right)

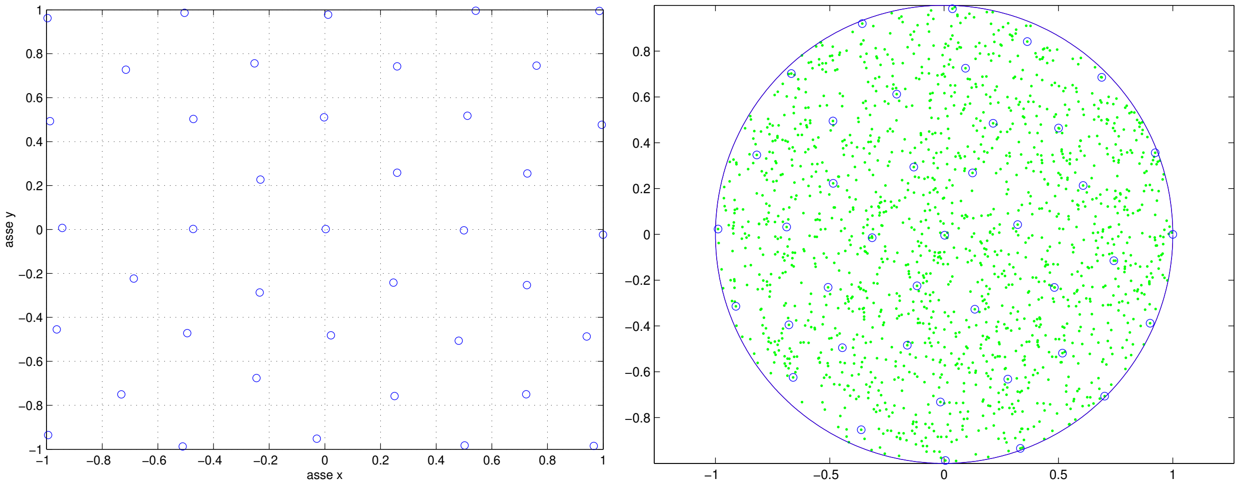


Figure 9: 40 Leja points on the square (left) and on the circle (right), both discretized by 2000 random points, computed by using Definition 3)

### 5.1 Leja-Bos points on 2-dimensional domains

A different definition of Leja points on 2-dimensional domains can be done by the following alternative definition (L. Bos in a private communication).

DEFINITION 3 *Let $E$ be a compact domain in $\mathbb{C} \equiv \mathbb{R}^2$ and let $z_0$ be arbitrarily chosen on $E$. Let us define the points $z_j,\ j = 1, 2, \ldots.$ as*

$$\min_{0 \le k \le n-1} \|z_n - z_k\|_2 = \max_{z \in E \setminus \{z_0, \ldots, z_{n-1}\}} \min_{0 \le k \le n-1} \|z - z_k\|_2 \,. \tag{14}$$

*The set $\{z_j\}$ is a set of Leja points for $E$.*

The function $\psi(z) = \min_{0 \le k \le n-1}(z - z_k)$ is not analytic, then the maximum principle can not be applied therefore the points $\{z_j,\ j = 0, 1, \ldots\}$ belong also to the interior of $E$.

**Conjecture 1** *If $z_0$ is chosen as $z_0 = \max_{z \in E} \|z\|_2$, then the Leja points defined by Definition 3 are asymptotically equidistributed.*

This conjecture is again due to L. Bos that sent me a private communication. So far, we were not able to prove this Conjecture. In the Figures 9 and 10 we show the distribution of Leja points computed by the Definition 3. These pictures provide numerical evidence of the Conjecture.

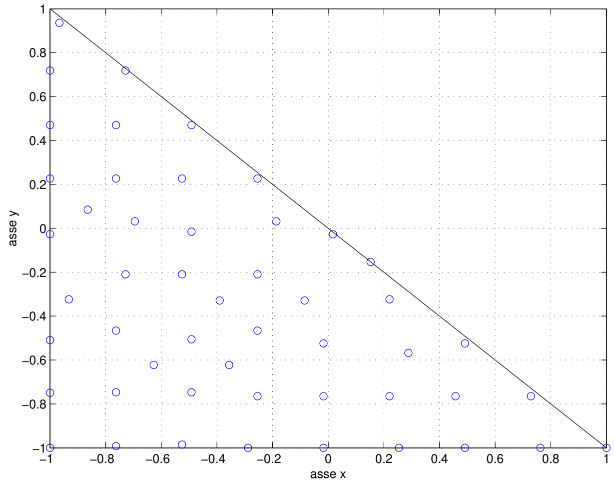


Figure 10: 50 Leja points on a right triangle, discretized by 2947 random points, computed by using Definition 3)

# 6 Applications

A typical application where Leja sequences have received a particular attention is in *interpolation* in 1 or 2 dimensions, by using the Newton form of the interpolant. In the 1-dimensional case, as proved in [9, 10], Leja points bounds the the growth of the condition number of the interpolant in Newton form. Actually this condition number grows slower than exponentially with the number of Leja points (cf. [9, Theor. 2.4, p. 238]). For the interested readers we suggest to refer to the beautiful paper [9] and the degree thesis [8] that provides more information and details.

