A shape preserving quasi-interpolation operator based on a new transcendental RBF

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Abstract

It is well-known that the univariate Multiquadric quasi-interpolation operator is constructed based on the piecewise linear interpolation by \(|x|\). In this paper, we first introduce a new transcendental RBF based on the hyperbolic tangent function as an smooth approximant to \(\phi(r) = r\) with higher accuracy and better convergence properties than the MQ RBF \(\sqrt{r^2 + c^2}\). Then the Wu–Schaback’s quasi-interpolation formula is rewritten using the proposed RBF. It preserves convexity and monotonicity. We prove that the proposed scheme converges with a rate of \(O(h^2)\). So it has a higher degree of smoothness. Some numerical experiments are given in order to demonstrate the efficiency and accuracy of the method.

Keywords: Radial basis functions (RBFs), quasi-interpolation, hyperbolic tangent function

1. Introduction

Given a set of \(n\) distinct (scattered) points \(\{x_j\}_{j=0}^n \in \Omega \subseteq \mathbb{R}^d\) and corresponding data values \(\{f_j\}_{j=0}^n \in \mathbb{R}\), a standard way to interpolate a function \(f \in C^1: \Omega \to \mathbb{R}\) is by using

\[
\mathcal{L}f(x) = \sum_{j=0}^n \lambda_j \mathcal{X}(x - x_j),
\]

with the coefficients \(\lambda_j\) determined by the interpolation conditions \(\mathcal{L}f(x_j) = f_j, \ j = 0, \ldots, n\), where \(\mathcal{X}(\cdot)\) is an interpolation kernel. Many authors use Radial Basis Functions (RBFs) to solve the interpolating problem (1), that is \(\mathcal{X}(x - x_j) = \phi(||x - x_j||), (\|\cdot\|\) is the Euclidean norm) with \(\phi: [0, \infty) \to \mathbb{R}\), is some radial function \([41]\). Then, the coefficients \(\lambda_j\) are determined solving a symmetric linear system \(A\lambda = f\), where \(A = [\phi(||x_i - x_j||)]_{0 \leq i, j \leq n}\). RBF method provides excellent interpolants for high dimensional scattered data sets. The corresponding theory had been extensively studied by many researchers (see e.g \([2, 25, 26, 27, 30, 31, 39, 41, 44, 46]\)). That is why in the last few decades, RBFs have been widely applied in a number of fields such as multivariate function approximation, neural networks and solution of differential and integral equations (see e.g \([6, 7, 10, 13, 17, 21, 22, 28, 34, 40, 47]\)). The Multiquadric (MQ) RBF

\[
\phi_j(x) = \sqrt{\|x - x_j\|^2 + c^2},
\]

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The main advantage of this formula is that it does not require the solution of any linear system. Wu–Schaback's MQ quasi-interpolation formula is based on the fact that the MQ degenerates to \( x - x_j \) as its coefficients. The drawback is that instead of \( c = O(h) \), one needs to use a smaller shape parameter \( c^2 \log c = O(h^2) \) in order to achieve quadratic convergence, resulting in a lower smoothness. Note that for \( c = 0 \), the basis functions given in quasi-interpolant \( L_{MQ} f \) are cardinal with respect to \( \{ x_j \}_{j=0}^n \). For a general quasi-interpolation operator \( L \) we can state the following definitions.
Definition 1.1. The quasi-interpolation operator \( L \) constructed at the data points \( \{(x_j, f_j)\} \), is called to be monotonicity-preserving, if the first order divided difference \( f[x_j, x_{j+1}] \) is nonnegative (non-positive) implies that \( (Lf)' \) is also nonnegative (non-positive).

Definition 1.2. The quasi-interpolation operator \( L \) constructed at the data points \( (x_j, f_j) \), is called to be convexity-preserving if the second order divided difference \( f[x_{j-1}, x_j, x_{j+1}] \) is nonnegative (non-positive, zero) implies that \( (Lf)'' \) is also nonnegative (non-positive, zero).

Since \( \sqrt{x^2 + c^2} \) tends to \( |x| \) as \( c \) tends to zero, and radial basis interpolation (as well as the quasi-interpolation) based on \( |x| \) is piecewise linear, Wu and Schaback claimed that the shape-preserving properties of piecewise linear interpolation can be expected to hold for quasi-interpolation with multiquadratics, too. Actually, they first showed that the quasi-interpolation operator of Beatson and Powell is indeed convexity preserving. Then they proved that the quasi-interpolation operator (3) is monotonicity and convexity preserving. In 2004, Ling [23] proposed a multilevel quasi-interpolation operator and proved that it converges with a rate of \( O(h^{2.5}) \log h \) as \( c = O(h) \). In 2009, Feng and Li [9] constructed a shape-preserving quasi-interpolation operator by shifts of cubic MQ functions proving that it can produce an error of \( O(h^2) \) as \( c = O(h) \). Wang et al. [38] proposed an improved univariate MQ quasi-interpolation operator, by using Hermite interpolating polynomials, with convergence rate heavily depending on the shape parameter \( c \). Jiang et al. [19] proposed two new multilevel univariate MQ quasi-interpolation operators with higher approximation order.

Ling proposed a multidimensional quasi-interpolation operator using the dimension-splitting multiquadric basis function approach [24], and Wu et al. modified their idea by using multivariate divided difference and the idea of the superposition [43].

