On the constrained mock-Chebyshev least-squares

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\textbf{A B S T R A C T}

The algebraic polynomial interpolation on \(n+1\) uniformly distributed nodes can be affected by the Runge phenomenon, also when the function \(f\) to be interpolated is analytic. Among all techniques that have been proposed to defeat this phenomenon, there is the mock-Chebyshev interpolation which produces a polynomial \(P\) that interpolates \(f\) on a subset of \(m+1\) of the given nodes whose elements mimic as well as possible the Chebyshev–Lobatto points of order \(m\). In this work we use the simultaneous approximation theory to produce a polynomial \(\hat{P}\) of degree \(r\), greater than \(m\), which still interpolates \(f\) on the \(m+1\) mock-Chebyshev nodes minimizing, at the same time, the approximation error in a least-squares sense on the other points of the sampling grid. We give indications on how to select the degree \(r\) in order to obtain polynomial approximant good in the uniform norm. Furthermore, we provide a sufficient condition under which the accuracy of the mock-Chebyshev interpolation in the uniform norm is improved. Numerical results are provided.

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1. Introduction

In many scientific disciplines, when we want to study a phenomenon, we can start in observing and recording what happens at regular instants of time. This provides a sample of information that we can use to give a more or less accurate approximation of the observed phenomenon. For this aim mathematical tools are needful. The first step is to imagine regular instants of time as a set of uniform distributed points and the sample of information as the evaluations of an unknown function. In this case a classical technique, used to associate to the discrete set of experimental data a continuous approximation of the phenomenon, is the algebraic polynomial interpolation. This technique has the well-known drawback that on uniformly distributed nodes might not converge, even if the considered function is regular. A classical example is given by Runge’s function

\[ f(t) = \frac{1}{1+25t^2}, \quad t \in [-1, 1] \]

on an equally spaced triangular array of nodes

\[ x_{0,0}, x_{0,1}, x_{1,1}, x_{0,2}, x_{1,2}, x_{2,2}, \ldots; x_{0,n}, x_{1,n}, x_{2,n}, \ldots, x_{n,n}; \ldots \]

where \(x_{i,n} = -1 + \frac{2}{n}i\) for \(i = 0, 1, \ldots, n, n \in \mathbb{N}_0\). In this case, the error made by interpolating \(f\) with polynomials has wild oscillations, a phenomenon known as \textit{Runge Phenomenon}. Many techniques have been proposed to defeat this phenomenon; just to mention some of them, the least-squares fitting by polynomials [1], the barycentric rational interpolation [2–4], its
extended version [5], the interpolation on subintervals [6]. A further technique exploited to cut down the Runge phenomenon is the so called mock-Chebyshev subset interpolation, which takes advantages of the optimality of the interpolation processes on Chebyshev–Lobatto nodes [7]. The main goal of this paper consists in a combination of this kind of interpolation with a simultaneous least-squares procedure (throughout the paper, simultaneous regression) aimed to improve the accuracy of the approximation of an analytic function; we will refer to this combination as constrained mock-Chebyshev least-squares.

The paper is structured as follows. In Section 2 we discuss some details on the mock-Chebyshev subset interpolation. The constrained mock-Chebyshev least-squares are introduced in Section 3 and deeply investigated in Sections 4 and 5 in which we deal with the choice of the degree of the simultaneous regression and with an estimation of the error in the uniform norm, respectively. Section 6 is devoted to some numerical results. Last Section contains the algorithm.

2. Mock-Chebyshev subset interpolation

Let \( f \) be an analytic function with singularities close to the interval \([-1, 1]\) and suppose that its evaluations are known on \( n + 1 \) equally spaced points of that interval. The idea that underlies the mock-Chebyshev subset interpolation is to interpolate \( f \) only on a proper subset, consisting of \( m + 1 \) of the given nodes, which "looks like" the Chebyshev–Lobatto grid of order \( m + 1 \). The result is that if we carefully choose \( m \), the convergence of the interpolation process on such a subset of nodes, for \( n \) which tends to infinity, will be preserved (cf. [8]). Some notations: from here onwards we will indicate the equispaced grid of cardinality \( n + 1 \) with the symbol \( X_n \), while the mock-Chebyshev subset of \( X_n \) of order \( m + 1 \) will be denoted by \( X'_m \). To understand how to properly choose \( m \) (see e.g. [9]), let us remember that the \( m + 1 \) Chebyshev–Lobatto nodes are defined as

\[
x^{\text{CL}}_j = \cos \left( \frac{j \pi}{m+1} \right), \quad j = 0, 1, \ldots, m.
\]

Let us expand \( x^{\text{CL}}_1 \) in Taylor series centered in zero

\[
x^{\text{CL}}_1 = -1 + \frac{\pi^2}{2m^2} + O \left( \frac{1}{m^2} \right) \leq -1 + \frac{\pi^2}{2m^2}.
\]

Being \( x^{\text{CL}}_0 = -1 \), the difference \( x^{\text{CL}}_1 - x^{\text{CL}}_0 \) is a \( O \left( \frac{1}{m^2} \right) \). In other words, this means that the \( m + 1 \) nodes of Chebyshev–Lobatto are distributed in \([-1, 1]\) with a density that is roughly quadratic in \( m \). So for \( n \) proportional to \( m^2 \) or \( m \) proportional to \( \sqrt{n} \), we can select among the given nodes a subset which mimic a sufficiently large Chebyshev–Lobatto grid. Let \( c \) be the constant of proportionality; a way to calculate it is to impose that the second node of the Chebyshev–Lobatto grid is as close as possible to the second node of the equispaced set \( X_n \)

\[-\cos \left( \frac{\pi}{m} \right) \approx -1 + \frac{2}{n}.
\]

This can be done in the following manner: by (2.1) we fix the largest integer \( m \) such that

\[ -1 + \frac{1}{n} < -1 + \frac{\pi^2}{2m^2} \]

so for

\[ m = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor \]

for sure \(-1 + \frac{2}{n}\) is the point of \( X_n \) closest to \( x^{\text{CL}}_1 \) (for an example, see Fig. 1). This choice of \( c < \frac{\pi}{\sqrt{2}} \) avoids the fact that the endpoints \(-1 \) and \( 1 \) can be selected more than once.

For analytic functions the polynomial interpolation on Chebyshev nodes converges geometrically and stably. The mock-Chebyshev interpolation is a stable procedure, but its rate of convergence is subgeometric. In [10] it has been shown that on equispaced nodes no stable method can converge geometrically.