## 6.1 A fast method for the evaluation of discrete integral transforms

Recently we used two fast and efficient methods for the evaluation of a discrete integral transform $(T_n\mathbf{u})_i = \sum_{j=1}^{n} w_j K(x_i, t_j, u_j), \quad i = 1, ..., q$ , where $n$ and $q$ are large, based on polynomial approximation of the action of the transform. The first resorted to *truncated Chebyshev series expansion* while the second to *interpolation at Leja sequences*. Both Chebyshev and Leja polynomial approximation reduce the complexity from $\mathcal{O}(nq)$ to $\mathcal{O}(n + q)$ in several applications, but the Leja approach turns out to be more efficient and shows speed-up ratios, with respect to direct evaluation, up to two orders of magnitude.

The *main idea* behind our approach is to *approximate* directly the *action* of the transform: computing the discrete transform $T_n\mathbf{u}$ can be seen in fact as evaluating the scalar function $\Phi_n$

$$\Phi_n(x) := \sum_{j=1}^{n} w_j\, K(x,t_j,u_j) \approx \Phi(x) := \int_a^b K(x,t,u(t))\,dt\ ,\quad x\in[a,b]\ , \tag{15}$$

at the target points $\{x_i\}$, $1 \le i \le q$. By *approximating globally* $\Phi_n$ in $[a,b]$ by a polynomial, say $P_{m,n}(x)$, whose construction and evaluation at the target points *costs* $\mathcal{O}(m(n+q))$ flops, with $m \ll n, q$, we obtain a *general purpose* fast method which exploits the *smoothing* effect of integration and works also in *nonlinear* instances.

EXAMPLE 11 We present two illustrative examples, where $[a,b]=[0,1]$ and $\{t_j\}=\{x_i\}$ (that is $q=n$). The results concerning both truncated Chebyshev series and interpolation at Leja sequences are taken from [11, 12]. For more examples, we invite the interested readers to look at these two papers.

1. *Uryshon kernel* (smooth and nonlinear):

$$K(x,t,u)=\frac{1}{x+t+0.3u},\quad x,t\in[0,1],\quad u>0\,, \tag{16}$$

   where the vectors $\mathbf{t}$ and $\mathbf{w}$ correspond to the Simpson quadrature rule at $n$ (odd) equispaced nodes. We chose $u(t)=1/(1+t)$. Then, the exact continuous transform is:

$$\Phi(x)=\frac{1}{2}\log\left(\frac{20x+23}{10x+3}\right)+\frac{5(x-1)}{r(x)}\cdot\left(\tanh^{-1}\left(\frac{5(x+3)}{r(x)}\right)-\tanh^{-1}\left(\frac{5(x+1)}{r(x)}\right)\right)\ ,$$

   where $r(x)=\sqrt{25x^2-50x-5}$.

2. *Weakly singular log kernel* (nonsmooth and linear). Here, $K(x,t,u)=\log(|x-t|)u(t)$, $\mathbf{t}$ and $\mathbf{w}$ correspond to the trapezoidal quadrature rule at $n$ (even) equispaced nodes, $u(t)\equiv 1$. In this case the compression is always dominated by the *discretization error* of the underlying quadrature formula, so that a *stalling* occurs at the size of this error. As stalling control parameter we take $\theta=0.25$.

   The exact continuous transform is $\Phi(x)=(1-x)(\log(x-1)-1)+x(\log(x)-1)$: thus the underlying discretization error can be computed.

In the Tables below we use the following notations ($X=\{x_i\}$ being the target points):

- *discretization error* : $DE=\|\Phi(X)-\Phi_n(X)\|_\infty/\|\Phi(X)\|_\infty$;
- *actual error* : $AE=\|\Phi_n(X)-P_{m,n}(X)\|_\infty/\|\Phi_n(X)\|_\infty$;

| $n$ | 129 | | 513 | | 2049 | |
|---|---|---|---|---|---|---|
| $DE$ | $1.8\,10^{-9}$ | | $6.9\,10^{-12}$ | | $3.0\,10^{-14}$ | |
| | (C) | (L) | (C) | (L) | (C) | (L) |
| $m$ | 17 | 13 | 17 | 13 | 17 | 13 |
| $EE$ | $7.2\,10^{-7}$ | $5.7\,10^{-7}$ | $7.2\,10^{-7}$ | $5.7\,10^{-7}$ | $7.2\,10^{-7}$ | $5.7\,10^{-7}$ |
| $AE$ | $1.1\,10^{-6}$ | $5.1\,10^{-7}$ | $1.1\,10^{-6}$ | $5.1\,10^{-7}$ | $1.1\,10^{-6}$ | $5.1\,10^{-7}$ |
| $SU$ | 3.3 | 9.2 | 13.2 | 35.6 | 52.5 | 146.3 |

Table 1: Compression of the discrete nonlinear Uryshon-type transform, $tol = 10^{-6}$, with Chebyshev (C) and Leja (L).

