Bounding the Lebesgue constant for Berrut's rational interpolant at general nodes

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Abstract

It has recently been shown that the Lebesgue constant for Berrut's rational interpolant at equidistant nodes grows logarithmically in the number of interpolation nodes. In this paper we show that the same holds for a very general class of well-spaced nodes and essentially any distribution of nodes that satisfy a certain regularity condition, including Chebyshev–Gauss–Lobatto nodes as well as extended Chebyshev nodes.

1 Introduction

For $n \in \mathbb{N}$, let $X_n = \{x_0, x_1, \ldots, x_n\}$ be a set of $n+1$ distinct *interpolation nodes* in the real interval $[a,b]$, and let $B_n = \{b_0, b_1, \ldots, b_n\}$ be a corresponding set of *cardinal basis functions*. The norm of the linear interpolation operator which maps $f \in \mathcal{C}^0[a,b]$ to $g = \sum_{j=0}^n b_j f(x_j)$ is also known as the *Lebesgue constant* [9, 10]

$$\Lambda(X_n, B_n) = \max_{a \le x \le b} \sum_{j=0}^n |b_j(x)|,$$

and one is interested in choices of nodes X_n and basis functions B_n such that $\Lambda(X_n, B_n)$ is small, as this leads to small bounds on the approximation error $\|f-g\|_\infty$, measured in the maximum norm.

In the case of polynomial interpolation, when B_n is the set of Lagrange basis polynomials with respect to X_n, it is well known that the Lebesgue constant grows logarithmically with n for Chebyshev nodes, but exponentially for equidistant nodes [4]. However, much smaller Lebesgue constants are observed for rational interpolation at equidistant nodes [5, 11]. In particular, Bos et al. [2, 3] show that the Lebesgue constant for Berrut's rational interpolant [1] with basis functions

$$b_i(x) = \frac{(-1)^i}{x - x_i} \Bigg/ \sum_{j=0}^n \frac{(-1)^j}{x - x_j}, \qquad i = 0, \ldots, n$$

grows logarithmically in the number of equidistant interpolation nodes, and that the same holds for the more general family of barycentric rational interpolants which was introduced by Floater and Hormann [6]. Moreover, Hormann et al. [7] show that this behaviour extends to *quasi-equidistant* nodes with a bounded global mesh ratio.

In this work we show that for essentially *any* reasonable choice of interpolation nodes X_n, the Lebesgue constant for Berrut's rational interpolant,

$$\Lambda(X_n) = \max_{a \le x \le b} \frac{\sum_{j=0}^n \frac{1}{|x - x_j|}}{\left|\sum_{j=0}^n \frac{(-1)^j}{x - x_j}\right|}, \tag{1}$$

is bounded from above by $c \ln(n)$ for some constant $c > 0$. More precisely, we derive this result for *well-spaced* nodes (Section 2), and show that such nodes can be generated by a distribution function satisfying a regularity condition (Section 3). We conclude the paper by discussing several examples (Section 4).

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1.1 Preliminaries

Without loss of generality we assume that the nodes in X_n are ordered,

$$x_0 < x_1 < \cdots < x_n,$$

and that $x_0 = 0$ and $x_n = 1$, because the Lebesgue constant in (1) is invariant with respect to linear transformations of the interpolations nodes. We further denote the length of the k-th *subinterval* by

$$h_k = x_{k+1} - x_k, \qquad k = 0, \ldots, n-1 \tag{2}$$

and restrict our discussion to the interpolation interval $[a, b] = [x_0, x_n] = [0, 1]$.

Following the approach in [2], let us introduce the functions

$$N_k(x) = (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{1}{|x - x_j|}$$

and

$$D_k(x) = (x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n} \frac{(-1)^j}{x - x_j} \right|$$

for $x_k < x < x_{k+1}$ and $k = 0, \ldots, n-1$, and observe that the Lebesgue constant (1) satisfies

$$\Lambda(X_n) = \max_{k=0,\ldots,n-1} \max_{x_k < x < x_{k+1}} \frac{N_k(x)}{D_k(x)}.$$