Gao and Wu [12] studied the quasi-interpolation for the linear functional data rather than the discrete function values. Moreover, MQ quasi-interpolation has been successfully applied in a wide range of fields. For example, in 2007, Wang and Wu [37] applied the operator (3) to tackle approximate implicitization of parametric curves. In 2011, Wu [42] presented a new approach to construct the so-called shape preserving interpolation curves based on MQ quasi-interpolation (3). Hon and Wu [16], Chen and Wu [3, 4], Jiang and Wang [18], and other researches provided some successful examples using MQ quasi-interpolation operators to solve different types of partial differential equations.

In this paper, in the next section we introduce a new quasi-interpolation operator based on the hyperbolic tangent function, that is the function
\[
g(x) = x \tanh \left( \frac{x}{c} \right) , \quad c > 0
\] (4)
which leads to a smooth and shape preserving interpolation operator with \( O(h^2) \) rate of convergence. In section 3, we discuss its accuracy providing an error estimate. Numerical experiments are presented in section 4 with the aim of comparing the accuracy of our quasi-interpolation scheme with that of Wu and Schaback’s, and also verifying the convergence rate of new quasi-interpolation operator by examples. The last section summarizes the conclusion and some further works.

2. Quasi-interpolation operator based on a new transcendental RBF

In this section, we first analyse a new approximation of \( |x| \) based on the hyperbolic tangent, with better accuracy than the MQ RBF \( \sqrt{x^2 + c^2} \). The general question is, are there any good approximations of the absolute value function which are smooth? One simple approximation is
MQ RBF $\sqrt{x^2 + c^2}$. Carlos Ramirez et al. [33] proved that $\sqrt{x^2 + c^2}$ is the most computationally efficient and smooth approximation of $|x|$, while S. Voronin et al. [35] proved the following Lemma.

**Lemma 2.1.** The approximation of $|x|$ by the multiquadrics $g(x) = \sqrt{x^2 + c^2}, c \in \mathbb{R}_+$ satisfies

$$
\left| |x| - \sqrt{x^2 + c^2} \right| \leq c,
$$

$$
|x| \leq \sqrt{x^2 + c^2}.
$$

As noticed by Gauss in [36], the hyperbolic tangent can be written using the continued fraction

$$
\tanh(x) = \frac{x}{1 + \frac{x^2}{3 + \frac{x^2}{5 + \cdots}}}.
$$

This fact shows immediately that the function $g(x) = x \tanh\left(\frac{x}{c}\right)$ is a nonnegative function that indeed can be used to approximate $|x|$.

Since for the hyperbolic tangent

$$
\lim_{c \to 0^+} \tanh\left(\frac{x}{c}\right) = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0.
\end{cases}
$$

we then have the approximation

$$
x \tanh\left(\frac{x}{c}\right) \approx |x|.
$$

Now, we show that the approximation of $|x|$ by $x \tanh\left(\frac{x}{c}\right)$ is more accurate than that given by the multiquadric.

**Lemma 2.2.** The approximation of $|x|$ by $g(x) = x \tanh\left(\frac{x}{c}\right), c \in \mathbb{R}_+$ satisfies

$$
\left| |x| - x \tanh\left(\frac{x}{c}\right) \right| \leq 0.28c < c, \quad (5)
$$

$$
x \tanh\left(\frac{x}{c}\right) \leq |x|. \quad (6)
$$

**Proof.** The proof of (5) is trivial for $x = 0$. Letting $h(x) = |x| - x \tanh\left(\frac{x}{c}\right)$ that, for $x > 0$, becomes $h(x) = x - x \tanh\left(\frac{x}{c}\right)$. The maxima and minima of $h$ are those that annihilate

$$
h'(x) = \left(\frac{x}{c}\right) \left(\tanh\left(\frac{x}{c}\right)^2 - \tanh\left(\frac{x}{c}\right) + 1 - \frac{x}{c}\right).
$$

Setting $\frac{x}{c} = t$, we have

$$
t \tanh^2(t) - \tanh(t) + (1 - t) = 0
$$

which reduces to solve $s(t) = t(\tanh(t) + 1) - 1 = 0$. The function $s$ on $t \geq 0$ is strictly increasing, with $s(0) = -1$. Hence there exits only one zero. By numerically solving it, we find the value of $t^* = 0.6392322714$ then $x^* = 0.6392322714c$. When $x < 0$, ans so $t < 0$, $s(t) < -1$, showing that the value $t^*$ is the only extremal value of $h$. Hence,

$$
h(x^*) = 0.2784645427c \approx 0.28c.
To prove 6, we have
\[ x \tanh \left( \frac{x}{c} \right) \leq |x| \iff x^2 \tanh^2 \left( \frac{x}{c} \right) \leq x^2, \]
\[ \iff \tanh^2 \left( \frac{x}{c} \right) \leq 1. \]

\[ \square \]

**Theorem 2.1.** *The approximation of* \(|x|\) *by* \(x \tanh \left( \frac{x}{c} \right)\) *is more accurate than that with* \(\sqrt{x^2 + c^2}\).

**Proof.** It is clear that
\[ \cosh \left( \frac{x}{c} \right) > \frac{x}{c}. \]
Since \(\cosh(x)\) is an even function we have
\[ \cosh^2 \left( \frac{x}{c} \right) > \frac{x^2}{c^2}, \]
then
\[ x^2 \text{sech}^2 \left( \frac{x}{c} \right) < c^2, \]
which in turn gives
\[ x^2 - x^2 \tanh^2 \left( \frac{x}{c} \right) < c^2. \]
Then
\[ x^2 - x^2 \tanh^2 \left( \frac{x}{c} \right) < c^2 = (x^2 + c^2) - x^2. \]

\[ \square \]

Moreover, the function \(x \tanh \left( \frac{x}{c} \right)\) converges to \(|x|\) faster than \(\sqrt{x^2 + c^2}\) to \(|x|\) by decreasing \(c\), as stated in the next Theorem.