- *estimated error* at degree $m$: $EE = E_m$, is

$$E_m = \frac{\|P_{m,n}(\Delta) - \Phi_n(\Delta)\|_\infty}{\|\Phi_n(\Delta)\|_\infty}\;, \tag{17}$$

  where $\Delta$ is a fixed (small) set of *control* points in $[a, b]$ as for Leja-like interpolation, while for the Chebyshev series approach we resorted to the remainder estimate in [11];

- *estimated speed-up* : $SU = n/m$, $m$ being the displayed exit degree, such that $E_m \leq tol$ or a stalling has been detected. When the error estimate (17) is used, then $SU = SU_\Delta = \frac{n^2}{nm+C_\Delta n+C_\Delta m} \approx \frac{n}{m+C_\Delta}$, $C_\Delta$ being the cardinality of the control set $\Delta$ (see Table 1 where $\Delta$ is the set of the first ten Chebyshev points in $[0, 1]$).

| $n$ | 128 | | 512 | | 2048 | |
|---|---|---|---|---|---|---|
| $DE$ | $3.2\,10^{-2}$ | | $9.5\,10^{-3}$ | | $2.8\,10^{-3}$ | |
| | (C) | (L) | (C) | (L) | (C) | (L) |
| $m$ | 11 | 9 | 26 | 9 | 61 | 9 |
| $EE$ | $8.2\,10^{-3}$ | $1.5\,10^{-2}$ | $2.9\,10^{-3}$ | $8.8\,10^{-3}$ | $7.1\,10^{-4}$ | $5.9\,10^{-3}$ |
| $AE$ | $4.0\,10^{-2}$ | $3.7\,10^{-2}$ | $1.0\,10^{-2}$ | $1.1\,10^{-2}$ | $3.1\,10^{-3}$ | $4.4\,10^{-3}$ |
| $SU$ | 5.8 | 6.7 | 7.9 | 26.9 | 12.3 | 107.8 |

Table 2: compression of the discrete "weakly-singular log" linear transform; here, the exit is forced by a stalling of the estimated error.

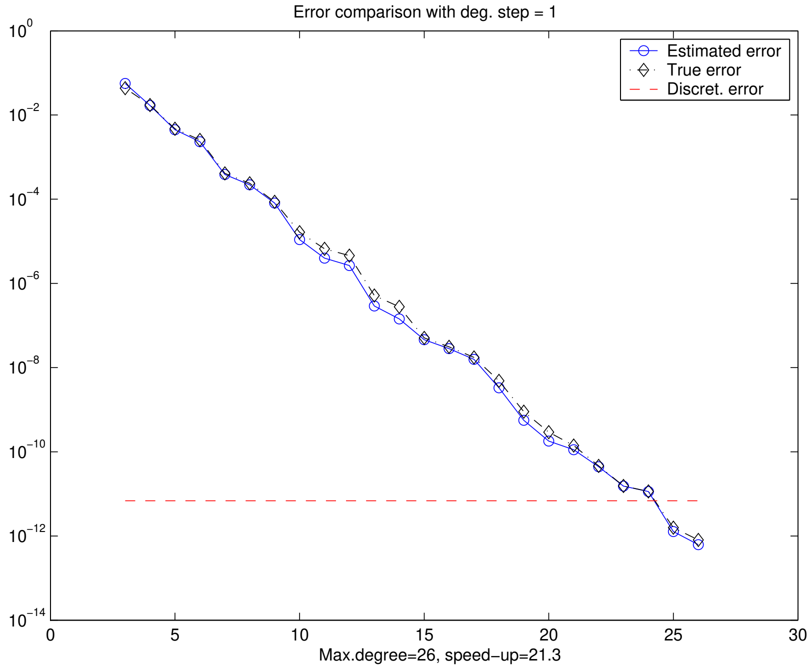


Figure 11: Behavior of the error in Leja-like compression of the discrete Uryshon-type nonlinear kernel, $n = 513$.

