Kernel-based Image Reconstruction from Scattered Radon Data by Positive Definite Functions

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Abstract Computerized tomography requires suitable numerical methods for the approximation of a bivariate target function \( f \) from a finite set of discrete Radon data, each of whose data samples represents one line integral of \( f \). In standard reconstruction methods, specific assumptions concerning the distribution of the sample lines are usually made, e.g. by parallel line geometry. In relevant applications of medical image reconstruction, however, such assumptions are often too restrictive. In this case, one would rather prefer to work with reconstruction methods allowing for arbitrary distributions of scattered sample lines.

This paper proposes a novel kernel-based algebraic reconstruction method for image reconstruction from scattered Radon data. Our reconstruction relies on generalized Hermite-Birkhoff interpolation by positive definite kernel functions in combination with a well-adapted regularization of the Radon transform. This leads to a very flexible image reconstruction method, which works for arbitrary distributions of Radon lines, unlike in classical Fourier-based methods relying on the filtered back projection formula. The good performance of the proposed kernel-based image reconstruction method is supported by numerical examples and comparisons.

Keywords Image reconstruction · computerized tomography · algebraic reconstruction techniques · Radon transform · positive definite kernels

1 Introduction

Computed Axial Tomography (CAT or CT) is a powerful technique to generate images from measurements of X-ray scans. One X-ray scan typically consists of several million of data samples, each of which corresponds to an X-ray beam passing through the computational domain, traveling from an emitter to a detector. The sensors of the operational CT scanner (positioned at the emitter and at the detector) then measures, for each X-ray beam, the loss of energy, resulting from the X-ray beam passing through the medium. The loss of energy reflects the ability of the medium to absorb energy, and so it depends on its specific structure and material properties. The amount of absorption can be described as a function of the computational domain \( \Omega \), termed attenuation coefficient function, \( f : \Omega \to [0, \infty) \).

Medical imaging is one relevant application for CT, where the primary goal is to reconstruct the unknown attenuation coefficient function \( f \) from given X-ray scans in order to generate clinically useful medical images. This requires robust numerical algorithms to reconstruct characteristic features of medical images at sufficiently high accuracy, on the one hand, and at sufficiently small computational costs, on the other hand. For details concerning the acquisition of X-ray scans, their underlying mathematical models, and standard computational methods for medical image reconstruction, we refer to the textbook [5] of Feeman.

To describe the mathematical problem of medical image reconstruction from X-ray scans, let us first re-
garding the Radon transform $Rf$ of $f$, given by
\[ Rf(t, \theta) = \int_{\mathbb{R}} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) \, ds \quad (1) \]
for $t \in \mathbb{R}$ and $\theta \in [0, \pi)$, where we assume $f : \Omega \to \mathbb{R}$ to be a bivariate function on a compact domain $\Omega \subset \mathbb{R}^2$, or, in other words, we assume that $f$ is compactly supported on $\mathbb{R}^2$, where we extend $f$ to $\mathbb{R}^2$ by letting $f \equiv 0$ outside $\Omega$. In many relevant application scenarios, we may assume that the image domain $\Omega$ is the unit square, i.e., $\Omega = [0,1]^2$, but this restriction is rather immaterial for our following discussion. We will merely assume $f \in L^1$, so that for any pair of $t \in \mathbb{R}$ and $\theta \in [0, \pi)$ the Radon integral in (1) is well-defined.

We remark that the Radon transform $Rf(t, \theta)$ gives, for any fixed pair $(t, \theta) \in \mathbb{R} \times [0, \pi)$, a line integral for $f$ over a specific straight line $\ell \equiv \ell_{t,\theta}$. In order to see this, let $\ell_{t,\theta} \subset \mathbb{R}^2$ denote the unique straight line, which is perpendicular to unit vector $n_0 = (\cos \theta, \sin \theta)$ and which passes through point $p = (t \cos \theta, t \sin \theta) = (n_0)$. In this case, the line $\ell_{t,\theta}$ can be parameterized as
\[(x_1(s), x_2(s)) = (t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta). \quad (2)\]
By this specific choice for a parameterization of $\ell_{t,\theta}$ in (2), we immediately see that, for $t \in \mathbb{R}$ and $\theta \in [0, \pi)$,
\[ \int_{\ell_{t,\theta}} f(x) \, dx = Rf(t, \theta), \]
where we let $x = (x_1, x_2)$, and so the line integral of $f$ over $\ell_{t,\theta}$ coincides with the Radon transform (1) of $f$ at $(t, \theta)$.

On the other hand, any straight line $\ell$ in the plane can be described by a unique pair $(t, \theta)$ of a radial parameter $t \in \mathbb{R}$ and an angular parameter $\theta \in [0, \pi)$ satisfying $\ell \equiv \ell_{t,\theta}$. In this way, the Radon transform $Rf$ of $f$ can be viewed as a transformation, which maps any bivariate function $f \in L^1$ (in Cartesian coordinates) onto a bivariate function $Rf$ (in polar coordinates), where the image $Rf$ contains all line integrals of $f$ over the set of straight lines in the plane.