3. Constrained mock-Chebyshev least-squares

In performing the mock-Chebyshev interpolation we know the evaluations of \( f \) on the whole set \( X_n \), but actually we only use the information corresponding to the elements of \( X'_m \). Indeed, in [9] the \( n - m \) remaining nodes are definitively discarded and the corresponding evaluations are lost. Our idea is to use those nodes, whose set will be denoted by \( X''_{n-m} = \{ x''_{n-m}, x''_{n-m-1}, \ldots, x''_0 \} \), to improve the accuracy of the approximation through a simultaneous regression. More precisely, let \( f \) be an analytic function on \([-1, 1]\) and let \( P^r = \{ P \in P^r : P(x''_i) = f(x''_i), \ i = 0, 1, \ldots, m \} \) where \( P^r \) is the space of polynomials of degree \( \leq r \) and \( m < r \leq n \). We search for the solution of the following constrained least-squares problem [11–13]

\[
\min_{P \in P^r} \| f - P \|_2^2
\]

where \( \| \cdot \|_2 \) is the discrete 2-norm on \( X''_{n-m} \).
Theorem 3.1. The constrained least-squares problem (3.1) has a unique solution.

Proof. Let us denote by $P_{X'}$ the interpolating polynomial for $f$ on $X_m$. It is not difficult to verify that a generic polynomial $P \in \mathcal{P}_{r-m}$ is of the form $P(t) = P_{X'}(t) + Q(t)\omega_m(t)$ with $\omega_m(t) = \prod_{i=0}^{m-1}(t - x_i)$ and $Q(t)$ an arbitrary polynomial of degree $r-m-1$. The problem (3.1) then becomes

$$
\min_{Q \in \mathcal{P}_{r-m-1}} \|f - (P_{X'} + Q \omega_m)\|^2 = \min_{Q \in \mathcal{P}_{r-m-1}} \sum_{k=1}^{n-m} \left\{ f(x''_{k,n-m}) - P_{X'}(x''_{k,n-m}) - Q(x''_{k,n-m}) \omega_m(x''_{k,n-m}) \right\}^2
$$

By introducing the following discrete weighted 2-norm

$$
\|u\|_{2,\omega_m^2} = \left( \sum_{k=1}^{n-m} w_k u^2(x''_{k,n-m}) \right)^{\frac{1}{2}}
$$

where $w_k = \omega_m^2(x''_{k,n-m})$ for $k = 1, \ldots, n-m$ and by defining $\hat{f}$ as

$$
\hat{f}(t) := \frac{f(t) - P_{X'}(t)}{\omega_m(t)}, \quad t \in [-1, 1],
$$

the problem (3.1) can be reduced to the following classical least-squares problem

$$
\min_{Q \in \mathcal{P}_{r-m-1}} \left\| \hat{f} - Q \right\|_{2,\omega_m^2}^2
$$

which has a unique solution. \hfill \blacksquare

Denoting by $\hat{Q}_{X'}(t)$ the solution of (3.3), the desired polynomial approximant is

$$
\hat{P}_X(t) = P_{X'}(t) + \hat{Q}_{X'}(t)\omega_m(t).
$$

To write $\hat{P}_X$ explicitly, let us introduce the discrete inner product associated to the norm $\|\cdot\|_{2,\omega_m^2}$

$$
(u, v)_{\omega_m^2} = \sum_{k=1}^{n-m} w_k u(x''_{k,n-m})v(x''_{k,n-m})
$$

and let $\{\pi_i(t, \omega_m^2)\}_{i=0}^{r-m-1}$ be a basis of $\mathcal{P}_{r-m-1}$ orthogonal with respect to the previous product. We can express $\hat{Q}_{X'}(t)$ with respect to that basis as

$$
\hat{Q}_{X'}(t) = \sum_{i=0}^{r-m-1} q_i \pi_i(t), \quad q_i = \frac{(\hat{f}, \pi_i)_{\omega_m^2}}{(\pi_i, \pi_i)_{\omega_m^2}}.
$$
Then $\hat{P}_X(t)$ becomes explicitly
\[ \hat{P}_X(t) = P_X(t) + \left( \sum_{i=0}^{r-m-1} q_i \pi_i(t) \right) \prod_{i=0}^{m} (t - x_{i,m}). \]

**Theorem 3.2.** In the discrete 2-norm on $X''_{n-m}$ the inequality
\[ \| f - \hat{P}_X \|_2 < \| f - P_X \|_2 \]
holds.

**Proof.** The choice of an orthogonal basis for $\mathcal{P}^{r-m-1}$ allows us to express the error $\hat{f} - \hat{Q}_{X''}$ in the $\| \cdot \|_{2,\omega_m}$ norm as follows:
\[ \| \hat{f} - \hat{Q}_{X''} \|_{2,\omega_m} = \left\{ \| f - P_X \|_2^2 - \sum_{i=0}^{r-m-1} q_i^2 \| \pi_i \|_{2,\omega_m}^2 \right\}^{\frac{1}{2}}, \quad q_i = \frac{(f, \pi_i)_{\omega_m}}{\langle \pi_i, \pi_i \rangle_{\omega_m}}. \]

Therefore the error $f - \hat{P}_X$ in the 2-norm is
\[ \| f - \hat{P}_X \|_2 = \left\{ \| f - P_X \|_2^2 - \sum_{i=0}^{r-m-1} \tilde{q}_i^2 \| \pi_i \|_2 \right\}^{\frac{1}{2}}, \quad \tilde{q}_i = \frac{(f - P_X, \pi_i \omega_m)}{\langle \pi_i \omega_m, \pi_i \omega_m \rangle}. \]

In other words, the error made by using the constrained mock-Chebyshev least-squares method is, in the 2-norm, strictly smaller than the error produced if we perform only to the mock-Chebyshev subset interpolation.

### 4. The degree of simultaneous regression

As shown in the previous section we approximate the function $f$ with a least-squares polynomial that satisfies interpolation conditions on a mock-Chebyshev subset of the given nodes. We have not specified yet how to choose the degree of the constructed approximant $P_X$. When this degree increases up to the total number of nodes the approximation gets worse, since the combined approximant approaches the interpolating polynomial.