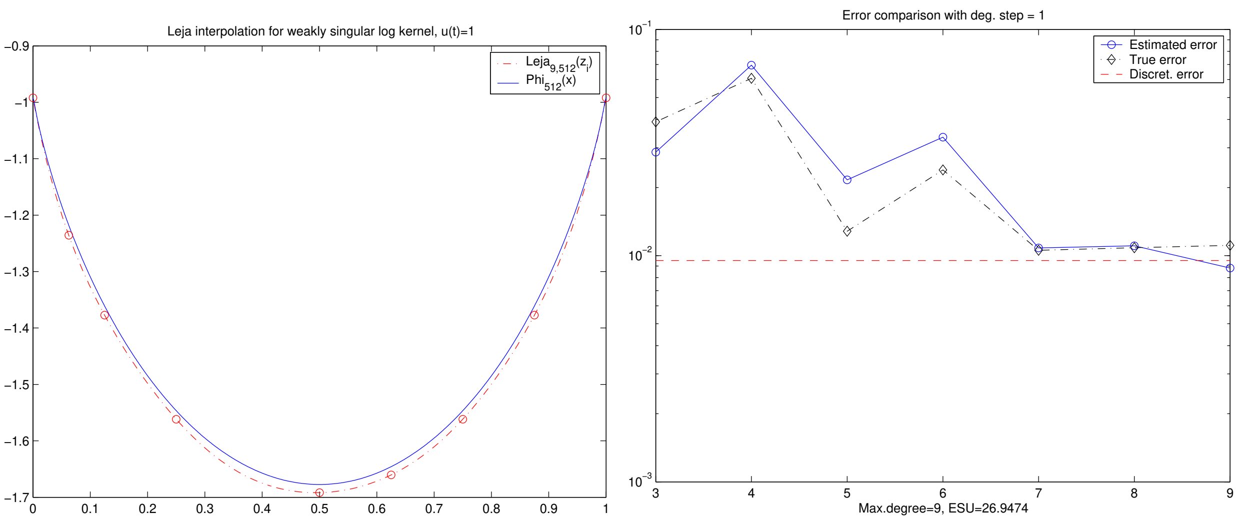


Figure 12: Leja-like compression of the discrete "weakly-singular log" linear transform, $n = 512$. *Left:* the exact transform with the interpolant at the leja points. *Right:* error estimate (17).

| $n$ | $\Lambda_{L_1}$ | $\Lambda_{L_2}$ | $\Lambda_{L_3}$ | $\Lambda_{L_b}$ |
|---|---|---|---|---|
| 1 | 4.486 | 4.6687 | 2.8727 | 1.3999 |
| 3 | 4.89e+2 | 1.02e+4 | 86.578 | 40.273 |
| 5 | 8.46e+3 | 5.43e+5 | 2.24e+3 | 32.981 |
| 7 | 5.68e+3 | 3.40e+5 | 8.17e+4 | 0.33e+3 |
| 8 | 1.03e+3 | 1.72e+5 | 1.04e+5 | 0.36e+3 |

Table 3: Lebesgue constants for different sequences of Leja points

### 6.2 Bivariate interpolation at Leja points on the square

For 2-dimensional square domains we have done some numerical experiments both in *polynomial interpolation* and *radial basis interpolation* using Leja sequences computed using different weight functions and also using the Definition 3. The first conclusion is that the Leja sequence defined by Definition 3 behaves nicely both from the interpolation error point of view and the stability of the interpolant. In what follows without loss of generality we restrict ourselves to the unit square $E = [0,1] \times [0,1]$.

In the experiments the weight functions and the corresponding Leja sequences are as follows:

1. $w_1(z) = \exp(-30\sqrt{x^2+y^2})$: we call $L_1$ the corresponding Leja sequence;
2. $w_2(z) = \exp(-30\,||x|+|y||)$: we call $L_2$ the corresponding Leja sequence;
3. $w_3(z) = \exp(-30\,||x|-|y||)$: we call $L_3$ the corresponding Leja sequence;
4. Leja-Bos points: we call $L_b$ the corresponding Leja sequence.

Moreover, the radial functions we used are:

1. $\Phi_1(x,y) = \exp(-\|x-y\|_2^2)$: the gaussian;
2. $\Phi_2(x,y) = \sqrt{1+\|x-y\|_2^2}$: the multiquadrics;
3. $\Phi_3(x,y) = \frac{1}{\sqrt{1+\|x-y\|_2^2}}$: the inverse multiquadrics.

In Table 3 we list the values of the Lebesgue constants $\Lambda_{L_*}$ corresponding to Leja sequences $L_1$, $L_2$, $L_3$ and $L_b$.