Due to the seminal work [12] of Johann Radon (in 1917), any (sufficiently regular) function $f$ can be reconstructed from its Radon transform $Rf$. The inversion of the Radon transform is given by the filtered back projection (FBP) formula (see [5, Chapter 6]),
\[ f(x) = \frac{1}{2} \mathcal{B} \{ F^{-1} \left[ |S| F(Rf)(S, \theta) \right] \} (x), \quad (3) \]
where $\mathcal{F}$ is, for any fixed angle $\theta$, the univariate Fourier transform w.r.t. the radial variable $t$, and so is $F^{-1}$ the univariate inverse Fourier transform w.r.t. the frequency variable $S$. Moreover, the back projection $\mathcal{B}$ is, for any function $h \equiv h(t, \theta)$ (in polar coordinates), given by the average
\[ \mathcal{B}h(x) = \frac{1}{\pi} \int_0^\pi h(x_1 \cos \theta + x_2 \sin \theta, \theta) \, d\theta \]
of $h(t, \theta)$ over the angular variable $\theta$, where we let $t = x_1 \cos \theta + x_2 \sin \theta = x \cdot n_0$ according to the one-to-one relation between the Cartesian coordinates $x = (x_1, x_2)$ and the polar coordinates $(t, \theta)$, as described above along with the parameterization of the lines $\ell_{t,\theta}$ in (2). For basic details concerning the derivation of the filtered back projection formula, we refer to [5], and for a more comprehensive mathematical treatment of the Radon transform and its inversion, we refer to the textbooks [7,10].

In practical application scenarios, however, only a finite set of Radon data,
\[ \mathcal{R}_\mathcal{L}(f) = \{ Rf(t_k, \theta_k) \}_{k=1}^m, \quad (5) \]
given as integrals of $f$ over a finite set of $m$ pairwise distinct lines,
\[ \mathcal{L} = \{ \ell_{t_k,\theta_k} : (t_k, \theta_k) \in \mathbb{R} \times [0, \infty) \text{ for } k = 1, \ldots, m \}, \]
is available. In this case, an approximate reconstruction of $f$ from Radon data $\mathcal{R}_\mathcal{L}f$ is sought. In standard techniques of medical imaging, the reconstruction of $f$ is accomplished by using a suitable discretization of the FBP in (3). For this class of Fourier-based reconstruction methods, the discrete lines in $\mathcal{L}$, over which the line integrals of $f$ are known, are usually required to be regularly spaced in the plane, e.g. by assuming parallel beam geometry or fan beam geometry (see [5] for particular assumptions on the geometry of $\mathcal{L}$).

In many clinical scenarios of data acquisition, however, we may face a limited range of angles (e.g. in mammography), or a limited dosage of X-ray expositions, so that the Radon data are partly corrupt or incomplete. In such relevant cases, the Radon data in (5) are scattered, i.e., the distribution of lines in $\mathcal{L}$ is essentially not regular but scattered, in which case standard Fourier methods, such as the Fourier-based FBP discretization in (3), do no longer apply. This requires more flexible approximation methods which work for arbitrary geometries of (scattered) Radon lines $\mathcal{L}$.

To approximate $f$ from scattered Radon data $\mathcal{R}_\mathcal{L}f$, algebraic reconstruction techniques (ART) [6] can be applied. The concept of ART is essentially different from that of Fourier-based reconstructions: in the setting of ART one fixes a set $\mathcal{G} = \{ g_j \}_{j=1}^n$ of basis functions beforehand to solve the reconstruction problem
\[ \mathcal{R}_\mathcal{L}(g) = \mathcal{R}_\mathcal{L}(f) \quad (6) \]
by using a linear combination
\[ g = \sum_{j=1}^{n} c_j g_j \]
of the basis functions in \( \mathcal{G} \). This ansatz amounts to solving the linear system
\[ Ac = b \]
for the unknown coefficients \( c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \) of \( g \), where the \( m \times n \) matrix \( A \) has the form
\[ A = (Rg_j(t_k, \theta_k))_{k=1, \ldots, m; j=1, \ldots, n} \in \mathbb{R}^{m \times n} \]
and the right hand side \( b \) is given by the \( m \) Radon observations \( b_k = Rf(t_k, \theta_k) \), for \( k = 1, \ldots, m \).

Unless the number \( m \) of Radon samples coincides with the number \( n \) of coefficients, the linear system in (7) is either overdetermined, for \( m > n \), or underdetermined, for \( n > m \). In case of an overdetermined system, the classical method of linear least squares approximation [2] is applied to minimize the residual (Euclidean) norm \( \|Ac - b\| \), whereas for an underdetermined system the iterative method of Kaczmarz [5, Section 9.3] is a standard tool to compute an approximate solution \( c \) satisfying \( Ac \approx b \). We remark that in either case the linear system in (7) is not guaranteed to have a unique solution, not even in the rare case, where \( m = n \). In fact, the latter is due to the Mairhuber theorem [4, Chapter 1] from multivariate approximation theory.