**Theorem 2.2.** *If* \(c \to 0^+\) *then* \(x \tanh \left( \frac{x}{c} \right) - |x| = o \left( \sqrt{x^2 + c^2} - |x| \right)\).

In fact,
\[ \lim_{c \to 0^+} \frac{x \tanh \left( \frac{x}{c} \right) - |x|}{\sqrt{x^2 + c^2} - |x|} = 0 \quad (7) \]

In order to illustrate the superiority of the new hyperbolic approximation to \(|x|\), \(L_\infty\) error norm
\[ \max_{1 \leq i \leq n} |g(x_i) - |x_i||, \]
and the rate of convergence
\[ r_c = \log \left( \frac{E_{c_i}}{E_{c_{i-1}}} \right) / \log \left( \frac{c_i}{c_{i-1}} \right), \]
for both approximants \(x \tanh \left( \frac{x}{c} \right)\) and \(\sqrt{x^2 + c^2}\) are reported in Table 1, for \(n = 100, 200, 400\) equally spaced points in \([-10, 10]\). Table 1 shows that \(x \tanh \left( \frac{x}{c} \right)\) approximates \(|x|\) much better than \(\sqrt{x^2 + c^2}\) while Table 1 and the logarithmic scale plots 1 show that the approximant
\[ x \tanh \left( \frac{x}{c} \right) \] has exponential rate of convergence to \(|x|\) as \(c \to 0\) instead of \(O(c^2)\) provided by \(\sqrt{x^2 + c^2}\).

Table 1: \(L_\infty\) errors and convergence rates for both approximants of \(|x|\) for different values of \(c\).

| \(n\) | \(c\) | \(|x| - \sqrt{x^2 + c^2}|\) \(L_\infty\) error \(r_c\) | \(|x| - x \tanh \left( \frac{x}{c} \right)|\) \(L_\infty\) error \(r_c\) |
|------|-----|-----------------|------------------|
| 100  | 0.1 | 4.1127e-02 — 2.3656e-02 | — 3.4922e-03 2.759988057 |
|      | 0.05| 1.1658e-02 1.813823944 | 1.9342e-08 11.65767264 |
|      | 0.025| 3.0478e-03 1.940431754 | 6.2490e-05 5.80436034 |
|      | 0.0125| 7.7050e-04 1.9830901373 | 1.9342e-08 11.65767264 |
|      | 0.00625| 1.9317e-04 1.995923901 | 1.8457e-05 23.32106557 |

| 200  | 0.1 | 6.1656e-02 — 2.6930e-02 | — 1.875e-02 1.181286716 |
|      | 0.05| 2.0637e-02 1.579218611 | 1.7723e-03 2.744232777 |
|      | 0.025| 5.8753e-03 1.812498837 | 5.9488e-03 2.744232777 |
|      | 0.0125| 1.5314e-03 1.939811357 | 3.2376e-05 5.774554268 |
|      | 0.00625| 3.8718e-04 1.983774825 | 1.0436e-08 11.59941019 |

| 400  | 0.1 | 7.8030e-02 — 2.7348e-02 | — 1.875e-02 1.181286716 |
|      | 0.05| 3.0867e-02 1.337963627 | 1.3456e-02 1.023185719 |
|      | 0.025| 1.0337e-02 1.578247724 | 1.3456e-02 1.023185719 |
|      | 0.0125| 2.9442e-03 1.811869965 | 3.5488e-03 1.17759030 |
|      | 0.00625| 7.6754e-04 1.939561834 | 1.6476e-05 11.59941019 |

2.1. New transcendental RBF

Let us introduce the following globally supported and infinitely differentiable transcendental RBF

\[
\phi(r) = r \tanh \left( \frac{r}{c} \right),
\]
abbreviated by RTH, where \( r = \|x - x_j\| \) and \( \| \cdot \| \) is the Euclidean norm \( \mathbb{R}^d \).

The parameter \( c > 0 \) is called shape parameter whose optimal value for getting accurate numerical solutions and good conditioning of the collocation matrix, can be found usually numerically.

**Theorem 2.3.** The RTH RBF is conditionally negative definite of order 1 on every \( \mathbb{R}^d \).

**Proof.** We show that \( \psi(r) = -\phi(r) \) is conditionally positive definite of order 1. We have \( \psi(r) = f(s) = -\sqrt{s} \tanh \left( \frac{\sqrt{s}}{c} \right) \), where \( s = r^2 \). Now for

\[
g(s) = -f'(s) = \frac{1}{2} s^{-\frac{1}{2}} \tanh \left( \frac{\sqrt{s}}{c} \right) + \frac{1}{2c} \left( 1 - \tanh^2 \left( \frac{\sqrt{s}}{c} \right) \right),
\]

we have

\[
(-1)^l g^{(l)}(s) \geq 0, \quad \text{for all } l \in \mathbb{N}_0 \text{ and all } s > 0.
\]

So \( -f'(s) \) is completely monotone on \((0, \infty)\). Now, since \( f \notin \Pi_m^d \), the claim is proved according to Micchelli’s theorem \([27]\).

**Remark 2.1.** Since \( \phi \) is conditionally negative definite of order 1 and \( \phi(0) = 0 \), then the matrix \( A = [\phi(||x_i - x_j||)]_{1 \leq i,j \leq n} \) has one positive and \( n - 1 \) negative eigenvalues and in particular it is invertible.