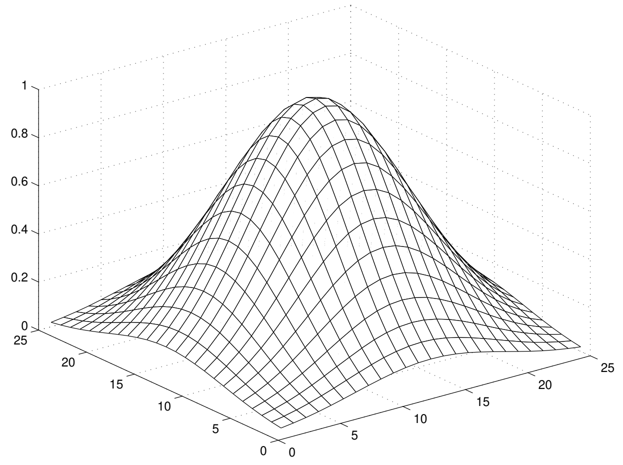


Figure 13: Function $f(x) = \exp\left(-8\,(\,(x-0.5)^2 + (y-0.5)^2\,)\right)$

EXAMPLE 12 We want to interpolate by polynomials $P_n(x,y)$ the function

$$f(x,y) = \exp(-8((x-0.5)^2 + (y-0.5)^2))$$

on $E = [0,1] \times [0,1]$ using Leja points $L_1,\ L_2,\ L_b$. In Table 4 we summaries the errors $\|R_n(f)\|_\infty$ in the sup-norm for $n \le 10$.

| $\|R_n(f)\|_{\infty,*}$ | $n=2$ | $n=4$ | $n=7$ | $n=10$ |
|---|---|---|---|---|
| $\|R_n(f)\|_{\infty,L_1}$ | 2.0340 | 0.4249 | 0.1764 | 0.1253 |
| $\|R_n(f)\|_{\infty,L_2}$ | 0.7926 | 4.8785 | 0.4911 | 0.1172 |
| $\|R_n(f)\|_{\infty,L_b}$ | 0.5589 | 0.2298 | 0.6392 | 0.1038 |

Table 4: Interpolation error (in the sup-norm) for the function in Example 12 on different Leja points

EXAMPLE 13 The function $f(x,y)$ is as in Example 12, while instead of polynomials, we use radial basis function interpolation. The interpolation problem consists in finding a function

$$g_\alpha(x) := \sum_{j=1}^{M} \alpha_j \Phi(x, x_j),\ x \in E \subset I\!R^2 \tag{18}$$

such that $g_\alpha(x_j) = f(x_j),\ j = 1,..,M$. In Tables 5-7 we display the sup-norm errors computed for different polynomials degrees $M = \binom{n+2}{2}$. The results shows that for this

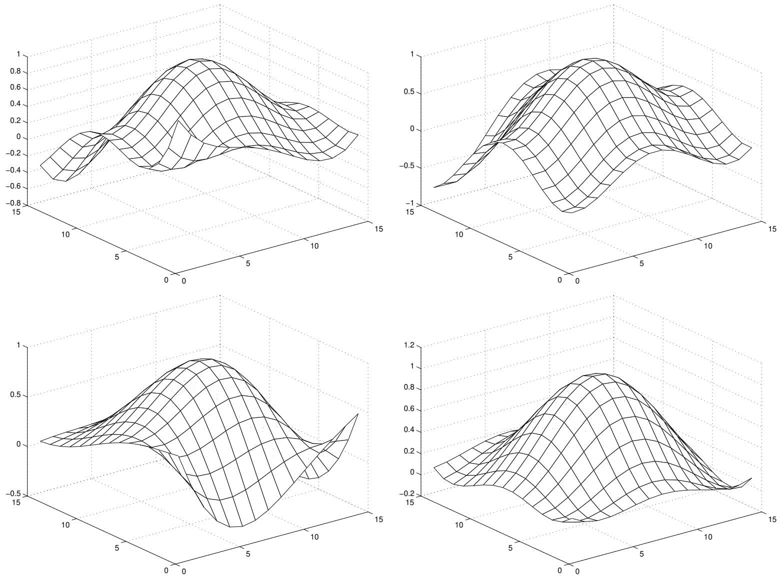

Figure 14: RBF interpolant of $f$ by means of $\Phi_1$ (gaussian) on 15 Leja points with weight functions $w_1$, $w_2$, $w_3$ and according Definition 2

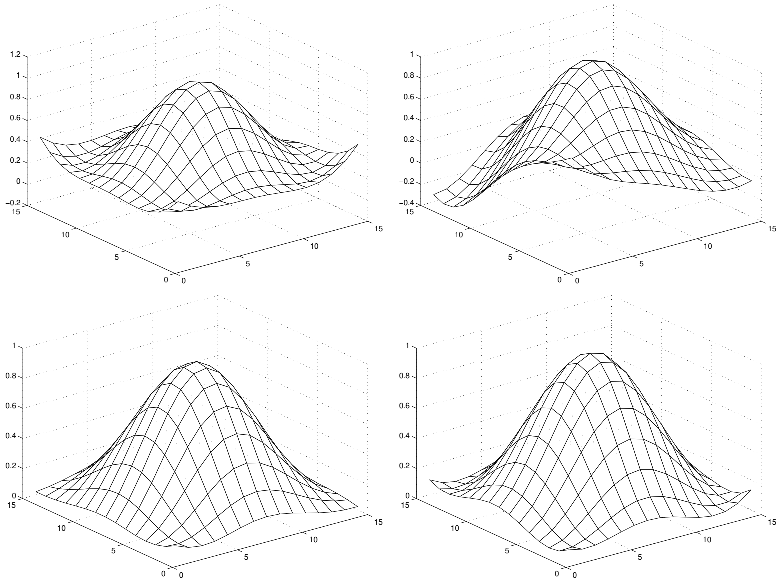

Figure 15: RBF interpolant of $f$ by means of $\Phi_2$ (multiquadrics) on 15 Leja points with weight functions $w_1$, $w_2$, $w_3$ and according Definition 2