**Theorem 4.1.** Let $r$ be the degree of $\hat{P}_X$ and let us denote by $P_X$ the interpolating polynomial of $f$ on $X_n$. If $r = n$ then
\[ \hat{P}_X \equiv P_X. \]

**Proof.** Recalling that
\[ \hat{P}_X(t) = P_X(t) + \hat{Q}_{X''}(t) \omega_m(t), \]
if $\hat{P}_X$ is an $n$ degree polynomial, the regression polynomial $\hat{Q}_{X''}$ must be a $n - m - 1$ degree polynomial. Since the least-squares set $X''_{n-m}$ has cardinality $n - m$, $\hat{Q}_{X''}$ is the interpolating polynomial for $\hat{f}$ on $X''_{n-m}$ that is
\[ \hat{Q}_{X''}(x''_{k,n-m}) = \hat{f}(x''_{k,n-m}), \quad k = 1, \ldots, n - m. \]

From the previous relation, it follows that
\[ \hat{P}_X(x''_{k,n-m}) = P_X(x''_{k,n-m}) + \hat{Q}_{X''}(x''_{k,n-m}) \omega_m(x''_{k,n-m}) \]
\[ = P_X(x''_{k,n-m}) + \hat{f}(x''_{k,n-m}) \omega_m(x''_{k,n-m}) \]
\[ = P_X(x''_{k,n-m}) + \frac{\hat{f}(x''_{k,n-m}) - P_X(x''_{k,n-m})}{\omega_m(x''_{k,n-m})} \omega_m(x''_{k,n-m}) \]
\[ = \hat{f}(x''_{k,n-m}) \]

that is $\hat{P}_X$ interpolates $f$ on $X''_{n-m}$. However, by construction $\hat{P}_X$ interpolates also $f$ on $X'_m$, then it coincides with the interpolating polynomial for $f$ on $X_n$ by the uniqueness of the interpolating polynomial of degree $n$ on $X_n$. \[ \]
of \( q \) (internal) nodes of \([-1, 1]\)

\[
z_k = -1 + \frac{2k - 1}{q}, \quad k = 1, \ldots, q.
\]

(4.1)

the degree \( p \) of the least-squares polynomial should be selected so that there is a subset of cardinality \( p + 1 \) of the equispaced set which is close, in the mock-Chebyshev sense, to the \( p + 1 \) Chebyshev grid. Actually, the result presented in [14] is more general since it deals with the least-squares approximation of a function on a Jordan curve in the complex plane. To explain the outlines of Reichel’s idea we use his notation. Let \( \Gamma \) be a Jordan curve or Jordan arc in the complex plane and let \( \Omega \) be the open set bounded by \( \Gamma \). If \( \Gamma \) is a Jordan arc then \( \Omega \) is void. Let \( \{z_{k,q}\}_{k=1}^q \) be a set of \( q \) distinct nodes on \( \Gamma \). For a given function \( \varphi \) on \( \Gamma \), let \( I_{p,q} \varphi \) denote the least-squares polynomial of degree \( \leq p \) with respect to the semi-norm

\[
\| \varphi \| := (\varphi, \varphi)^{\frac{1}{2}}
\]

defined through the inner product

\[
(\varphi, \psi) := \sum_{k=1}^q \varphi(z_{k,q})\overline{\psi(z_{k,q})}.
\]

Moreover, let \( I_{p,q} \) be the interpolating polynomial of \( \varphi \) at \( p + 1 \) distinct points \( \{w_{k,p}\}_{k=0}^p \) on \( \Gamma \). We write \( I_{p} \prec I_{p,q} \) if

\[
\{w_{k,p}\}_{k=0}^p \subset \{z_{k,q}\}_{k=1}^q.
\]

We equip the domain and the range of \( I_{p,q} \) and \( I_p \) with the uniform norm on \( \Gamma \)

\[
\|\varphi\|_{\Gamma} = \sup_{z \in \Gamma} |\varphi(z)|
\]

and we denote the induced operator norm with the symbol \( \|\cdot\|_{\Gamma} \). Finally, we define

\[
E_p(\varphi) := \inf_{Q_{p,q} \in \Omega} \| \varphi - Q_p \|_{\Gamma}.
\]

The following theorem [14, Theorem 2.1] bounds the norm of the least-squares projection \( I_{p,q} \) in terms of the norm of the interpolation projection \( I_p \).

**Theorem 4.2.** Let \( I_{p,q} \) and \( I_p \) be defined on the set of continuous function on \( \Gamma \cup \Omega \) and analytic in \( \Omega \). Then

\[
\| I_{p,q} \| \leq \| I_p \| \left( 1 + \sqrt{q} \sup_{\|\varphi\|_{\Gamma} = 1} E_p(\varphi) \right), \quad \forall I_p \prec I_{p,q}, \forall q \geq p.
\]

(4.2)

By means of examples, it has been shown that also when \( p \) is fixed the \( \sqrt{q} \) growth of the right-hand side of (4.2) can be achieved. This suggests to make further assumptions on the distribution of the interpolation nodes and on the smoothness of the function. Generally, we will assume that \( p \) is an increasing function of \( q \). Using Jackson’s theorem [15, p. 147] the following corollary [14, Corollary 2.1] shows that additional smoothness of the function to be approximated decreases the growth of \( \| I_{p,q} \| \) with \( q, p(q) \).

**Corollary 4.3.** Let \( \Gamma' = [-1, 1] \) and let \( F_{d,k,\Gamma'} := \{ \varphi : \varphi \in C^k[-1, 1], \| \frac{d^k \varphi}{dx^k} \|_{\Gamma'} \leq d \} \) be the domain of \( I_{p,q} \). Then for some constant \( D \) depending on the constant \( d \) and on the integer \( k \)

\[
\| I_{p,q} \| \leq \| I_p \| \left( 1 + D\sqrt{q(p + 1)^{-k}} \right), \quad \forall I_p \prec I_{p,q}.
\]

The next step is to determine a bound for \( \min_{p \leq p,q} \| I_p \| \). We do not discuss in detail the estimates calculated for \( \| I_p \| \) in [14], but only mention that a useful bound for \( \min_{p \leq p,q} \| I_p \| \) is obtained when the interpolation points are Fejér points or points close to Fejér points. Let us recall that for a generic curve \( \Gamma \) the Fejér points are defined as the image on \( \Gamma' \) of the unit circle through a particular conformal mapping [14]. In particular, if \( \Gamma' = [-1, 1] \) the Chebyshev points are Fejér points [14, Example 3.1]. The estimates obtained for \( \| I_p \| \) in [14] suggest the following least-squares approximation method:

**Criterion 4.4.** Let \( \Gamma' = [-1, 1] \). Given a function \( \varphi \in F_{d,k,\Gamma'} \) and \( q \) least-squares nodes \( \{z_{k,q}\}_{k=1}^q \) on \( \Gamma \), choose the degree of the approximating polynomial \( I_{p,q} \varphi \) as the greatest \( p \) such that \( p + 1 \) points are close to \( p + 1 \) Fejér points.

When the \( q \) nodes are equispaced like in (4.1) this means that the degree \( p \) of the least-squares approximant should be selected so that there are \( p + 1 \) points among the equispaced ones which are close to the \( p + 1 \) Chebyshev nodes. In other words, \( p \) should be selected in the mock-Chebyshev sense.