Our general goal is to establish the bounds

$$N_k(x) \le c_1 h_k \ln(n)$$

and

$$D_k(x) \ge c_2 h_k,$$

where the constants $c_1 > 0$ and $c_2 > 0$ are independent of x, k, and n. To this end, it helps to recall from [2] that

$$N_k(x) = h_k + (x - x_k)(x_{k+1} - x) \left(\sum_{j=0}^{k-1} \frac{1}{x - x_j} + \sum_{j=k+2}^{n} \frac{1}{x_j - x} \right) \tag{3}$$

and from [7] that

$$\begin{aligned} D_k(x) &= (x - x_k)(x_{k+1} - x) \left[\cdots + \left(-\frac{1}{x - x_{k-3}} + \frac{1}{x - x_{k-2}} \right) + \left(-\frac{1}{x - x_{k-1}} + \frac{1}{x - x_k} \right) \right. \\ &\qquad \left. + \left(\frac{1}{x - x_{k+1}} - \frac{1}{x - x_{k+2}} \right) + \left(\frac{1}{x - x_{k+3}} - \frac{1}{x - x_{k+4}} \right) + \cdots \right] \\ &\ge (x - x_k)(x_{k+1} - x) \left[\left(-\frac{1}{x - x_{k-1}} + \frac{1}{x - x_k} \right) + \left(\frac{1}{x_{k+1} - x} - \frac{1}{x_{k+2} - x} \right) \right] \\ &= h_k - (x - x_k)(x_{k+1} - x) \left(\frac{1}{x - x_{k-1}} + \frac{1}{x_{k+2} - x} \right), \end{aligned} \tag{4}$$

because all paired terms are positive for $x_k < x < x_{k+1}$. We remark that one of the two terms in the last factor is missing in the cases $k = 0$ and $k = n-1$, but as the factor will be smaller then, we can safely ignore this detail in the subsequent discussion.

2 Well-spaced nodes

Definition 2.1. For each $n \in \mathbb{N}$, let X_n be a set of interpolation nodes. We then say that $X = (X_n)_{n \in \mathbb{N}}$ is a *family of well-spaced nodes*, if there exist constants $C, R \ge 1$, independent of n, such that the three conditions

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} \le \frac{C}{k + 1 - j}, \qquad j = 0, \ldots, k, \quad k = 0, \ldots, n-1, \tag{5}$$

$$\frac{x_{k+1}-x_k}{x_j-x_k} \le \frac{C}{j-k}, \qquad j=k+1,\dots,n, \quad k=0,\dots,n-1, \tag{6}$$

$$\frac{1}{R} \le \frac{x_{k+1}-x_k}{x_k-x_{k-1}} \le R, \qquad k=1,\dots,n-1 \tag{7}$$

hold for each set of nodes X_n.

This definition includes equidistant nodes (with smallest possible constants $C = R = 1$), but it also covers Chebyshev–Gauss–Lobatto nodes and extended Chebyshev nodes, as shown in Section 4.

Theorem 2.2. *If $X = (X_n)_{n\in\mathbb{N}}$ is a family of well-spaced nodes, then there exists a constant $c > 0$ such that*

$$\Lambda(X_n) \le c\ln(n)$$

for any $n \ge 2$.

Proof. We will actually show, more precisely, that

$$\Lambda(X_n) \le (R+1)(1+2C\ln(n)), \tag{8}$$

where R and C are the constants defining X as well-spaced in Definition 2.1. The claim then follows by letting $c = 2(R+1)(C+1)$.

Let $x_k < x < x_{k+1}$ for some k. From (3) we get

$$\begin{aligned} N_k(x) &= h_k + (x_{k+1}-x)\sum_{j=0}^{k-1}\frac{x-x_k}{x-x_j} + (x-x_k)\sum_{j=k+2}^{n}\frac{x_{k+1}-x}{x_j-x} \\ &\le h_k + (x_{k+1}-x_k)\sum_{j=0}^{k-1}\frac{x_{k+1}-x_k}{x_{k+1}-x_j} + (x_{k+1}-x_k)\sum_{j=k+2}^{n}\frac{x_{k+1}-x_k}{x_j-x_k} \\ &= h_k\left(1+\sum_{j=0}^{k-1}\frac{x_{k+1}-x_k}{x_{k+1}-x_j} + \sum_{j=k+2}^{n}\frac{x_{k+1}-x_k}{x_j-x_k}\right). \end{aligned}$$