In this paper, we propose a kernel-based algebraic reconstruction method, whose solution is always unique. To this end, we use positive definite kernel functions to obtain a square system of the form (7) with a symmetric and positive definite matrix \( A \), where the basis functions in \( \mathcal{G} = \{g_j\}_{j=1}^{n} \) do essentially depend on the given Radon functionals \( R \). Our proposed reconstruction scheme relies on the theory of kernel-based multivariate interpolation from generalized Hermite-Birkhoff data [8], where we adapt this particular interpolation scheme to the special case of reconstruction from scattered Radon data. But in order to ensure well-defined entries in the reconstruction matrix \( A \), we need to apply weighted Radon transforms for the purpose of regularization, as explained later in this paper. This way we obtain a very flexible reconstruction scheme, which works for arbitrary scattered Radon data.

The outline of this paper is as follows. In the following Section 2, we briefly review generalized Hermite-Birkhoff interpolation, where we show how to adapt this particular reconstruction method to scattered Radon data. This is followed by a discussion on the regularization of the Radon transform in Section 4, where we work with weighted Radon transforms. In Section 5, numerical results are finally provided for illustration. Our numerical experiments of Section 5 include comparisons between Fourier-based reconstructions relying on the FBP formula (3) and the kernel-based ART proposed in this paper, where we apply the two reconstructions methods to three popular phantoms. Moreover, we study their behaviour for noisy Radon data.

2 Generalized Hermite-Birkhoff Interpolation

To solve the reconstruction problem (6), we consider applying Hermite-Birkhoff interpolation [8]. To explain the general framework of this particular interpolation method, let \( A = \{\lambda_1, \ldots, \lambda_n\} \) denote a set of linearly independent linear functionals. Moreover, suppose we are given a vector \( f_A = (\lambda_1(f), \ldots, \lambda_n(f))^T \in \mathbb{R}^n \) of samples taken from an unknown function \( f \). Now the solution of the general Hermite-Birkhoff reconstruction problem requires finding a function \( g \) satisfying the interpolation conditions \( g_A = f_A \), i.e.,
\[ \lambda_k(g) = \lambda_k(f) \quad \text{for all } k = 1, \ldots, n. \] (8)

This general framework covers our reconstruction problem (6), when the linear functionals \( \lambda_k \) are defined as
\[ \lambda_k(f) := Rf(t_k, \theta_k) \quad \text{for } k = 1, \ldots, n. \] (9)

By the interpolation conditions in (8), we obtain \( n \) linear equations of the form
\[ \sum_{j=1}^{n} c_j \lambda_k(g_j) = \lambda_k(f) \quad \text{for } k = 1, \ldots, n, \]
corresponding to the linear system in (7), and by the choice of our particular ansatz, the number of data matches the number of basis functions, i.e., \( n = m \).

To obtain a well-posed interpolation scheme, we assume that the basis functions \( g_j \) are of the form
\[ g_j(x) = \lambda^x_j K(x, y) \quad \text{for } j = 1, \ldots, n, \] (10)
where \( \lambda^x_j K(x, y) \) denotes action of the functional \( \lambda_j \) to \( K \) w.r.t. variable \( y \in \mathbb{R}^2 \). Moreover, the kernel function \( K \) is required to be symmetric, i.e.,
\[ K(x, y) = K(y, x) \quad \text{for all } x, y \in \mathbb{R}^2, \]
and positive definite.

Rather than dwelling much on explaining positive definite functions, we remark that for the purposes of this paper it is sufficient to say that a symmetric function \( K \) is positive definite, iff the matrix
\[ A_{\Lambda,K} = (\lambda^x_j \lambda^y_k K(x, y))_{1 \leq j, k \leq n} \in \mathbb{R}^{n \times n} \] (11)
is symmetric positive definite for any set \( A = \{ \lambda_j \}_{j=1}^n \) of linearly independent functional values \( \lambda_j \). For a comprehensive account to the construction and characterization of positive definite kernels, we refer the reader to [9,13].

To make relevant examples for positive definite kernels, we resort to radially symmetric kernels, in which case \( K \) is assumed to have the form

\[
K(x, y) = \phi(\|x - y\|) \quad \text{for } x, y \in \mathbb{R}^2,
\]

where \( \phi(\cdot) \) is radial w.r.t. the Euclidean norm \( \| \cdot \| \).

Classical choices for radial kernels are the Gaussians

\[
\phi_\varepsilon(\|x\|) = e^{-(\varepsilon\|x\|)^2} \quad \text{for } x \in \mathbb{R}^2 \text{ and } \varepsilon > 0,
\]

which are for any \( \varepsilon > 0 \) positive definite, i.e.,

\[
K(x, y) = \exp(-\varepsilon\|x - y\|^2)
\]

is positive definite. Other popular examples for radial positive definite kernels are the inverse multiquadrics

\[
\phi_\varepsilon(\|x\|) = (1 + (\varepsilon\|x\|)^2)^{-1/2} \quad \text{for } x \in \mathbb{R}^2 \text{ and } \varepsilon > 0,
\]

and Askey’s compactly supported radial characteristic functions [1],

\[
\phi_\varepsilon(\|x\|) = (1 - \varepsilon\|x\|)^\beta \quad \text{for } \|x\| < 1/\varepsilon \\
0 \quad \text{for } \|x\| \geq 1/\varepsilon
\]

which are positive definite for any \( \beta > 3/2 \).