In the case of simultaneous regression the least-squares nodes are those of \( X_{n-m}'' \) and therefore they are not equally spaced. However, when the cardinality of \( X_{n-m}'' \) is sufficiently large we can approximate an equispaced grid with width \( \geq 2h \), \( h = \frac{1}{n} \) using nodes belonging to \( X_{n-m}'' \). In fact, the maximum distance between two consecutive nodes of \( X_{n-m}'' \) is at most \( 2h \). To be aware of it, let us observe that the interval \( I = [x_{1,n-m}, x_{n-m,n-m}] \) according to the mock-Chebyshev extraction is
properly contained in $[-1, 1]$ and symmetric with respect to the origin. Because of the choice of $m$ the first and the second node of $X''_m$ are equal to $x_{0,n}$ and $x_{1,n}$, respectively, i.e. $X''_m = \{x_{0,n}, x_{1,n}, \ldots\}$. Moreover, we have

**Lemma 4.5.** The first three nodes of $X_n$ belong to $X''_m$, i.e.

$$X''_m = \{x_{0,n}, x_{1,n}, x_{2,n}, \ldots\}. \tag{4.3}$$

**Proof.** To prove that $x_{2,n}$ together with $x_{0,n}, x_{1,n}$ has been taken during the mock-Chebyshev extraction, we need to in Taylor series the difference between the second and the third Chebyshev–Lobatto node

$$x^{CL}_2 - x^{CL}_1 = -\cos \left( \frac{2\pi}{m} \right) + \cos \left( \frac{\pi}{m} \right) = -2 \sin \left( \frac{3\pi}{2m} \right) \sin \left( -\frac{\pi}{2m} \right) = 2 \frac{\pi}{2m} \frac{3\pi}{2m} + O \left( \frac{\pi^4}{m^4} \right) < 2 \frac{\pi}{2m} \frac{3\pi}{2m}.$$ 

Recalling that $m$ is given by (2.2) the previous difference can be rounded up by $\frac{3}{n}$ and the thesis follows (see Fig. 2).  

**Lemma 4.6.** For $n > 7$, $x_{3,n}$ does not belong to $X''_m$, i.e.

$$x_{3,n} \notin X''_{n-m}, \quad n > 7. \tag{4.4}$$

**Proof.** Let us expand $x^{CL}_3$ in Taylor series

$$x^{CL}_3 = -\cos \left( \frac{3\pi}{m} \right) = -1 + \frac{9\pi^2}{2m^2} - \frac{81\pi^4}{24m^4} + O \left( \frac{\pi^6}{m^6} \right) > -1 + \frac{9}{n} - \frac{27}{2n^2}$$

and check for which values of $n \in \mathbb{N}$ the following inequality holds

$$-1 + \frac{9}{n} - \frac{27}{2n^2} > -1 + \frac{7}{n}.$$ 

We obtain that

$$n > \frac{27}{4}$$

and therefore $|x^{CL}_3 + 1 - \frac{6}{n}| > |x^{CL}_3 + 1 - \frac{6}{n}|$.  

**Proposition 4.7.** For sufficiently large $n$ the following inequality

$$\max\left| x^{''}_{j-1,n-m} - x^{''}_{j,n-m} \right| \leq 2h$$

holds.

**Proof.** The thesis is equivalent to the fact that among the nodes of $X''_m$ belonging to $I = \left[ -1 + \frac{6}{n}, 1 - \frac{6}{n} \right]$ there are not two consecutive nodes of $X_n$. By Lemmas 4.5 and 4.6 the nodes of the $m + 1$ Chebyshev–Lobatto grid which are contained in $I$ are

$$x^{CL}_j = -\cos \left( \frac{\pi j}{m} \right), \quad j = 3, \ldots, m - 3.$$ 

It is well-known that the nodes (4.3) are more dense near the endpoints of $I$ and less near its center, therefore it is sufficient to verify that the distance between $x^{CL}_3$ and $x^{CL}_4$ is greater than $2h$. Let us expand in Taylor series $x^{CL}_4 - x^{CL}_3$

$$x^{CL}_4 - x^{CL}_3 = -\cos \left( \frac{4\pi}{m} \right) + \cos \left( \frac{3\pi}{m} \right) = -2 \sin \left( \frac{7\pi}{2m} \right) \sin \left( -\frac{\pi}{2m} \right)$$

$$= 2 \left( \frac{7\pi}{2m} - \left( \frac{7\pi}{2m} \right)^3 \frac{3}{6} + O \left( \frac{(7\pi)^5}{2m} \right) \right) \left( \frac{\pi}{2m} - \left( \frac{\pi}{2m} \right)^3 \frac{1}{6} + O \left( \frac{(\pi)^5}{2m} \right) \right)$$

$$= \frac{7\pi^2}{2m^2} - \frac{175\pi^4}{24m^4} + O \left( \frac{\pi^6}{m^6} \right).$$
round downward by
\[
\frac{7}{n} - \frac{175}{6n^2} < x_4^C - x_3^C
\]
and impose that
\[
\frac{4}{n} < \frac{7}{n} - \frac{175}{6n^2}.
\]
From the previous inequality it follows that
\[
n > \frac{175}{18} \simeq 9.72
\]
and the thesis holds.

At this point we can apply the results presented in [14] to the simultaneous regression. Taking into account that the grid
(4.1) is equispaced in \([-1 + \frac{1}{q}, 1 - \frac{1}{q}]\) with width \(\frac{2}{q}\), we note that, for \(n\) sufficiently large, we can approximate such a grid
with \(q = \frac{n}{6} = \frac{1}{3h}\) and nodes coming from \(X_{n-m}'\). We denote this grid with \(\tilde{X}_{n-m}'\). The choice for the degree of the simultaneous
regression which gives good approximation in the uniform norm is therefore
\[
p = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor. \tag{4.4}
\]
As a consequence, the degree \(r\) of the polynomial \(\hat{P}_X\) which gives, under appropriate conditions, good approximation to \(f\) is
\[
r = m + p = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor + \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor. \tag{4.5}
\]
The issue of the next section is to establish such conditions. The following proposition gives an upper and lower bound for \(r\).

**Proposition 4.8.** Let \(r\) as in (4.5), then
\[
\left(1 + \frac{1}{\sqrt{6}}\right)m - 2 < r < \left(1 + \frac{1}{\sqrt{6}}\right)(m + 1). \tag{4.6}
\]

**Proof.** The right-hand side of (4.6) is easily proved as follows
\[
r = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor + \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor \leq \left(1 + \frac{1}{\sqrt{6}}\right) \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor
\]
\[
< \left(1 + \frac{1}{\sqrt{6}}\right) \frac{\pi}{\sqrt{2}} \sqrt{n} \left(1 + \frac{1}{\sqrt{6}}\right) \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor
\]
\[
= \left(1 + \frac{1}{\sqrt{6}}\right)(m + 1).
\]
Similarly, the following inequalities prove the left-hand side of (4.6)
\[
r = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor + \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor - 1 \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor - 1
\]
\[
\geq \left(1 + \frac{1}{\sqrt{6}}\right) \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor - 2 > \left(1 + \frac{1}{\sqrt{6}}\right) \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor - 2
\]
\[
> \left(1 + \frac{1}{\sqrt{6}}\right)m - 2. \quad \blacksquare
\]
Let us observe that since the degree of the mock-Chebyshev interpolation and the degree of the regression are chosen in
the same way, we can obtain the previous result applying to \(X_{n-m}'\) the idea explained in [9], that is imposing that

\[
- \cos \left(\frac{\pi}{p}\right) \simeq -1 + \frac{6}{p}.
\]
It is a straightforward calculus to prove that \(p\) will be like in (4.4).
5. Uniform norm estimation