It then follows from conditions (5) and (6) in Definition 2.1 that

$$N_k(x) \le h_k\left(1+\sum_{j=0}^{k-1}\frac{C}{k+1-j} + \sum_{j=k+2}^{n}\frac{C}{j-k}\right) \le h_k\left(1+2C\sum_{j=2}^{n}\frac{1}{j}\right) \le h_k\big(1+2C\ln n\big) \le c_1 h_k\ln(n)$$

with the constant $c_1 = 2(C+1) > 0$.

To bound $D_k(x)$ from below, we consider the function

$$\begin{aligned} g(x) &= (x-x_k)(x_{k+1}-x)\left(\frac{1}{x-x_{k-1}}+\frac{1}{x_{k+2}-x}\right) \\ &= \frac{(x-x_k)(x_{k+1}-x)}{(x-x_{k-1})(x_{k+2}-x)}(x_{k+2}-x_{k-1}) \\ &= \frac{1}{\left(1+\frac{x_k-x_{k-1}}{x-x_k}\right)\left(1+\frac{x_{k+2}-x_{k+1}}{x_{k+1}-x}\right)}(x_{k+2}-x_{k-1}) \\ &= \frac{h_{k-1}+h_k+h_{k+1}}{\left(1+\frac{h_{k-1}}{x-x_k}\right)\left(1+\frac{h_{k+1}}{x_{k+1}-x}\right)}. \end{aligned}$$

Let us assume that $h_{k-1} \le h_{k+1}$ and note that the other case can be treated similarly, due to the apparent symmetry of $g(x)$ with respect to h_{k-1} and h_{k+1}. It is then straightforward to verify that we can bound $g(x)$ from above by replacing h_{k+1} with h_{k-1},

$$g(x) \le \frac{h_{k-1}+h_k+h_{k-1}}{\left(1+\frac{h_{k-1}}{x-x_k}\right)\left(1+\frac{h_{k-1}}{x_{k+1}-x}\right)},$$

and that this upper bound attains its maximum at $x^* = (x_k + x_{k+1})/2$. Hence,

$$g(x) \le \frac{h_{k-1} + h_k + h_{k-1}}{\left(1 + \frac{h_{k-1}}{x^* - x_k}\right)\left(1 + \frac{h_{k-1}}{x_{k+1} - x^*}\right)} = \frac{2h_{k-1} + h_k}{\left(1 + \frac{h_{k-1}}{h_k/2}\right)\left(1 + \frac{h_{k-1}}{h_k/2}\right)} = \frac{{h_k}^2(2h_{k-1} + h_k)}{(2h_{k-1} + h_k)^2} = \frac{{h_k}^2}{2h_{k-1} + h_k}.$$

It then follows from (4) and the upper bound in condition (7) that

$$D_k(x) \ge h_k - \frac{{h_k}^2}{2h_{k-1} + h_k} = \frac{2h_{k-1}h_k}{2h_{k-1} + h_k} \ge \frac{h_{k-1}h_k}{h_{k-1} + h_k} = \frac{h_k}{1 + h_k/h_{k-1}} \ge \frac{h_k}{1 + R} = c_2 h_k$$

for the constant $c_2 = 1/(R + 1) > 0$. Note that the lower bound in condition (7) is required to ensure $h_k/h_{k+1} \le R$ for $k = 0, \dots, n-2$, which in turn is needed in the other case when $h_{k-1} > h_{k+1}$. □

3 Generating well-spaced nodes

Theorem 2.2 guarantees that the Lebesgue constant for Berrut's rational interpolant grows logarithmically in the number of interpolation nodes in case they are well-spaced, but the question remains how general the latter property really is. It turns out, though, that being well-spaced is nothing special, as the three conditions in Definition 2.1 are satisfied by essentially any reasonable choice of interpolation nodes. To this end, let us consider families of nodes which are derived by equidistantly sampling some given function.

Definition 3.1. We say that a function $F \in C[0,1]$ is a *distribution function* if it is a strictly increasing bijection on the interval $[0,1]$.