In the sequel, we consider \( d = 1 \), since our work is confined to the univariate case. We have seen before that the RTH RBF is an smooth approximant to \( \tau(r) = r \) with higher accuracy and better convergence properties than the MQ RBF \( \sqrt{r^2 + c^2} \), by decreasing shape parameter \( c \). In Figure 2, we have plotted both RTH basis

\[
\phi_j(x) = (x - x_j) \tanh \left( \frac{x - x_j}{c} \right),
\]

and MQ basis (2) centered at \( x_j = 0 \). It can be noted from Figure 2 that the RTH RBF approaches to \( |x| \) faster than the MQ RBF, even with larger shape parameters. Moreover, in RTH RBF \( \phi_j(x_j) = 0 \) independent of the value of \( c \), but MQ requires that \( c = 0 \). This property of the RTH RBF leads to getting more accurate results in corresponding quasi-interpolants.

![Figure 2: Plots of Multiquadric RBF (left), and RTH RBF (right) for different values of shape parameter \( c \).](image)
The first and second derivatives of the RTH RBF (8) are of the form

\[
\phi_j'(x) = \tanh \left( \frac{x - x_j}{c} \right) + \frac{(x - x_j)}{c} \left( 1 - \left( \tanh \left( \frac{x - x_j}{c} \right) \right)^2 \right),
\]

\[
\phi_j''(x) = 2 \frac{1}{c} \left( 1 - \left( \tanh \left( \frac{x - x_j}{c} \right) \right)^2 \right) - 2 \frac{(x - x_j)}{c^2} \tanh \left( \frac{x - x_j}{c} \right) \left( 1 - \left( \tanh \left( \frac{x - x_j}{c} \right) \right)^2 \right),
\]

and are plotted in Figure 3 for \( c = 1 \).

![Graphs of RTH RBF, its first derivative, and its second derivative.](image)

**Figure 3:** RTH RBF (a), first derivative (b), and second derivative of RTH RBF (c) with shape parameter \( c = 1 \).

In Tables 2 and 3, we summarized the properties of both MQ and RTH RBFS, where \( \xi = 1.199678640 \), is obtained numerically by calculating the roots of the second derivative.

### Table 2: Comparing both RBFs.

<table>
<thead>
<tr>
<th>Name</th>
<th>( \phi_j(x) )</th>
<th>( \lim_{x \to x_j} \phi_j(x) )</th>
<th>( \lim_{c \to 0} \phi_j(x) )</th>
<th>( \lim_{x \to \pm \infty} \phi_j'(x) )</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>MQ RBF</td>
<td>( \sqrt{c^2 + (x - x_j)^2} )</td>
<td>( c )</td>
<td>(</td>
<td>x - x_j</td>
<td>)</td>
</tr>
<tr>
<td>RTH RBF</td>
<td>((x - x_j) \tanh \left( \frac{x - x_j}{c} \right))</td>
<td>0</td>
<td>(</td>
<td>x - x_j</td>
<td>)</td>
</tr>
</tbody>
</table>

### Table 3: Comparing both RBFs.

<table>
<thead>
<tr>
<th>Name</th>
<th>( \phi_j(x) )</th>
<th>( \phi_j'(x) )</th>
<th>( \phi_j''(x) )</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>MQ RBF</td>
<td>( \sqrt{c^2 + (x - x_j)^2} )</td>
<td>Strictly increasing</td>
<td>( \geq 0 )</td>
<td>( x \in (-\infty, \infty) )</td>
</tr>
<tr>
<td>RTH RBF</td>
<td>((x - x_j) \tanh \left( \frac{x - x_j}{c} \right))</td>
<td>Strictly increasing</td>
<td>( \geq 0 )</td>
<td>( x \in [-c\xi, c\xi] )</td>
</tr>
</tbody>
</table>
2.2. Quasi-interpolation operator

The quasi-interpolation operator of a function \( f : [a, b] \to \mathbb{R} \) with RTH RBF on the scattered points

\[
a = x_0 < x_2 < \cdots < x_n = b \quad h := \max_{2 \leq j \leq n} (x_j - x_{j-1}),
\]

has the form

\[
(L_{RTH} f)(x) = f_0 \alpha_0(x) + f_1 \alpha_1(x) + \sum_{j=2}^{n-2} f_j \psi_j(x) + f_{n-1} \alpha_{n-1}(x) + f_n \alpha_n(x)
\]

where

\[
\alpha_0(x) = \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},
\]

\[
\alpha_1(x) = \frac{\phi_2(x) - \phi_1(x) - \phi_1(x) - (x - x_0)}{2(x_2 - x_1) - 2(x_1 - x_0)},
\]

\[
\alpha_{n-1}(x) = \frac{(x_n - x) - \phi_{n-1}(x) - \phi_{n-1}(x) - \phi_{n-2}(x)}{2(x_n - x_{n-1}) - 2(x_{n-1} - x_{n-2})},
\]

\[
\alpha_n(x) = \frac{1}{2} + \frac{\phi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})},
\]

\[
\phi_j(x) = (x - x_j) \tanh \left( \frac{x - x_j}{c} \right), \quad j = 1, \ldots, n - 1, \ c \in \mathbb{R}_+,
\]

\[
\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad 2 \leq j \leq n - 2.
\]