In the following discussion, we show that Gaussian kernels are suitable choices for \( \phi_\varepsilon \), especially for reconstructions from Radon data, where we need to assemble the reconstruction matrix \( A \equiv A_{\Lambda,K} \) in (11) by applying the Radon transform \( R \) on the chosen kernel \( K \).

Our preference for the Gaussian kernel is also supported by our extensive numerical experiments in [14], where we have observed that the approximation quality of Gaussian reconstruction is superior to that of other commonly used positive definite kernels. This complies with available error estimates for (plain) Lagrange interpolation by Gaussian kernels [15, Section 11.3], which are shown to provide arbitrary convergence orders.

3 Construction of Gaussian Basis Functions

Let \( K(x, y) = \phi_\varepsilon(\|x - y\|) \) denote the Gaussian kernel in (12). To discuss the assembly of the reconstruction matrix \( A \) in (11), let us first compute the Gaussian basis functions \( g_j(x) = R Y K(x, y)(t_j, \theta_j) \) in (10).

To this end, note that \( \phi_\varepsilon(\tau) \in L^1([0, \infty)) \), and so the Radon transform

\[
RY K(x, y)(t, \theta) = \int_\mathbb{R} K(x, y(s, t, \theta)) \, ds
\]

where \( y(s, t, \theta) = (t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta)) \), is for any combination of \((t, \theta) \in \mathbb{R} \times [0, \pi) \) and \( x \in \mathbb{R}^2 \) well-defined, i.e., the Gaussian basis functions \( g_j \) are well-defined. Note that this is in contrast to reconstruction by inverse multiquadrics, where the integral in (13) is singular for any \((t, \theta) \in \mathbb{R} \times [0, \pi) \) and \( x \in \mathbb{R}^2 \).

For the following of our analysis it will be convenient to use the shift property of the Radon transform.

**Lemma 1** Let \( K(x, y) = \phi(\|x - y\|) \) with \( \phi \in L^1(\mathbb{R}) \).

Then, for any \( x \in \mathbb{R}^2 \) the Radon transform \( \mathcal{R}^x K(x, y) \) at \((t, \theta) \in \mathbb{R} \times [0, \pi) \) can be expressed as

\[
\mathcal{R}^x K(x, y)(t, \theta) = (\mathcal{R} K(0, y))(t - x \cdot n_\theta, \theta),
\]

where \( n_\theta = (\cos \theta, \sin \theta) \).

Although this is a rather standard fact about Radon transforms [7], we provide a short proof of this property.

**Proof** First note that identity (14) can be rewritten as

\[
\int_{t,\theta} \phi(\|y - x\|) \, dy = \int_{t-x \cdot n_\theta} \phi(\|z\|) \, dz.
\]

To see this is true, further note that \( y = (y_1, y_2) \in \ell_{t,\theta} \) implies

\[
t = y \cdot n_\theta = y_1 \cos \theta + y_2 \sin \theta
\]

from (4). But this in turn yields \( y - x \in \ell_{t-x \cdot n_\theta,\theta} \), since

\[
t - x \cdot n_\theta = y \cdot n_\theta - x \cdot n_\theta = (y - x) \cdot n_\theta.
\]

In other words, for \( y \in \ell_{t,\theta} \), we see that \( \ell_{t-x \cdot n_\theta,\theta} \) is, for fixed \( \theta \), the unique straight line passing through \( y - x \).

Now we are in a position to compute the Gaussian basis functions \( g_j = \mathcal{R}^x y K(x, y)(t_j, \theta_j) \) in (10).

**Theorem 1** For any \((t, \theta) \in \mathbb{R} \times [0, \pi) \), the Gaussian basis function \( g \equiv g(t, \theta) \) is given as

\[
g(x) = (\mathcal{R}^x Y K(x, y))(t, \theta) = \frac{\sqrt{\pi}}{\varepsilon} e^{-\varepsilon^2(t - x \cdot n_\theta)^2},
\]

where \( n_\theta = (\cos \theta, \sin \theta) \).