Given a distribution function F and some $n \in \mathbb{N}$, we define the associated interpolation nodes $X_n = X_n(F)$ by setting

$$x_k := F(k/n), \qquad k = 0, \dots, n. \tag{9}$$

By Definition 3.1, these nodes are ordered and satisfy our assumption $x_0 = 0$ and $x_1 = 1$. As shown in Section 4.2, there exist distribution functions that yield node sets for which the Lebesgue constant in (1) grows faster than logarithmically with n, but this behaviour can be ruled out by the following regularity condition.

Definition 3.2. We say that a distribution function F is *regular*, if $F \in C^1[0,1]$ and F' has a finite number of zeros $T = \{t_1, t_2, \dots, t_l\} \subset [0,1]$ with finite multiplicities. That is, $F'(t) > 0$ for $t \in [0,1] \setminus T$ and there exist positive real numbers $r_1, r_2, \dots, r_l$, such that $G_j(t) = F'(t)/|t - t_j|^{r_j}$ is continuous with $\lim_{t \to t_j} G_j(t) > 0$ for $j = 1, \dots, l$.

We now prove two properties of regular distribution functions, which later help to establish the three conditions from Definition 2.1. Note that some of the technical details in the proofs are provided in the Appendix.

Proposition 3.3. *Let F be a regular distribution function. Then there exists a constant $C > 0$ such that*

$$\frac{F[x,y]}{F[x,z]} \le C$$

for all $x, y, z \in [0,1]$ with $x > y \ge z$.

Proof. Since $F \in C^1[0,1]$, the divided difference $F[u,v]$ is a continuous bivariate function. By Definition 3.2, $F[u,v] \ge 0$ for any $(u,v) \in [0,1]^2$ and $F[u,v] = 0$ if and only if $u = v \in T$. Hence, the trivariate function

$$G(x,y,z) = \frac{F[x,y]}{F[x,z]}$$

is continuous on $[0,1]^3 \setminus E$, where $E = \{(x,y,x) \in [0,1]^3 : x \in T\}$.

To construct a "safety margin" around the singularities of G at E, let us fix $\delta > 0$, such that the open intervals $I_j = (t_j - \delta, t_j + \delta)$ are mutually disjoint for $j = 1, \dots, l$, and consider the set

$$D := [0,1]^3 \setminus \bigcup_{j=1}^{l} (I_j \times [0,1] \times I_j).$$

As G is continuous, it has an upper bound C_0 on the compact set D, and we need only analyse what happens near the set E.

To this end, we first note that Definition 3.2 guarantees

$$m = \min_{j=1,\dots,l} \left(\inf_{t \in I_j} \frac{F'(t)}{|t-t_j|^{r_j}} \right) \qquad \text{and} \qquad M = \max_{j=1,\dots,l} \left(\sup_{t \in I_j} \frac{F'(t)}{|t-t_j|^{r_j}} \right) \tag{10}$$

to be positive real numbers. Suppose now that $x, z \in I_j \cap [0,1]$ for some $j \in \{1, \dots, l\}$ and $x > y \geq z$. Then,

$$F[x,y] = \frac{1}{x-y} \int_y^x F'(t)dt \leq \frac{M}{x-y} \int_y^x |t-t_j|^{r_j} \, dt = \frac{M}{x-y} \int_{y-t_j}^{x-t_j} |t|^{r_j} \, dt$$

and

$$F[x,z] = \frac{1}{x-z} \int_z^x F'(t)dt \geq \frac{m}{x-z} \int_z^x |t-t_j|^{r_j} \, dt = \frac{m}{x-z} \int_{z-t_j}^{x-t_j} |t|^{r_j} \, dt,$$

so that

$$G(x,y,z) = \frac{F[x,y]}{F[x,z]} \leq \frac{M}{m} 2(r_j+1) =: C_j$$

by Lemma A.2. The statement then follows by letting $C = \max(C_0, C_1, \dots, C_l)$. □

Proposition 3.4. *Let F be a regular distribution function. Then there exist some $\varepsilon > 0$ and a constant $R > 0$ such that*

$$\frac{1}{R} \leq \frac{F[x,x+s]}{F[x-s,x]} \leq R$$

for all $s \in [0,\varepsilon]$ and $x \in [s, 1-s]$.