The formula (10) can be rewritten as

\[
(L_{RTH} f)(x) = \frac{1}{2} \sum_{j=1}^{n-1} f[x_{j-1}, x_j, x_{j+1}](x_{j+1} - x_{j-1}) \phi_j(x) + \frac{f_0 + f_n}{2} + \frac{1}{2} f[x_0, x_1](x - x_0) - \frac{1}{2} f[x_{n-1}, x_n](x_n - x).
\]

Let \( \phi_1(x) = |x - x_1|, \ \phi_0(x) = |x - x_0|, \ \phi_n(x) = |x - x_n| \) and \( \phi_{n+1}(x) = |x - x_{n+1}| \), then for \( x \in [x_0, x_n] \), the operator \( L_{RTH} \) can be rearranged as

\[
(L_{RTH} f)(x) = \sum_{j=0}^{n} f_j \psi_j(x),
\]

where

\[
\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad j = 0, \ldots, n,
\]

and \( x_{-1} < x_0, \ x_{n+1} > x_n \).

**Remark 2.2.** From relation (11), it is clear that the quasi-interpolation operator \( L_{RTH} \) reproduces the linear polynomials on \([x_0, x_n]\), that is

\[
\sum_{j=0}^{n} (ax_j + b)\psi_j(x) = ax + b, \quad a, b \in \mathbb{R},
\]
from which we also get $\sum_{j=0}^{n} \psi_j(x) = 1$ at any point $x \in [x_0, x_n]$.

In order to prove the shape-preserving property of the quasi-interpolation operator (10), we give some important definitions and theorems from differential geometry (cf. e.g. [29]).

**Definition 2.1.** A differentiable plane curve $\alpha : (a, b) \to \mathbb{R}^2$ is said to be regular if its derivative never vanishes. That is
\[ \forall t \in (a, b), \quad \alpha'(t) = \left( \frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt} \right) \neq (0, 0). \]

**Theorem 2.4.** Let $C$ be a regular plane curve given by $\alpha(t)$. Then the curvature $\kappa$ of $C$ at $t$ is given by
\[ \kappa[\alpha](t) = \|\alpha'(t) \times \alpha''(t)\| / \|\alpha'(t)\|^3. \]

**Definition 2.2.** Let $f \in C^2[a, b]$. The curvature of the plane curve $y = f(x)$ is given by
\[ \kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}. \]

**Theorem 2.5** (Fundamental theorem of plane curves). Let $\alpha, \gamma : (a, b) \to \mathbb{R}^2$ be regular plane curves such that $\kappa[\alpha](t) = \kappa[\gamma](t)$ for all $t \in (a, b)$. Then there is an orientation-preserving Euclidean motion $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\gamma = F \circ \alpha$.

**Corollary 2.5.1.** Two unit-speed plane curves which have the same curvature differ only by a Euclidean motion.

**Theorem 2.6.** The quasi-interpolation operator $L_{RT_H}$ constructed by data points $\{(x_j, f_j)\}$, is monotonicity and convexity-preserving for $c$ small enough.

**Proof.** According to the Corollary 2.5.1, it suffices to show that
\[ \lim_{c \to 0} |\kappa_{MQ}(x) - \kappa_{RT_H}(x)| = 0. \]

Let $x \neq x_j$, otherwise both quasi-interpolants (3) and (10) do not have first and second derivatives as $c$ approaches 0. Now, according to definition 2.2, we have
\[ \kappa_{MQ}(x) = \frac{|(L_{MQ}f)''(x)|}{\left(1 + ((L_{MQ}f)'(x))^2\right)^{3/2}}. \]

Since for MQ RBF,
\[ \phi_j''(x) = \frac{c^2}{(c^2 + (x - x_j)^2)^{3/2}}, \]

then
\[ \lim_{c \to 0} \phi_j''(x) = 0. \]
Moreover

\[(\mathcal{L}_{MQ}f)^\prime\prime(x) = \frac{1}{2} \sum_{j=1}^{n-1} \left[ \frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_{j} - x_{j-1}} \right] \phi_j''(x), \]

then

\[\lim_{c \to 0} \kappa_{\mathcal{L}_{MQ}}(x) = 0,\]

which leads to

\[\forall \epsilon > 0 \quad \exists \delta_1 > 0; \quad |c| < \delta_1 \Rightarrow |\kappa_{\mathcal{L}_{MQ}}(x)| < \epsilon.\]

Similarly, for RTH RBF, we have

\[\lim_{c \to 0} \phi_j''(x) = 0,\]

then

\[\lim_{c \to 0} \kappa_{\mathcal{L}_{RTH}}(x) = 0,\]

which leads to

\[\forall \epsilon > 0 \quad \exists \delta_2 > 0; \quad |c| < \delta_2 \Rightarrow |\kappa_{\mathcal{L}_{RTH}}(x)| < \epsilon.\]

The proof completes by considering \(\delta = \min\{\delta_1, \delta_2\}\).

3. Accuracy of the quasi-interpolation operator \(\mathcal{L}_{RTH}\)

In this section, we give an approximation order for the quasi-interpolation operator \(\mathcal{L}_{RTH}\).

**Theorem 3.1.** Assume \(f''\) is Lipschitz continuous. The quasi-interpolation operator \(\mathcal{L}_{RTH}f\), at the point set (9) as \(h \to 0\), converges as follows

\[\|f - \mathcal{L}_{RTH}f\|_{\infty} \leq kh^2,\]  \hspace{1cm} (14)

where \(k\) is independent of \(h\) and \(c\).