**Proof** For the Radon transform \( \mathcal{R} K_0 \) of \( K_0(y) = K(0, y) \) at \((t, \theta) \in \mathbb{R} \times [0, \pi) \) we get

\[
\mathcal{R} K_0(t, \theta) = \int_{\ell_{t,\theta}} K_0(y) \, dy
\]

where

\[
K_0(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) \, ds
\]

\[
= \int_{\mathbb{R}} e^{-\varepsilon^2(t \cos \theta - s \sin \theta)^2 - \varepsilon^2(t \sin \theta + s \cos \theta)^2} \, ds
\]

\[
= \int_{\mathbb{R}} e^{-\varepsilon^2(t^2 + s^2)} \, ds = e^{-\varepsilon^2 t^2} \int_{\mathbb{R}} e^{-\varepsilon^2 s^2} \, ds
\]

\[
= \frac{\sqrt{\pi}}{\varepsilon} e^{-\varepsilon^2 t^2}.
\]
which in turn yields the Gaussian basis function
\[
(R^c K(x, y))(t, \theta) = (RK_0)(t - x \cdot n_\theta, \theta) = \frac{\sqrt{\pi}}{\varepsilon} e^{-t^2(x - n_\theta)^2}
\]
from the Radon transform’s shift property, Lemma 1.

Now let us turn to the entries of the reconstruction matrix \( A = (a_{kj})_{1 \leq j, k \leq n} \) in (11). Note that the matrix entry \( a_{kj} \) is given by the application of the Radon transform \( R \equiv R(t_k, \theta_k) \) to the basis function \( g_j = R^c K(\cdot, y) (t_j, \theta_j) \), so that
\[
a_{kj} = R^c[(R^c K(x, y))(t_j, \theta_j)](t_k, \theta_k).
\]

We can compute the entries of matrix \( A \) as follows.

**Theorem 2** For the Gaussian kernel \( \phi_\varepsilon \), the entries (16) of the reconstruction matrix \( A \) are given as
\[
a_{kj} = \frac{\pi}{\varepsilon^2 \sin(\theta_k - \theta_j)}
\]

**Proof** We let \((t, \theta) := (t_j, \theta_j)\) and \((r, \varphi) := (t_k, \theta_k)\) for the sake of notational simplicity. Recalling the form of the Gaussian basis function in (15), we can compute the entry \( a_{kj} \) by
\[
R^c \left[ e^{-t^2(x - n_\theta)^2} \right] (r, \varphi) = \frac{\pi}{\varepsilon} \int_R e^{-|t - r\cos \varphi - s\sin \varphi|^2 - (r \sin \varphi + s \cos \varphi)^2} ds
\]
\[
= \frac{\pi}{\varepsilon} \int_R e^{-|t - r\cos \varphi - s\sin \varphi|^2} ds
\]
\[
= \frac{\pi}{\varepsilon} \int_R e^{-|a + b|^2} ds,
\]
where we let \( a := \sin(\varphi - \theta) \) and \( b := t - r \cos(\varphi - \theta) \). Hence, for \( a \neq 0 \), we get
\[
a_{kj} = \frac{\pi}{\varepsilon} \int_R e^{-\varepsilon^2(t - x_n \theta)^2} ds
\]
\[
= \frac{\pi}{\varepsilon} \int_R e^{-\varepsilon^2|a + b|^2} ds = \frac{\pi}{\varepsilon^2 a}
\]
\[
= \frac{\pi}{\varepsilon^2 \sin(\varphi - \theta)}
\]
as stated. \( \square \)

Note that the above entries \( a_{kj} \) do not depend on the radial parameters, \( t_j \) and \( t_k \), but only on the angular parameters, \( \theta_j \) and \( \theta_k \). This is due to the radial symmetry of the Gaussian Kernel \( K \).

For different angles \( \theta_j, \theta_k \in [0, \pi] \), \( \theta_j \neq \theta_k \), the entry \( a_{kj} \) of \( A \) in (17) is well-defined. However, for coincident angles \( \theta_j = \theta_k \), the entry \( a_{kj} \) is singular, i.e.,

\[
4 \text{ Regularization by Gaussian Weights}
\]

To obtain non-singular entries in the reconstruction matrix \( A \) in (11), we require suitable regularizations for the Radon transform \( R \). To this end, we work with weighted Radon transforms of the form
\[
R_w f(t, \theta) := R(f w)(t, \theta) = \int_{t \in \ell} f(x) w(x) dx
\]
where \( w \equiv w(x) \) is a fixed weight function. This yields a standard regularization for the Radon transform \( R \), which also arises in relevant applications of single photon emission computed tomography (SPECT), where \( R_w \) is called attenuated Radon transform [11].

As for the choice of the weight \( w \), we prefer to use the radially symmetric Gaussian function
\[
w_\nu(x) = \exp(-\nu \|x\|^2) \quad \text{for } \nu > 0.
\]

This particular choice is well-motivated, since the product \( \phi_\varepsilon \cdot w_\nu \) of the two positive definite Gaussians, \( \phi_\varepsilon \) and \( w_\nu \), is positive definite, due to the classical Bochner theorem [3]. Indeed, according to Bochner’s theorem, any function \( f \in L^1 \) with positive Fourier transform \( \hat{f} \) is positive definite. Hence, for any pair of functions \( f, g \in L^1 \) with positive Fourier transforms \( \hat{f}, \hat{g} \), their product \( f * g \) is positive definite, since their convolution \( f * \hat{g} \) (giving their Fourier transform) is positive.