| $\|R_n(f)\|_{\infty,*}$ | $n=6$ | $n=15$ | $n=28$ | $n=66$ |
|---|---|---|---|---|
| $\|R_n(f)\|_{\infty,L_1}$ | 1.1366 | 0.7325 | 0.4467 | 0.0039 |
| $\|R_n(f)\|_{\infty,L_2}$ | 0.7058 | 2.16675 | 2.4392 | 0.0216 |
| $\|R_n(f)\|_{\infty,L_b}$ | 0.4247 | 0.3754 | 0.1669 | 0.0106 |

Table 5: Interpolation error (in the sup-norm) for the function in Example 12 on different Leja points using radial basis interpolation based on the gaussian $\Phi_1(x, y)$

| $\|R_n(f)\|_{\infty,*}$ | $n=6$ | $n=15$ | $n=28$ | $n=66$ |
|---|---|---|---|---|
| $\|R_n(f)\|_{\infty,L_1}$ | 0.6130 | 0.1806 | 0.0085 | 1.3e-4 |
| $\|R_n(f)\|_{\infty,L_2}$ | 0.7782 | 0.2046 | 0.064 | 2.64e-4 |
| $\|R_n(f)\|_{\infty,L_b}$ | 0.5911 | 0.1580 | 0.023 | 2.95e-5 |

Table 6: Interpolation error (in the sup-norm) for the function in Example 12 on different Leja points using radial basis interpolation based on the multiquadrics $\Phi_2(x, y)$

interpolation problem, the more satisfactory points seems to be the Leja points computed according Definition 3. In Figures 14-16 we display the RBF interpolants of $f$ on 15 Leja points.

More examples and plots can be found in the thesis [8].

# 7 Open problems

Let us consider the *Van der Corput-Schaake inequality* (cf. [13]) for trigonometric polynomials $T(\theta)$ of degree $\leq m$ with $|T(\theta)| \leq 1$, $\forall\theta$:

$$|T'(\theta)| \leq m\sqrt{1 - T^2(\theta)}\ . \tag{19}$$

or equivalently

$$\left|\frac{T'(\theta)}{\sqrt{1 - T^2(\theta)}}\right| \leq m \tag{20}$$

The latter also says:

$$\left|\frac{d}{d\theta}\cos^{-1}(T(\theta))\right| \leq m\ . \tag{21}$$

Fekete points on $E = [-1, 1]$ has the *arccosine distribution*, that is the distance $|\cos^{-1}(a) - \cos^{-1}(b)| = |\theta_a - \theta_b|$ is constant for every couple $a, b \in E$.

| $\|R_n(f)\|_{\infty,*}$ | $n=6$ | $n=15$ | $n=28$ | $n=66$ |
|---|---|---|---|---|
| $\|R_n(f)\|_{\infty,L_1}$ | 1.6527 | 0.6042 | 0.0130 | 1.3e-4 |
| $\|R_n(f)\|_{\infty,L_2}$ | 1.2468 | 0.3892 | 0.0754 | 6.1e-4 |
| $\|R_n(f)\|_{\infty,L_b}$ | 0.4339 | 0.3630 | 0.00172 | 4.83e-5 |

Table 7: Interpolation error (in the sup-norm) for the function in Example 12 on different Leja points using radial basis interpolation based on the inverse multiquadrics $\Phi_3(x,y)$

**Lemma 1** *Take $a,b \in [-1,1]$. Then*

$$|\cos^{-1}(a) - \cos^{-1}(b)| = \sup_{\|P\|_{[-1,1]}\le 1} (\deg P)^{-1} \left|\cos^{-1}(P(a)) - \cos^{-1}(P(b))\right| . \tag{22}$$

**Proof.** Let $T(\theta) = P(\cos(\theta))$, $a = \cos(\theta_a)$ and $b = \cos(\theta_b)$. Then

$$|\cos^{-1}(T(\theta_a)) - \cos^{-1}(T(\theta_b))| = \left|\int_{\theta_a}^{\theta_b} \cos^{-1}(T(\theta))d\theta\right| \le m \int_{\theta_a}^{\theta_b} d\theta \le m|\theta_b - \theta_a| .$$

Since $\cos^{-1}(a) = \theta_a$ and $\cos^{-1}(b) = \theta_b|$, then

$$|\cos^{-1}(T(\theta_a)) - \cos^{-1}(T(\theta_b))| \le m|\cos^{-1}(a) - \cos^{-1}(b)| .$$

Then

$$|\cos^{-1}(a) - \cos^{-1}(b)| \le \max_{\|P\|_{[-1,1]}\le 1} (\deg P)^{-1} \left|\cos^{-1}(P(a)) - \cos^{-1}(P(b))\right| .$$

This concludes the proof. □

Then we have a new formulation for the distance that can be generalized in every dimension.