We have determined the degree $p$ as in (4.4) for the polynomial $\hat{Q}_c$ which, according to Reichel’s theory, gives good approximation in the uniform norm. Now, we want to calculate an estimation for the norm error $E_{\hat{P}_c}(f) = \|f - \hat{P}_c\|_\infty$ in the uniform norm. Let $\hat{P}_c : C[-1, 1] \to P^r$ the projection operator which associates to a continuous function in $[-1, 1]$ its constrained mock-Chebyshev polynomial and let $\hat{Q}_c : C[-1, 1] \to P^{r-m-1}$ the projection operator which associates to a continuous function in $[-1, 1]$ its least-squares polynomial in the norm $\|\cdot\|_{2,a_0^2}$. As in the proof of Theorem 4.2 and Corollary 4.3 we get an estimate for the operator norm $\|\hat{Q}_c\|$. 

**Theorem 5.1.** Let $\varphi \in C[-1, 1]$ and $I_p \varphi$ be the interpolating polynomial of $\varphi$ on the $p + 1$ mock-Chebyshev subset $X''_p = \{x''_{k,p}\}_{k=0}^p$ of $\hat{X}''_{n-m}$. Then

$$
\|\hat{Q}_c \varphi - \varphi\|_{2,a_0^2} \leq \|Q_p \varphi - \varphi\|_{2,a_0^2}.
$$

**Proof.** Let $Q_p \varphi$ be the polynomial of degree $\leq p$ such that $E_p(\varphi) \leq \|Q_p \varphi - \varphi\|_\infty$. By (3.3)

$$
\|\hat{Q}_c \varphi - \varphi\|_{2,a_0^2} \leq \|Q_p \varphi - \varphi\|_{2,a_0^2}.
$$

On the other hand,

$$
\|Q_p \varphi - \varphi\|_{2,a_0^2} = \left(\sum_{k=1}^{n-m} w_k \left(Q_p^* (x''_{k,n-m}) - \varphi(x''_{k,n-m})\right)^2\right)^{1/2}
$$

$$
\leq \left(\sum_{k=1}^{n-m} w_k\right)^{1/2} \|Q_p \varphi - \varphi\|_\infty
$$

$$
= \left(\sum_{k=1}^{n-m} w_k\right)^{1/2} E_p(\varphi).
$$

(5.1)

Let $l_k(t)$, $k = 0, \ldots, p$ be the elementary Lagrangian polynomials associated with $X''_p$, that is

$$
l_k(t) = \sum_{j=0}^{p} \varphi(x''_{j,p}) l_j(t).
$$

Let us express $\hat{Q}_c \varphi$ in the same basis as

$$
\hat{Q}_c \varphi(t) = \sum_{j=0}^{p} \alpha_j l_j(t),
$$

for some coefficients $\alpha_j$. From (5.1) it follows that

$$
\sqrt{\tilde{w}_j} |\alpha_j - \varphi(x''_{j,p})| \leq \|\hat{Q}_c \varphi - \varphi\|_{2,a_0^2} \leq \left(\sum_{k=1}^{n-m} w_k\right)^{1/2} E_p(\varphi), \quad j = 0, \ldots, p.
$$

where $\tilde{w}_j, j = 0, \ldots, p$ are the positive weights corresponding to the nodes $\{x''_{k,p}\}_{k=0}^p$ and then

$$
|\alpha_j - \varphi(x''_{j,p})| \leq \frac{\left(\sum_{k=1}^{n-m} w_k\right)^{1/2}}{\sqrt{\tilde{w}_j}} E_p(\varphi).
$$
Proof. Let us start from the following relations

\[
E_{\hat{f}}(f) = \left\| f - P_f f - \hat{Q}_{\chi''}\omega_m \right\|_\infty \\
= \left\| f - P_f f \omega_m - \hat{Q}_{\chi''} \left( f - P_f f \omega_m \right) \omega_m \right\|_\infty \\
\leq E_{\hat{Q}_{\chi''}} \left( f - P_f f \omega_m \right) \|\omega_m\|_\infty,
\]

where \(E_{\hat{Q}_{\chi''}} \left( f - P_f f \omega_m \right)\) is the uniform norm error made in approximating \(\hat{f}\) with its least-squares polynomial in the norm \(\|\cdot\|_{2,\omega_m}\). Since \(\hat{Q}_{\chi''}\) is a projection operator which reproduces the polynomials the following inequality holds

\[
E_{\hat{Q}_{\chi''}} \left( f - P_f f \omega_m \right) \leq \left( 1 + \left\| \hat{Q}_{\chi''} \right\| \right) E_p(\hat{f})
\]

where \(E_p(\hat{f}) = \min_{Q \in \mathcal{P}_p} \| \hat{f} - Q \|_\infty\). Therefore

\[
E_{\hat{f}}(f) \leq \left( 1 + \left\| \hat{Q}_{\chi''} \right\| \right) E_p(\hat{f}) \|\omega_m\|_\infty
\]

which applying Corollary 5.2 to \(f\) gives the thesis. ■
Theorem 5.3 gives a sufficient condition to improve in the uniform norm the accuracy of the mock-Chebyshev interpolation through the constrained mock-Chebyshev least-squares.

Corollary 5.4. Let \( f \in C^{m+1}[-1, 1] \). If

\[
\left( 1 + \| f \|_p \right) \left( 1 + D \left( \min_{j=0,\ldots,p} \sqrt{w_j} (p+1)^{-p} \right) \right) E_{\hat{P}}(\hat{f}) < \frac{\| f^{(m+1)} \|}{(m+1)!}
\]

then

\[
E_{\hat{P}_X}(\hat{f}) < E_{\hat{P}_{X'}}(f)
\]

where \( E_{\hat{P}_{X'}}(f) = \| f - P_{X'} \|_{\infty} \).