**Proof.** Let \(t(y)\) be the local Taylor approximation of \(f\) at \(y\), that is

\[t(y) = f(x) + f'(x)(y - x), \quad x \in [a, b] \]

According to Remark 2.2, we get

\[\sum_{j=0}^{n} (x - x_j)\psi_j(x) = 0, \quad \sum_{j=0}^{n} \psi_j(x) = 1.\]

Then we get

\[\sum_{j=0}^{n} t(x_j)\psi_j(x) = \sum_{j=0}^{n} \left[ f(x) + f'(x)(x_j - x) \right] \psi_j(x)\]

\[= f(x) \sum_{j=0}^{n} \psi_j(x) + f'(x) \sum_{j=0}^{n} (x - x_j)\psi_j(x)\]

\[= f(x).\]
4.1. Test problem 1

Since \( f''(x) \) is Lipschitz continuous, then for every \( x_1, x_2 \in [a, b] \),
\[
|f''(x_1) - f''(x_2)| \leq c_0 |x_1 - x_2|, 
\]
where \( 0 < c_0 = \text{ess sup}_{a \leq x \leq b} |f''(x)| \). Now according to (11), we have
\[
|\mathcal{L}_{RTH}(f)(x) - f(x)| = \left| \sum_{j=0}^{n} (f(x_j) - t(x_j)) \psi_j(x) \right| 
\leq \frac{1}{2} \sum_{j=1}^{n-1} |f[x_{j-1}, x_j, x_{j+1}] - t[x_{j-1}, x_j, x_{j+1}]| (x_{j+1} - x_{j-1}) \psi_j(x) 
\quad + c_1(x - x_0)^2 + c_2(x_n - x)^2 
\leq \frac{1}{4} \sum_{j=1}^{n-1} |f''(\xi) - f''(\eta)||\psi_j(x)| (x_{j+1} - x_{j-1}) 
\quad + c_1(x - x_0)^2 + c_2(x_n - x)^2 
\leq \frac{1}{2} c_0 h \sum_{j=1}^{n-1} |x - x_j|(x_{j+1} - x_{j-1}) 
\quad + c_1(x - x_0)^2 + c_2(x_n - x)^2 
\leq 4c_0 h^3 + c_0 h \left( \int_{|x-t|>h} |x-t| \, dt + O(h) \right) 
\quad + c_1(x - x_0)^2 + c_2(x_n - x)^2 
\leq k_1 h^3 + k_2 h^2 + k_3 (b-a)^2 
\leq kh^2 
\]
\[\Box\]

4. Numerical results

In this section, we compare the accuracy of the quasi-interpolation operator \( \mathcal{L}_{RTH} \) with that of Wu and Schaback, \( \mathcal{L}_{MQ} \) (defined in (3)) for the approximation of five functions. We take equidistant center points and choose different shape parameters \( c \) and also different step sizes \( h \). The maximum absolute error norm is then computed for comparing approximation accuracy. The rate of convergence is also computed by
\[
r_h = \frac{\ln \left( \frac{E_{h}}{E_{h_{i-1}}} \right)}{\ln \left( \frac{h}{h_{i-1}} \right)},
\]
where \( E_{h_i} \) indicates the error of the quasi-interpolant \( \mathcal{L}_{RTH} f \) corresponding to the parameter \( h_i \). In all tests, we chose \( m = 200 \) equidistant evaluation points.

4.1. Test problem 1

In the first test problem, we apply the RTH quasi-interpolation to approximate the function (cf. [8])
\[
f_1(x) = \frac{\sinh(x)}{1 + \cosh(x)}, \quad x \in [-3, 3].
\]
The results are shown in Tables 4-6. In Tables 4, 5, and 6, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|L_{\text{RTH}} f - f\|_{\infty}$ and $\|L_{\text{MQ}} f - f\|_{\infty}$. In Table 7, we set $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate $r_h$ of $L_{\text{RTH}} f$ with the variation of $h$.

Table 4: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 1.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$|L_{MQ} f - f|_{\infty}$</td>
<td>$9.3 \times 10^{-3}$</td>
<td>$3.1 \times 10^{-3}$</td>
<td>$1.1 \times 10^{-3}$</td>
<td>$3.8 \times 10^{-4}$</td>
<td>$2.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>$|L_{RTH} f - f|_{\infty}$</td>
<td>$2.9 \times 10^{-3}$</td>
<td>$6.2 \times 10^{-4}$</td>
<td>$7.1 \times 10^{-5}$</td>
<td>$2.3 \times 10^{-4}$</td>
<td>$2.4 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 5: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 1.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.02</th>
<th>0.01</th>
<th>0.005</th>
<th>0.002</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$|L_{MQ} f - f|_{\infty}$</td>
<td>$1.8 \times 10^{-4}$</td>
<td>$5.3 \times 10^{-5}$</td>
<td>$1.6 \times 10^{-5}$</td>
<td>$3.7 \times 10^{-6}$</td>
<td>$1.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>$|L_{RTH} f - f|_{\infty}$</td>
<td>$3.0 \times 10^{-5}$</td>
<td>$6.3 \times 10^{-6}$</td>
<td>$7.2 \times 10^{-7}$</td>
<td>$1.7 \times 10^{-9}$</td>
<td>$7.9 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