In this way, the resulting reconstruction problem
\[
R_w g(t_k, \theta_k) = R f(t_k, \theta_k) \quad \text{for all } k = 1, \ldots, n,
\]
assuming (10) for the form of the Gaussian basis functions, leads to a well-posed reconstruction scheme with a positive definite reconstruction matrix
\[
A \equiv A_w = (R_w g_j(t_k, \theta_k))_{1 \leq j, k \leq n} \in \mathbb{R}^{n \times n}.
\]

We compute the entries of \( A \) analytically as follows.

**Theorem 3** For the Gaussian kernel \( \phi_\varepsilon \), \( \varepsilon > 0 \), and the Gaussian weight \( w_\nu \), \( \nu > 0 \), the entries (20) of the reconstruction matrix \( A_w = (a_{kj})_{1 \leq j, k \leq n} \) are given as
\[
a_{kj} = \pi \frac{\varepsilon}{\varepsilon^2 \alpha_{kj}^2 + \nu^2} \exp \left[ -\nu^2 \left( \frac{t_k^2}{\alpha_{kj}^2 + \nu^2} + \frac{\varepsilon^2 \beta_{kj}^2}{\varepsilon^2 \alpha_{kj}^2 + \nu^2} \right) \right],
\]
where \( \beta_{kj} = t_k - t_j \cos(\theta_k - \theta_j) \) and \( \alpha_{kj} = \sin(\theta_k - \theta_j) \).
Proof For notational convenience, we let \((t, \theta) = (t_j, \theta_k)\) and \((r, \varphi') = (t_k, \theta_k)\). Recalling the representation of the Gaussian basis function \(g \equiv g(t, \theta)\) in Theorem 1, we can compute the weighted Radon transform \(R_wg\) by following along the lines of our proof in Theorem 2, which yields
\[
(R_w g) (r, \varphi') = \int_{\mathbb{R}} g(x(s)) e^{-\nu^2 |x|^2} \, ds
\]
\[
= \frac{\sqrt{\pi}}{\nu} \int_{\mathbb{R}} e^{-\nu^2 (a^2 + b^2)} e^{-\nu^2 (r^2 + s^2)} ds,
\]
where we let \(a = \sin(\varphi - \theta)\) and \(b = t - r \cos(\varphi - \theta)\), and where we have used the identity \(\|x(s)\|^2 = r^2 + s^2\) for \(x(s) = (r \cos \varphi - s \sin \varphi, r \sin \varphi + s \cos \varphi)\).

This immediately yields the entries of \(A_w\) by
\[
(R_w g)(r, \varphi') = \frac{\sqrt{\pi}}{\nu} e^{-\nu^2 t^2 - \nu^2 r^2} \int_{\mathbb{R}} e^{-\nu^2 a^2 + \nu^2 b^2} ds
\]
\[
= \frac{\sqrt{\pi}}{\nu} e^{-\nu^2 t^2 - \nu^2 r^2} + \frac{\nu^2 a^2 b^2}{\nu^2 a^2 + \nu^2 b^2} \int_{\mathbb{R}} e^{-\nu^2 a^2 + \nu^2 b^2} ds
\]
\[
= \frac{\pi}{\sqrt{\nu^2 a^2 + \nu^2 b^2}} e^{-\nu^2 (r^2 + \frac{\nu^2 a^2}{\nu^2 a^2 + \nu^2 b^2})}.
\]
\(\square\)

Note that for any \(\nu > 0\), the entries \(a_{kj}\) of \(A_w\) are well-defined, even for coincident angles \(\theta_j\) and \(\theta_k\). Indeed, for \(\theta_j = \theta_k\), we have \(a_{kj} = 0\), \(\beta_{kj} = t_k - t_j\), and so in this case
\[
a_{kj} = \frac{\pi}{\nu^2} \exp\left[-\nu^2 \left(\frac{\nu^2 a^2}{\nu^2 a^2 + \nu^2 b^2} + \frac{\nu^2 a^2}{\nu^2 a^2 + \nu^2 b^2}\right)\right].
\]

5 Numerical Results

We have implemented the proposed kernel-based reconstruction method, relying on the Gaussian kernel \(\phi_\nu\) and on the Gaussian weight function \(w_\nu\). To illustrate the performance of our reconstruction method, the following subsections include numerical results concerning

- comparisons with Fourier-based reconstructions;
- reconstructions from scattered Radon data;
- reconstructions from noisy Radon data.

In our previous investigations [14], more comprehensive numerical results and comparisons are presented, where different combinations of positive definite kernels \(\phi_\nu\) and weights \(w_\nu\) are used. According to our numerical comparisons in [14], the approximation behaviour of the Gaussian kernel is superior to that of the inverse multiquadrics and the radial characteristic functions. This complies with well-known approximation results concerning plain Lagrange interpolation, where Gaussian kernels lead to accuracy rates of arbitrary order (see [15, Section 11.3] for details).