Proof. Let us recall that the error in the Lagrange interpolation can be bounded as follows

\[
E_{P_{X'}}(f) \leq \frac{\| f^{(m+1)} \|}{(m+1)!} \| \omega_m \|_{\infty}.
\]

From Theorem 5.3 we get the thesis. \( \blacksquare \)

Corollary 5.5. If \( f = p_r \) with \( p_r \in \mathcal{P}^{m+p} \), then

\[
\hat{P}_X f = f.
\]

Proof. If \( f = p_r \) with \( r \leq m \)

\[
\hat{f}(t) = \frac{p_r(t) - P_{X'}p_r(t)}{\omega_m(t)} = \frac{p_r^{(m+1)}(\xi_t)}{(m+1)!} \equiv 0, \quad \xi_t \in (-1, 1).
\]

If \( f = p_m \) with \( m < r \leq m + p \)

\[
\hat{f}(t) = \frac{p_r(t) - P_{X'}p_r(t)}{\omega_m(t)}
\]

is a polynomial of degree \( r - (m + 1) \). In both cases \( E_{\hat{P}}(\hat{f}) = 0 \) and the right-hand side of (5.3) is zero. \( \blacksquare \)

6. Numerical results

We finally carried out a series of numerical tests to compare, in the uniform norm, the approximation of the constrained mock-Chebyshev least-squares and the mock-Chebyshev interpolation. A first set of test functions includes the following ones (the first three functions were already considered in [16]):

\[
\begin{align*}
f_1(t) & = \sqrt{|t|}, \\
f_2(t) & = \frac{1}{1 + 25t^2}, \quad t \in [-1, 1], \\
f_3(t) & = \frac{10^{-15}}{10^{-15} + 25t^2}, \\
f_4(t) & = |t|,
\end{align*}
\]

The function \( f_1 \) is Hölder continuous with exponent 1/2, the function \( f_2 \) is a modification of \( f_2 \) obtained by introducing the exponential \( 10^{-15} \) in order to squash \( f_2 \) on \( x \) and \( y \) axes, the function \( f_4 \) is of class \( C^1 \). The errors are computed as the maximum absolute value of the difference between the approximant and the exact function at 10 001 equispaced points in \([-1, 1]\). Let us rename with \( p \) the degree of the simultaneous regression polynomial \( \hat{Q}_{X'} \). In Table 1 \( p \) ranges from \( p = 28 \) to \( p = 100 \). We denote with \( p^* \) the degree of the simultaneous regression which, according to the theory explained above, gives good approximation in the uniform norm. Table 1 allows to compare the two errors of interest in the case of \( n + 1 = 1001 \) equispaced interpolation nodes. At the top of the table, in green, is highlighted the error \( E_{\hat{P}}(\hat{f}) \) in correspondence of the degree \( p^* \). In red is highlighted the minimum possible error \( E_{\hat{P}_{X'}}(f) \) in the range \([1, n - m - 1]\). At the bottom, in blue, is represented the error \( E_{\hat{P}_{X'}}(f) \). As we can see, the constrained mock-Chebyshev least-squares improve the accuracy of the approximation.
Table 1
Comparison between $E_{\hat{P}}(f_i)$ and $E_{P X}(f_i)$ for $n = 1000$. In this case $m = 70$, $p^* = 28$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$E_{\hat{P}}(f_1)$</th>
<th>$E_{P X}(f_1)$</th>
<th>$E_{\hat{P}}(f_3)$</th>
<th>$E_{P X}(f_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>29</td>
<td>7.8950585e-002</td>
<td>8.5899644e-009</td>
<td>9.9994769e-009</td>
<td>5.4308526e-005</td>
</tr>
<tr>
<td>33</td>
<td>7.7268588e-002</td>
<td>6.2480174e-009</td>
<td>9.9994276e-009</td>
<td>4.8879707e-005</td>
</tr>
<tr>
<td>35</td>
<td>7.7593678e-002</td>
<td>7.886833e-009</td>
<td>9.9994378e-009</td>
<td>4.6558802e-005</td>
</tr>
<tr>
<td>37</td>
<td>7.6667394e-002</td>
<td>5.8488638e-009</td>
<td>9.999408e-009</td>
<td>4.8512907e-005</td>
</tr>
<tr>
<td>38</td>
<td>7.6667394e-002</td>
<td>5.8470333e-009</td>
<td>9.9994083e-009</td>
<td>5.0625425e-005</td>
</tr>
<tr>
<td>47</td>
<td>7.5926645e-002</td>
<td>7.2563035e-009</td>
<td>9.9993836e-009</td>
<td>7.866277e-005</td>
</tr>
<tr>
<td>49</td>
<td>7.6081471e-002</td>
<td>8.018418e-009</td>
<td>9.999392e-009</td>
<td>8.310850e-005</td>
</tr>
<tr>
<td>61</td>
<td>1.051466e-001</td>
<td>1.1889342e-008</td>
<td>9.9993920e-009</td>
<td>1.2342492e-004</td>
</tr>
<tr>
<td>99</td>
<td>3.5158374e-001</td>
<td>2.9978376e-008</td>
<td>3.0304570e+000</td>
<td>3.8933185e-004</td>
</tr>
<tr>
<td>100</td>
<td>3.5157737e-001</td>
<td>2.9977317e-008</td>
<td>3.0304057e+000</td>
<td>4.0022634e-004</td>
</tr>
</tbody>
</table>

of the mock-Chebyshev interpolation. We note that in correspondence of the degree $p^*$ we obtain an improvement of the accuracy of approximation. More in detail, for $f_1$ there is an interval for $p$ in which the approximation obtained with our method is better than the one coming from the mock-Chebyshev interpolation. In this case the improvement involves only the coefficients. When the function to be approximated is the Runge function, our approximation is everywhere more accurate for $p$ ranging from 1 to 100. In particular, there is a range for $p$ in which we get 2 digits of precision more than the mock-Chebyshev interpolation and $p^*$ lies in this range. For $f_2$ our approximation is, up to a certain value, better but almost the same of the approximation obtained with the mock-Chebyshev interpolation and then gets little worse. In the case of $f_3$ there is an interval for $p$ in which we get 1 digit of precision more than the mock-Chebyshev interpolation.

We have done further tests using the Runge function and the following ones:

$$f_5(t) = \frac{1}{t^2 - (1 + 0.5)},$$

$$f_6(t) = \frac{1}{t^4 + \left(\frac{\sqrt{5}}{5} - 1\right)t^2 + \left(\frac{13}{50}\right)^2}, \quad t \in [-1, 1],$$

$$f_7(t) = \frac{1}{t^4 + \left(\frac{2}{50}\right)^2},$$

which, as the Runge function, are analytic in the interval $[-1, 1]$. The function $f_5$ has poles at $\pm \sqrt{1 + 0.5}$, while the function $f_6$ has poles at $\pm \frac{1}{2} \pm i \frac{1}{10}$ and $\pm \frac{1}{2} \pm i \frac{1}{10}$, and the function $f_7$ has poles at $\pm \frac{1}{\sqrt{2}} \pm i \frac{1}{5 \sqrt{2}}$ and $\pm \frac{1}{\sqrt{2}} \pm i \frac{1}{5 \sqrt{2}}$.