Table 6: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 1.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.002</th>
<th>0.001</th>
<th>0.0005</th>
<th>0.0002</th>
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<tr>
<td>$h$</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$|L_{MQ} f - f|_{\infty}$</td>
<td>$2.7 \times 10^{-6}$</td>
<td>$7.5 \times 10^{-7}$</td>
<td>$2.1 \times 10^{-7}$</td>
<td>$4.6 \times 10^{-8}$</td>
<td>$1.6 \times 10^{-8}$</td>
</tr>
<tr>
<td>$|L_{RTH} f - f|_{\infty}$</td>
<td>$3.0 \times 10^{-7}$</td>
<td>$6.3 \times 10^{-8}$</td>
<td>$7.2 \times 10^{-9}$</td>
<td>$1.7 \times 10^{-11}$</td>
<td>$1.1 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

Table 7: Convergence rates of $L_{\text{RTH}} f$ by using $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$ for the test problem 1.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
<td>0.025</td>
<td>0.0125</td>
</tr>
<tr>
<td>$|L_{RTH} f - f|_{\infty}$</td>
<td>$9.5 \times 10^{-4}$</td>
<td>$2.4 \times 10^{-4}$</td>
<td>$5.4 \times 10^{-5}$</td>
<td>$5.1 \times 10^{-6}$</td>
<td>$1.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>$r_h$</td>
<td>-</td>
<td>1.9855</td>
<td>2.1657</td>
<td>3.4056</td>
<td>2.3028</td>
</tr>
</tbody>
</table>

4.2. Test problem 2

In this experiment we apply the RTH quasi-interpolation to approximate the function (again considered in [8])

$$f_2(x) = \sin \left( \frac{x}{2} \right) - 2 \cos(x) + 4 \sin(\pi x), \quad x \in [-4, 4].$$  (15)

The comparison results are shown in Tables 8-10. In Tables 8, 9, and 10, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|L_{\text{RTH}} f - f\|_{\infty}$ and $\|L_{\text{MQ}} f - f\|_{\infty}$. In Table 11, we set $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate $r_h$ of $L_{\text{RTH}} f$ with the variation of $h$.

Table 8: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 2.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$|L_{MQ} f - f|_{\infty}$</td>
<td>$1.2$</td>
<td>$4.5 \times 10^{-1}$</td>
<td>$1.7 \times 10^{-1}$</td>
<td>$7.1 \times 10^{-2}$</td>
<td>$5.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$|L_{RTH} f - f|_{\infty}$</td>
<td>$4.5 \times 10^{-1}$</td>
<td>$1.2 \times 10^{-1}$</td>
<td>$1.4 \times 10^{-2}$</td>
<td>$4.5 \times 10^{-2}$</td>
<td>$4.9 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
For approximating by the RTH quasi-interpolation operator. The comparison results are shown in Table 12-14. In Tables 12, 13, and 14, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|L_{RTH}f - f\|_\infty$ and $\|L_{MQ}f - f\|_\infty$. In Table 15, we set $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate $r_h$ of $L_{RTH}f$ on varying $h$.

Table 12: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 3.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$|L_{MQ}f - f|_\infty$</td>
<td>$4.9 \times 10^{-1}$</td>
<td>$2.0 \times 10^{-1}$</td>
<td>$7.4 \times 10^{-2}$</td>
<td>$3.1 \times 10^{-2}$</td>
<td>$2.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$|L_{RTH}f - f|_\infty$</td>
<td>$2.2 \times 10^{-1}$</td>
<td>$5.5 \times 10^{-2}$</td>
<td>$6.4 \times 10^{-3}$</td>
<td>$2.0 \times 10^{-2}$</td>
<td>$2.1 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 13: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 3.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.02</th>
<th>0.01</th>
<th>0.005</th>
<th>0.002</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$|L_{MQ}f - f|_\infty$</td>
<td>$3.0 \times 10^{-2}$</td>
<td>$9.2 \times 10^{-3}$</td>
<td>$2.9 \times 10^{-3}$</td>
<td>$7.1 \times 10^{-4}$</td>
<td>$2.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>$|L_{RTH}f - f|_\infty$</td>
<td>$6.4 \times 10^{-3}$</td>
<td>$1.4 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-4}$</td>
<td>$3.7 \times 10^{-7}$</td>
<td>$1.7 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

Table 14: Comparison of approximation accuracy of RTH and MQ quasi-interpolation for the test problem 3.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.002</th>
<th>0.001</th>
<th>0.0005</th>
<th>0.0002</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$|L_{MQ}f - f|_\infty$</td>
<td>$2.1 \times 10^{-4}$</td>
<td>$6.0 \times 10^{-5}$</td>
<td>$1.8 \times 10^{-5}$</td>
<td>$3.9 \times 10^{-6}$</td>
<td>$1.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>$|L_{RTH}f - f|_\infty$</td>
<td>$2.8 \times 10^{-5}$</td>
<td>$5.9 \times 10^{-6}$</td>
<td>$6.7 \times 10^{-7}$</td>
<td>$1.6 \times 10^{-9}$</td>
<td>$7.5 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

4.3. Test problem 3

Consider the function (see again [8])

$$f_3(x) = 10e^{-x^2} + x^2, \quad x \in [-3, 3],$$

for approximating by the RTH quasi-interpolation operator. The comparison results are shown in Tables 12-14. In Tables 12, 13, and 14, we set $h = 0.1, 0.01, 0.001$, respectively, and $c = 2h, h, 0.5h, 0.2h, 0.1h$, then we compute the $\|L_{RTH}f - f\|_\infty$ and $\|L_{MQ}f - f\|_\infty$. In Table 15, we set $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$, to observe the convergence rate $r_h$ of $L_{RTH}f$ on varying $h$.