Definition 4 *Let $a,b \in E \subset I\!R^n$. Then*

$$\mu_E(a,b) = \sup_{\|P\|_E\le 1} (\deg P)^{-1} \left|\cos^{-1}(P(a)) - \cos^{-1}(P(b))\right| . \tag{23}$$

*is called the* **Dubiner** *metric on E (cf. [14]).*

Example 14 $E = [-1,1]$. The Dubiner metric is $\mu_E(a,b) = |\cos^{-1}(a) - \cos^{-1}(b)|$ and all "good" set of points for interpolation, like Chebyshev, Fekete, Leja and the zeros of orthogonal polynomials are asymptotically distributed according to this metric.

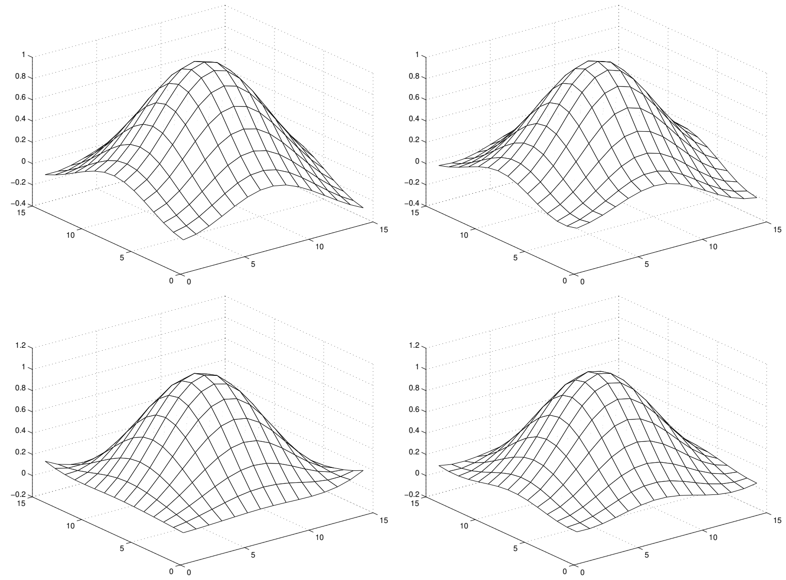

Figure 16: RBF interpolant of $f$ by means of $\Phi_3$ (inverse multiquadrics) on 15 Leja points with weight functions $w_1$, $w_2$, $w_3$ and according Definition 2

**Is this also true in higher dimensions?**

The expressions for the Dubiner metric on the square and the circle are known and from numerical experiments we can give a positive answer to the previous question.

DEFINITION 5 *Let $E = [0,1] \times [0,1]$ and $x = (x_1, x_2)$, $y = (y_1, y_2)$ two points of $E$. Then*

$$\mu_E(x,y) = \max\{\left|\cos^{-1}(x_1) - \cos^{-1}(y_1), \cos^{-1}(x_2) - \cos -1(y_2)\right|\} . \tag{24}$$

*is the* **Dubiner** *metric on the square.*

DEFINITION 6 *Let $E$ be the circle and $x = (x_1, x_2)$, $y = (y_1, y_2)$ two its points. Then*

$$\mu_E(x,y) = \left|\cos^{-1}\left(x_1 y_1 + x_2 y_2 + \sqrt{1 - x_1^2 - x_2^2}\sqrt{1 - y_1^2 - y_2^2}\right)\right| . \tag{25}$$

*is the* **Dubiner** *metric on the circle.*

How to prove that the Leja points on the square and the circle are asymptotically equidistributed w.r.t. to the Dubiner metric still an *open problem.* Certainly, by numerical experiments we were able to have an evidence that this can be true, as shown in Figure 17.