Fig. 3(a) compares the errors for $f_2$. The error in the constrained mock-Chebyshev least-squares is, for every $30 \leq n \leq 3530$, smaller than the error in the mock-Chebyshev interpolation. The number $n = 3530$ is due to the fact that the constrained mock-Chebyshev least-squares method reaches order $10^{-15}$ on $n + 1 = 3531$ equispaced nodes. The accuracy of the mock-Chebyshev interpolation on the same set of nodes is of order $10^{-12}$. Fig. 3(b) shows how the errors vary for the function $f_3$ when $20 \leq n \leq 292$. Also in this case the approximation provided by the constrained mock-Chebyshev least-squares is more accurate than the one provided by the mock-Chebyshev interpolation and again when the accuracy of the former is of order $10^{-15}$ the accuracy of the latter is of order $10^{-11}$. Fig. 3(c) shows the errors behavior for the function $f_6$ when $40 \leq n \leq 924$ and the results are similar than in the previous cases. Finally, Fig. 3(d) compares the errors for $f_7$. In this case, the maximum order of precision that can be reached by the constrained mock-Chebyshev method is $10^{-12}$.

The remaining part of the present Section is devoted to the comparison of the constrained mock-Chebyshev method with some Radial Basis Functions (RBFs), Hermite Function interpolation (cf. [17]) and Floater–Hormann barycentric
Fig. 3. (a) Comparison between $E_{\hat{P}}(f_2)$ (lower curve) and $E_{P}(f_2)$ (upper curve) for $30 \leq n \leq 3530$. When $n = 3530$, $\deg(\hat{P}Xf_2) = m + p^* = 131 + 53$; (b) Comparison between $E_{\hat{P}}(f_5)$ (lower curve) and $E_{P}(f_5)$ (upper curve) for $20 \leq n \leq 292$. When $n = 292$, $\deg(\hat{P}Xf_5) = m + p^* = 67 + 15$; (c) Comparison between $E_{\hat{P}}(f_6)$ (lower curve) and $E_{P}(f_6)$ (upper curve) for $40 \leq n \leq 924$. When $n = 924$, $\deg(\hat{P}Xf_6) = m + p^* = 196 + 80$.

interpolation. A difference between these techniques and the constrained mock-Chebyshev least-squares is the structure of the approximation. Indeed, only the constrained mock-Chebyshev least-squares is based on polynomials, while the other approximants belong to other classes of functions.

**Constrained mock-Chebyshev method vs. RBF interpolation**

Given $n$ points $\xi_1, \ldots, \xi_n$ in $[-1, 1]$ (called centers) and the corresponding values $f_i$ of a given function $f$ on them, an RBF interpolant for $f$ takes the form

$$S(t) = \sum_{i=1}^{n} \lambda_i \phi (|t - \xi_i|)$$

where $\phi(r)$ is a function defined for $r \geq 0$. The $\lambda_i$ are determined, as usual, by imposing the interpolation conditions $S(\xi_j) = f_j$, $j = 1, \ldots, n$. Popular choices for $\phi(r)$ are (cf. [18]):

- $\phi(r) = |r|^{2m+1}$, Monomials (MN),
- $\phi(r) = (1 - r)^{\frac{1}{2}} (1 + 4r)^{\frac{1}{2}}$, Wendland (W2),
- $\phi(r) = \frac{1}{\sqrt{1 + (\epsilon r)^2}}$, Inverse Multiquadric (IMQ),
- $\phi(r) = \exp(- (\epsilon r)^2)$, Gaussian (G),

$\epsilon$ is known as *shape parameter* since as $\epsilon \to 0$ RBFs become flat, while $\epsilon \to \infty$ makes the RBFs spiky. The first two are parameter-free and piecewise smooth, while Inverse Multiquadratics and Gaussians are infinitely smooth and depend on $\epsilon$. Although we will numerically compare the constrained mock-Chebyshev method with the RBF interpolants associated to every choice of $\phi$ listed above, from a theoretical point of view we focus our attention on the Gaussian RBFs (GRBFs). In [19] it has been proved that, when $\epsilon \to 0$, smooth RBF interpolants converge on the polynomial interpolants on the same
nodes. This means that, in such a flat limit case, as the polynomial interpolation also the RBF approximation on uniform grids suffers of the Runge phenomenon. Furthermore, in [20] the author showed that the GRBFs on equally spaced nodes and fixed parameter diverge when interpolating functions that have poles in the Runge region of polynomial interpolation. A way to avoid the Runge phenomenon when interpolating with GRBF is to vary the shape parameter with \( n \). Indeed, as suggested in [21], if we define \( \alpha = \frac{2}{n+1} \) for \( \alpha = O\left(\frac{1}{\sqrt{n}}\right) \) the Runge phenomenon disappears. Such a choice has a drawback since, as \( n \to \infty \), the condition number of the interpolation matrix increases exponentially. Hence, the GRBFs can defeat the Runge Phenomenon just as the constrained mock-Chebyshev least-squares, but being ill-conditioned they can be used only on few nodes. Ill-conditioning, mainly due to the basis of translates, can be reduced significantly by using stable bases, as discussed in [22].

Fig. 4(a) shows that, in approximating the Runge function \( f_2 \), the constrained mock-Chebyshev least-squares are, for initial values of \( n \), less accurate than the RBFs interpolants, while, as \( n \) increases, they become more accurate. To have an idea of the discrepancy, while the constrained mock-Chebyshev least-squares reach order \( 10^{-15} \), the order of the RBFs interpolants for large \( n \) ranges from \( 10^{-7} \) to \( 10^{-9} \). Similar results are obtained for the function \( f_6 \) as shown in Fig. 4(b). In performing this numerical test, for every fixed \( n \), we have determined the shape parameter of IMQ and GRBFs using the so called Trial & Error technique which consists in varying \( \epsilon \) into a fixed (discrete) range and choosing the “optimal” parameter as the one that produces the minimum error. Unfortunately this method requires a lot of CPU time for finding the “optimal” shape parameter. Other techniques are also available, as those described in [18, Chapter 17], but for our purposes the Trial & Error was a suitable way to estimate the optimal \( \epsilon \).

Constrained mock-Chebyshev method vs. Hermite function interpolation

For a given function \( f \) the Hermite function interpolant on \( n \) points \( \xi_1, \ldots, \xi_n \) in \([-1, 1]\) can be expressed in the first barycentric form as

\[
H(t) = \Omega(t) \sum_{j=1}^{n} \frac{\mu_j}{t - \xi_j} f(\xi_j), \quad \Omega(t) = \exp\left(-\frac{n(n - 1)}{2} \log(4) t^2\right) \prod_{i=1}^{n} (t - \xi_i), \quad \mu_j = \left(\frac{d\Omega}{dt}(\xi_j)\right)^{-1}
\]

where \( \gamma \) is a free parameter (optimal choices are 1 or slightly smaller). As stated in [17], the computational cost of the previous formula is \( O(n^2) \) which means that the Hermite function interpolation is cheaper than the GRBF interpolation. Furthermore, in the same paper the authors give numerical evidence that the Hermite function interpolation is substantially more accurate than the GRBF interpolation. However, as RBFs, also this kind of interpolation is strongly ill-conditioned and therefore its use must be limited to a maximum of about 250 interpolation points. In fact, as shown in Fig. 5(a), in which we compare the constrained mock-Chebyshev least-squares with the Hermite interpolation for \( f_2 \), the Hermite interpolation is more accurate than the constrained mock-Chebyshev least-squares only on few nodes, while the constrained mock-Chebyshev method becomes more precise from \( n \approx 400 \) and it reaches the machine precision for \( n = 3530 \). Fig. 5(b) shows again how the ill-conditioning of the Hermite function interpolant limits its best attainable accuracy in approximating \( f_6 \).