Table 15: Convergence rates of $L_{RTH}f$ by using $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$ for the test problem 2.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.02</th>
<th>0.01</th>
<th>0.005</th>
<th>0.002</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$|L_{MQ}f - f|_\infty$</td>
<td>$4.9 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-4}$</td>
<td>$4.1 \times 10^{-5}$</td>
<td>$9.0 \times 10^{-6}$</td>
<td>$3.3 \times 10^{-6}$</td>
</tr>
<tr>
<td>$|L_{RTH}f - f|_\infty$</td>
<td>$6.4 \times 10^{-5}$</td>
<td>$1.4 \times 10^{-5}$</td>
<td>$1.5 \times 10^{-6}$</td>
<td>$3.7 \times 10^{-9}$</td>
<td>$1.7 \times 10^{-13}$</td>
</tr>
</tbody>
</table>
Table 15: Convergence rates of $L_{RTH}f$ by using $c = 0.01, h = 0.2, 0.1, 0.05, 0.025, 0.0125$; Test Problem 3.

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
<td>0.025</td>
<td>0.0125</td>
</tr>
<tr>
<td>$|L_{RTH}f - f|_\infty$</td>
<td>$8.6 \times 10^{-2}$</td>
<td>$2.1 \times 10^{-2}$</td>
<td>$5.0 \times 10^{-3}$</td>
<td>$4.7 \times 10^{-4}$</td>
<td>$2.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>$r_h$</td>
<td>-</td>
<td>2.0085</td>
<td>2.0943</td>
<td>3.4210</td>
<td>2.0419</td>
</tr>
</tbody>
</table>

**Remark 4.1.** By analyzing the results in Tables 4-6, 8-10, and 12-14, we see that the accuracy of the $RTH$ quasi-interpolation scheme is dependent on the shape parameter $c$ and on step size $h$. Furthermore, the accuracy of the $RTH$ quasi-interpolation operator is better than that of MQ for the same values of $c$ and $h$. From Tables 7, 11, 15, we see that the convergence rate of $L_{RTH}$ reaches up to 2 which justifies our theoretical findings of Section 3. By these numerical experiments, we can say that the quasi-interpolation $L_{RTH}$ is a very attractive alternative, in terms of accuracy and convergence, to $L_{MQ}$.

4.4. Test problem 4 (Runge function)

Let us consider the Runge function on $[-1, 1]$, that is $f_4(x) = \frac{1}{1 + 25x^2}$. Figure 4 shows the exact and approximate values of $f_4$ for $c = 0.01, h = 0.1, 0.02$. In Figure 4, we see that the Runge phenomenon has disappeared by decreasing $h$. Relative errors are shown in Figure 5 using the RTH quasi-interpolation operator.

![Figure 4: RTH quasi-interpolation of $f_4(x) = \frac{1}{1 + 25x^2}$; $h = 0.1$ (a), $h = 0.02$ (b), and $c = 0.01$.](image)

![Figure 5: Relative errors: $c = 0.1$ (a), $c = 0.01$ (b), $c = 0.001$ (c), and $h = 0.02$; for the test problem 4.](image)
4.5. Test problem 5 (Gibbs Phenomenon)

It is well-known that any global or high order approximation method suffers from the Gibbs phenomenon if the function has a jump discontinuity in the given domain. In this test problem, we show that the RTH quasi-interpolation operator substantially mitigates the Gibbs phenomenon (cf. [5]).

\[ f_5(x) = \begin{cases} \frac{10}{3}x, & 0 \leq x \leq 0.3, \\ 1, & 0.3 \leq x \leq 0.6, \\ 0, & 0.6 < x \leq 1. \end{cases} \]

Figure 6 shows the exact and approximate values of \( f_5 \). In Figure 6, we see that the Gibbs oscillations are considerably attenuated by decreasing \( c \). Relative errors are reported in Figure 7.

![Figure 6](image1)

Figure 6: Approximations of \( f_5 \) with RTH quasi-interpolation; \( c = 0.1 \) (a), \( c = 0.01 \) (b), \( c = 0.001 \) (c), and \( h = 0.01 \).

![Figure 7](image2)

Figure 7: Relative errors: \( c = 0.1 \) (a), \( c = 0.01 \) (b), \( c = 0.001 \) (c), and \( h = 0.01 \); Test problem 5.

4.6. Test problem 6 (A piecewise analytic function)

As a final example, we consider the piecewise analytic function (cf. [20])

\[ f_6(x) = \begin{cases} \sin(x), & x < 0, \\ \cos(x), & x > 0, \end{cases} \]

with \( x \in [-1, 1] \). Figure 8 shows the exact and approximate values of \( f_6 \), where Gibbs oscillations are considerably attenuated by decreasing \( c \). Relative errors are shown in Figure 9.
5. Conclusion

In this paper, an efficient shape preserving quasi-interpolation operator with high degree of smoothness and very accurate results is proposed. It is based on the reformulation of Wu–Schaback’s quasi-interpolation operator by a new transcendental RBF of the form \( \phi(r) = r \tanh \left( \frac{r}{c} \right) \). The quasi-interpolation operator, called \( L_{\text{RTH}} \) has nice convergence properties, being \( \| L_{\text{RTH}} - f \|_\infty \leq k h^2 \), with \( h \) being the step size and \( k \) a positive constant independent on the shape parameter \( c \) and the step size \( h \) (cf. Theorem 3.1). Numerical experiments reveal that the proposed quasi-interpolation operator not only gives very accurate results but also it does not suffer of the Runge and Gibbs phenomena (see Test problems 4-6).

As a future work we are working in the application of the operator to real worlds problems, in particular to irregular surfaces approximation and image segmentation.

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References


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