### 5.1 Kernel-based vs Fourier-based Reconstruction

We compare the proposed kernel-based Gaussian reconstruction method with Fourier-based reconstructions, relying on the filtered back projection formula (3). To this end, we consider using three popular test cases of phantoms: the crescent-shaped phantom, the bull’s eye, and the Shepp-Logan phantom, as displayed in Figure 1.

In the performance of our comparisons, we need to work with regular distributions of Radon lines. This restriction is due to the Fourier-based method, whose implementation relies on a discretization of the FBP (3), requiring regular distributions of Radon lines.

To this end, we decided to work with a parallel beam geometry. In this case, by the discretization \(\theta_k = k\pi/N\), for \(k = 0, \ldots, N - 1\), of the angular variable \(\theta \in [0, \pi]\) and \(t_j = jd\) (for some fixed sampling spacing \(d > 0\)), where \(j = -M, \ldots, M\), for the radial variable \(t \in \mathbb{R}\), the data acquisition relies on \((2M + 1) \times N\) regularly distributed Radon lines \(t_{ij}, \theta_k\). Note that for any fixed angle \(\theta_k\), the Radon lines \(t_{ij}, \theta_k\), \(-M \leq j \leq M\), are parallel (at uniform distance \(d\) and symmetric about the origin), which explains the naming parallel beam geometry. For illustration, Figure 2 shows a set of \((2M + 1) \times N = 110\) Radon lines \(t_{ij}, \theta_k\) on parallel beam geometry, where we let \(N = 10\), \(M = 5\), and \(d = 0.2\).

![Fig. 2 Radon lines on parallel beam geometry. A set of 110 Radon lines \(t_{ij}, \theta_k\) is shown, using increments of \(\pi/10\) for the radial variable \(\theta_k\), i.e., \(N = 10\), with \(2M + 1 = 11\) different Radon lines per angle, letting \(t_j = jd\), for \(j = -5, \ldots, 5\), at sampling spacing \(d = 0.2\).](image)
In each test case, we let \( N = 45 \) and \( M = 40 \), so that \((2M + 1) \times N = 3645\) Radon samples are taken. In the kernel-based reconstruction, we let \( \varepsilon = 60 \) for the shape parameter of the Gaussian kernel \( \phi_{\varepsilon} \). For the weight parameters of the Gaussian weight \( w_{\nu} \), we have chosen \( \nu = 0.5 \) (for the crescent-shaped phantom), \( \nu = 0.4 \) (for bull’s eye), and \( \nu = 1.1 \) (for the Shepp-Logan phantom). The resulting reconstructions obtained from the two methods are displayed in Figure 3.

To measure the approximation quality of the reconstruction, we use the standard root mean square error

\[
\text{RMSE} = \sqrt{\frac{1}{J} \sum_{j=1}^{J} (f_j - g_j)^2},
\]

where \( J \) is the size of the image (i.e., the number of pixels) and \( \{f_j\}_{j=1}^{J}, \{g_j\}_{j=1}^{J} \) are the greyscale values at the pixels of the original image \( f \) and of the reconstructed image \( g \), respectively. The corresponding RMSE for the three test cases are shown in Table 1.

Given our numerical results, we can conclude that our kernel-based reconstruction method is competitive to the Fourier-based reconstruction method, by the visual quality of their reconstructions and by the RMSE.

### Table 1 Kernel-based vs Fourier-based reconstruction.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>Kernel-based</th>
<th>Fourier-based</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.10215</td>
<td>0.12053</td>
</tr>
<tr>
<td>bull’s eye</td>
<td>0.14233</td>
<td>0.13396</td>
</tr>
<tr>
<td>Shepp-Logan</td>
<td>0.15932</td>
<td>0.17661</td>
</tr>
</tbody>
</table>

5.2 Reconstruction from Scattered Radon Data

Let us turn to the reconstruction from scattered Radon data. In our numerical experiments, we have generated sets of unstructured Radon lines by random, so that the geometry of the sample lines is unstructured. Figure 4 shows one example for a set of 170 randomly distributed Radon lines in the computational domain.

We applied our kernel-based reconstruction method to a sequence of \( n = 2000, 5000, 10000, 20000 \) randomly chosen scattered Radon lines, taken from the phantoms bull’s eye and crescent-shaped.

Figure 5 shows the obtained sequences of reconstructions, where in each test case we have used fixed shape parameters \( \varepsilon = 50 \) (for the Gaussian kernel \( \phi_{\varepsilon} \)) and \( \nu = 0.7 \) (for the Gaussian weight \( w_{\nu} \)). The corresponding RMSE values are shown in Table 2.

Note that the geometry of the two phantoms are captured very well by our kernel-based reconstruction method. Although the geometry of the scattered Radon lines are visible in the displays of Figure 5, the basic features of the two phantoms are captured very well, where the reconstruction quality is improved as the number \( n \) of Radon samples increases.