Constrained mock-Chebyshev method vs. Floater–Hormann interpolation

A Floater–Hormann interpolant is a rational global approximant obtained by blending local interpolating polynomials. More precisely, given \( n + 1 \) distinct points \(-1 = x_0 < x_2 < \cdots < x_n = 1\) and fixed an integer \( d \) such that \( 0 \leq d \leq n \), a
Fig. 5. Comparison between $E_P(f_{2})(\cdot \ast)$ and the errors obtained in approximating $f_2 = \frac{1}{1 + 25t^2}$ with the Hermite function interpolant $(\cdot \cdot \cdot)$ for $20 \leq n \leq 3530$; Comparison between $E_P(f_{6})(\cdot \ast)$ and the error obtained in approximating $f_6 = \frac{1}{t^4(\frac{25}{2}t^2 - 1)^2 + \frac{1}{2}}$ with the Hermite function interpolant $(\cdot \cdot \cdot)$ for $40 \leq n \leq 924$.

Fig. 6. Comparison between $E_P(f_{7})(\cdot \ast)$ (upper curve), and the errors obtained in approximating $f_7 = \frac{1}{(\cdot \cdot \cdot)}$ with $(\cdot \cdot \cdot)$ (lower curve) for $20 \leq n \leq 7843$.

Floater–Hormann barycentric interpolant for $f$ can be written as

$$R(t) = \frac{\sum_{i=0}^{n-d} v_i(t) p_i(t)}{\sum_{i=0}^{n-d} v_i(t)}$$

where $p_i(t)$ is the polynomial of degree at most $d$ which interpolates $f$ in $x_i, \ldots, x_{i+d}$, $i = 0, \ldots, n - d$, while

$$v_i(t) = \frac{(-1)^i}{(t - x_i) \cdots (t - x_{i+d})}.$$  

This is a stable technique as confirmed by the study of the Lebesgue constant in [23]. Looking at Fig. 6, it is evident that, in approximating $f_7$, the Floater–Hormann interpolant reaches $10^{-12}$ on few nodes, but then stabilizes without gaining anymore precision. Such a limit seems to be related to the smoothness of the function and to the location of its poles within the Runge region. The error in the Floater–Hormann barycentric interpolation has been calculated using the Chebfun algorithms which for each value of $n$ choose the “best” blending parameter [24].

From previous comparisons we can conclude that the constrained mock-Chebyshev least-squares are a competitive polynomial strategy to defeat the Runge phenomenon. In this context, we can affirm that this method currently provides the best we can expect from polynomials.
7. Algorithm

Let us recall that, fixed $p$ as in (4.4), the polynomial $\hat{P}_X$ is given by

$$\hat{P}_X (t) = P_X (t) + \hat{Q}_{X'} (t) \omega_m (t)$$

where the polynomial $\hat{Q}_{X'}$ is the solution of the following least-squares problem

$$\min_{Q \in \mathcal{P}_p} \| f - P_{X'} - Q \omega_m \|_2^2 .$$

We can express the previous minimum problem in matrix-form as follows

$$\min_{c \in \mathbb{R}^{p+1}} \| Ac - b \|_2^2$$  \tag{7.1}

where $A = [\omega_m (x_m^i, n - m) \times (x_m^j, n - m)]_{i=1, \ldots, n-m}^{j=1, \ldots, p+1}$ is a real $(n - m) \times (p + 1)$ matrix, $c = [c_1, \ldots, c_{p+1}]^T$ is the vector of coefficients of $\hat{Q}_{X'}$ and $b = [P_X (x_m^i, n - m), \ldots, P_X (x_m^n, n - m)]^T$. Thus, the polynomial $\hat{P}_X$ can be computed using the following algorithm:

**Algorithm 1** Constrained mock-Chebyshev least-squares

**Input:** $X_n$, the set of $n + 1$ equispaced nodes in $[-1, 1]$ and the evaluations of $f$ at $X_n$

1. Determine the subset $X_m'$ of $X_n$ whose elements are the nearest to the $m + 1$ Chebyshev–Lobatto nodes and its complement $X_{n-m}';$
2. Compute the polynomial $P_{X'}$ of degree $m$ which interpolates $f$ on $X_m'$;
3. Compute the polynomial $\omega_m$;
4. Form the matrix $A$;
5. Solve $\min_{c \in \mathbb{R}^{p+1}} \| Ac - b \|_2^2$;

**Output:** $\hat{P}_X = P_{X'} + \hat{Q}_{X'} \omega_m$.

For the sake of better readability, in Algorithm 1 we have not specified that, when we deal with the computation of a polynomial (cf. Steps 2–3), we refer to its evaluations on a given array. To improve the performance of this algorithm we implemented Step 2 using the barycentric formula (cf. [25]). Such a formula is stable (cf. [26]) and its computational cost is $O(m^2) = O(n)$. The evaluations of $Q_{X'}$ and $\omega_m$ are performed using the Horner algorithm. Let us observe that Step 5 is the most expensive one. Since $A$ has full rank, if we solve (7.1) with the Householder QR factorization (which is a stable method) we need $2(n - m)(p + 1)^2 - 2(p + 1)^3 / 3$ flops (cf. [27]). Recalling that both $m$ and $p$ are proportional to $\sqrt{n}$, solving (7.1) requires $O(n^2)$ flops. Thus, the cost of the constrained mock-Chebyshev least-squares is $O(n^2)$.

8. Conclusion and perspective

In this work, we have combined the mock-Chebyshev interpolation with a simultaneous regression, to defeat the Runge Phenomenon for analytic functions with singularities close to the interval $[-1, 1]$. We have determined a degree for the simultaneous regression and a sufficient condition under which for such a degree the error of the constrained mock-Chebyshev method is, in the uniform norm, less than the error of the mock-Chebyshev interpolation. The proposed examples confirm that, in the uniform norm, the constrained mock-Chebyshev least-squares have better accuracy than the mock-Chebyshev interpolation. It might be interesting to extend this idea to the multivariate case on domains whose optimal distribution of nodes is known (cf. [28]).

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