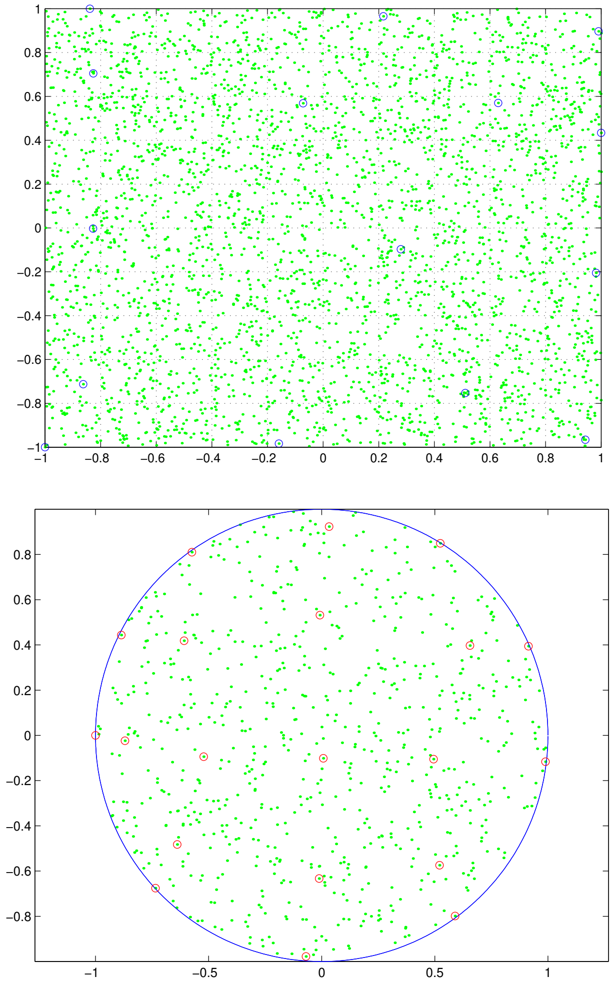


Figure 17: Leja points (small circles) on the square and the circle using Dubiner metric. The dots are random points from which we extracted Leja points.

Moreover, we did some calculations on the growth of the Lebesgue constant for the square on the Leja points distributed according to the Dubiner metric. In Table 8 we provide these results and the ones for the sequences we already computed (with $\Lambda_{Dub}$ we denoted the Lebesgue constant using the Dubiner metric on the square). The results shows that the sequence of Leja so determined are good candidates for interpolation.

| $n$ | $\Lambda_{L_1}$ | $\Lambda_{L_2}$ | $\Lambda_{L_3}$ | $\Lambda_{L_b}$ | $\Lambda_{Dub}$ |
|---|---|---|---|---|---|
| 1 | 4.486 | 4.6687 | 2.8727 | 1.3999 | 2.212 |
| 3 | 4.89e+2 | 1.02e+4 | 86.578 | 40.273 | 6.9719 |
| 5 | 8.46e+3 | 5.43e+5 | 2.24e+3 | 32.981 | 43.238 |
| 7 | 5.68e+3 | 3.40e+5 | 8.17e+4 | 0.33e+3 | 62.111 |
| 8 | 1.03e+3 | 1.72e+5 | 1.04e+5 | 0.36e+3 | 125.685 |

Table 8: Lebesgue constants for different sequences of Leja points on the square

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## References

[1] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganz-zahlingen Koefficzienten, *Math. Zeits* 17 (1923).

[2] L. Fejér, Bestimmung derjeningen Abszissen eines Intervalles, für welche die Quadratsummes der Grundfunktionen der Lagrangeschen Interpolation im Intervalle [-1,1] ein möglish kleines Maximum besitzt, *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. Ser. II*, 1:263–276 (1932).

[3] F. Leja, Sur certaines suites liées aux ensembles plans er leur application à la représentation conforme, *Ann. Polon. Math.* 4:8–13 (1957).

[4] A. Edrei, Sur les déterminants récurrents et les singularités d'une fonction donnée par so développement *Compositio Math.* 7:20–88 (1939).

[5] J. Baglama, D. Calvetti and L. Reichel, Fast Leja points, *Elect. Trans. Numer. Anal.* 7:124–140 (1998).

[6] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, New York, 1959.

[7] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, Berlin, 1997.

[8] F. Del Favero, Punti di Leja per l'interpolazione di funzioni, Degree Thesis (in Italian), University of Udine, 2002.

[9] L. Reichel, Newton interpolation at Leja points, *BIT Numerical Analysis* 30:332-346 (1990).

[10] B. Fisher, L. Reichel, Newton interpolation in Fejer and Chebyshev points, *Math. Comp.* 53 (187):265–278 (1989).

[11] S. De Marchi, M. Vianello, Approximating the approximant: a numerical code for polynomial compression of discrete integral operators, *Numer. Algorithms* 28:101–116 (2001).

[12] S. De Marchi, M. Vianello, Fast evaluation of discrete integral transforms by Chebyshev and Leja polynomial approximation." In CONSTRUCTIVE THEORY OF FUNCTIONS", Varna 2002, (B. Bojanov, Ed), DARBA, Sofia (2003), pp. 347–353.

[13] J.G. van der Corput, G. Schaake, Ungleichungen fur Polynome und trigonometriske Polynome, *Compositio Math.*, 2:321–361 (1935).

[14] M. Dubiner, Spectral methods on triangles and other domains, *J. Sci. Comp.*, 6(4):345–390 (1991).