Table 2 Reconstruction from scattered Radon lines.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
<th>20000</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.1516</td>
<td>0.1405</td>
<td>0.1431</td>
<td>0.1174</td>
</tr>
<tr>
<td>bull’s eye</td>
<td>0.1876</td>
<td>0.1721</td>
<td>0.2102</td>
<td>0.1893</td>
</tr>
</tbody>
</table>

5.3 Reconstruction from Noisy Data

5.3.1 Reconstruction on parallel beam geometry.

For further comparison between the Fourier-based and the kernel-based reconstruction method, we compare their behaviour on noisy Radon data. To this end, we use the test cases of Subsection 5.1, where we have
added Gaussian noise of zero mean and with variance \( \sigma = 0.001 \) to each of the three phantoms.

In the kernel-based reconstruction, we have used the same shape parameters \(- \varepsilon \) for the Gaussian kernel \( \phi_c \) and \( \nu \) for the Gaussian weight function \( w_\nu \) – as in Subsection 5.1. The resulting reconstructions are displayed in Figure 6, and their corresponding RMSE are shown in Table 3. Note that for any of the three test cases, the proposed kernel-based reconstruction is superior to the Fourier-based reconstruction by a smaller RMSE. But the visual quality of the two different reconstruction methods is comparable, cf. Figure 6.

### Table 3 Reconstruction on parallel beam geometry. Resulting RMSE values for reconstructions in Figure 6.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>Kernel-based</th>
<th>Fourier-based</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.1502</td>
<td>0.1933</td>
</tr>
<tr>
<td>bull’s eye</td>
<td>0.1796</td>
<td>0.2322</td>
</tr>
<tr>
<td>Shepp-Logan</td>
<td>0.1716</td>
<td>0.2041</td>
</tr>
</tbody>
</table>

#### 5.3.2 Reconstruction from scattered Radon data.

To further illustrate the good performance and flexibility of the proposed kernel-based reconstruction, let us turn to noisy scattered data. To this end, we rely on the examples of Subsection 5.2, where for each test case we have added Gaussian noise of zero mean and with variance \( \sigma = 0.001 \) the two phantoms. The reconstructions were computed by using the same shape parameters \( \varepsilon = 50 \) and \( \nu = 0.7 \) as in Subsection 5.2.

The resulting reconstructions for noisy and noise-free scattered Radon data are displayed in Figure 7. The corresponding RMSE values are in Table 4. The good visual quality of the reconstruction, in combination with their small RMSE values in Table 4, shows utility of the kernel-based reconstruction method yet once more.

### Table 4 Reconstruction from scattered Radon data. Resulting RMSE values for reconstructions in Figure 7.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>noisy</th>
<th>noise-free</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.2576</td>
<td>0.1820</td>
</tr>
<tr>
<td>bull’s eye</td>
<td>0.3140</td>
<td>0.2453</td>
</tr>
</tbody>
</table>

### 6 Final Remarks and Conclusion

We have developed a kernel-based method for image reconstruction from scattered Radon data. The proposed reconstruction method relies on Hermite-Birkhoff interpolation by positive definite Gaussian kernels in combination with a regularization of the Radon transform by Gaussian weights. Our kernel-based reconstruction method is shown to be competitive with Fourier-based reconstructions from Radon data on parallel beam geometry. The proposed kernel-based image reconstruction method is very flexible, since it can unconditionally be applied to arbitrary scattered Radon data, unlike classical Fourier-based reconstructions. The good performance of the kernel-based method is also demonstrated by numerical examples concerning noisy Radon data.

### References


Fig. 1 **Three popular phantoms.** Crescent-shaped phantom, bull’s eye, and Shepp-Logan (each of size $256 \times 256$).

Fig. 3 **Kernel-based vs Fourier-based reconstruction.** For the phantoms crescent-shaped, bull’s eye, and Shepp-Logan, their Fourier-based reconstructions (1st row) and kernel-based reconstructions (2nd row) are shown. The six reconstructions are computed from $(2M + 1) \times N = 3645$ Radon samples (where we let $N = 45$ and $M = 40$) on parallel beam geometry, respectively. The corresponding root mean square errors (RMSE) are displayed in Table 3.
Fig. 5 Kernel-based reconstruction from scattered Radon data. For the test cases bull’s eye (first row) and crescent-shaped phantom (second row), a sequence of scattered Radon data is taken. The figure shows the corresponding reconstructions for an increasing number $n = 2000, 5000, 10000, 20000$ of scattered Radon samples (from left to right), obtained from the proposed kernel-based reconstruction method. The corresponding root mean square errors (RMSE) are displayed in Table 2.
Fig. 6 Reconstruction from noisy Radon data on parallel beam geometry. For comparison between the Fourier-based method (1st row) and the kernel-based method (2nd row), their reconstructions are shown for three test cases: crescent-shaped phantom (1st col), bull’s eye (2nd col), Shepp-Logan phantom (3rd col). The corresponding RMSE values are in Table 3.

Fig. 7 Reconstruction from scattered Radon data. For the purpose of comparison between kernel-based reconstruction from noise-free Radon data (1st row) and noisy Radon data (2nd row), the resulting reconstructions are shown for two test cases: the crescent-shaped phantom (1st column) and bull’s eye (2nd column). The corresponding RMSE values are in Table 4.