Proof. By arguments similar to the ones used in the proof of Proposition 3.3, the bivariate function

$$G(x,s) = \frac{F[x,x+s]}{F[x-s,x]}$$

is continuous, apart from the singularities at $\{(x,0) : x \in T\}$. Note that for $x \notin T$, we have $F'(x) > 0$, hence $G(x,0) = F'(x)/F'(x) = 1$. We consider again the mutually disjoint intervals $I_j = (t_j - \delta, t_j + \delta)$ for $j = 1, \dots, l$ around the zeros of F', fix $\varepsilon = \delta/2$, and define

$$D := \{(x,s) : s \in [0,\varepsilon], x \in [s, 1-s] \setminus \cup_{j=1}^l I_j\}.$$

As G is continuous and positive on the compact set D, there exists some $R_0 > 0$ such that

$$\frac{1}{R_0} \leq G(x,s) \leq R_0, \qquad (x,s) \in D,$$

and it remains to analyse the behaviour of G near the singularities.

To this end, let $s \in [0,\varepsilon]$ and $x \in I_j \cap [s, 1-s]$ for some $j \in \{1, \dots, l\}$. Then,

$$G(x,s) = \frac{F(x+s) - F(x)}{F(x) - F(x-s)} = \frac{\int_x^{x+s} F'(t)dt}{\int_{x-s}^x F'(t)dt} \leq \frac{M}{m} \frac{\int_x^{x+s} |t|^{r_j} \, dt}{\int_{x-s}^x |t|^{r_j} \, dt}$$

with the constants $m, M > 0$ from (10), and further

$$G(x,s) \leq \frac{M}{m} 3^{r_j+1} =: R_j$$

by Lemma A.4. Similarly, we also have

$$G(x,s) \geq \frac{m}{M} \frac{\int_x^{x+s} |t|^{r_j} \, dt}{\int_{x-s}^x |t|^{r_j} \, dt} \geq \frac{m}{M} 3^{-(r_j+1)} = \frac{1}{R_j},$$

so that the statement follows by letting $R = \max(R_0, R_1, \dots, R_l)$. □

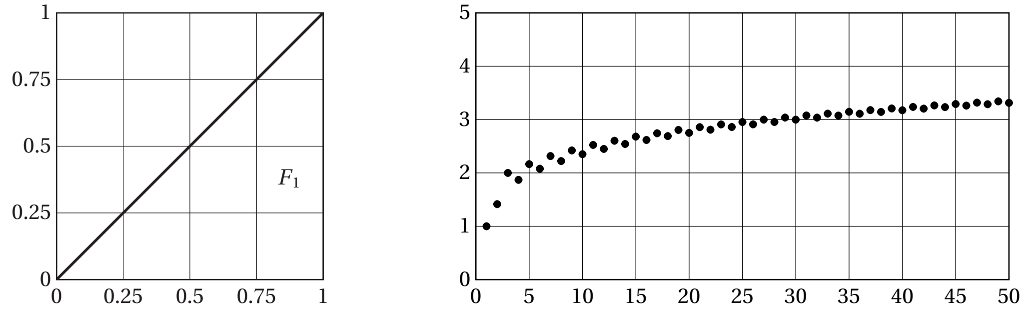

Figure 1: Lebesgue constant for Berrut's rational interpolant at $n+1$ equidistant interpolation nodes, generated by the regular distribution function $F_1(t) = t$ for $1 \le n \le 50$.

Theorem 3.5. *Let F be a regular distribution function and X_n the associated interpolation nodes from* (9) *for any $n \in \mathbb{N}$. Then the family of nodes $X = (X_n)_{n\in\mathbb{N}}$ is well-spaced.*

Proof. We must show the three conditions of well-spaced nodes in Definition 2.1. For condition (5), note that we may write

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} = \frac{F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)}{F\left(\frac{k+1}{n}\right) - F\left(\frac{j}{n}\right)} = \frac{\frac{1}{n} F\left[\frac{k}{n}, \frac{k+1}{n}\right]}{\frac{k+1-j}{n} F\left[\frac{j}{n}, \frac{k+1}{n}\right]} = \frac{F\left[\frac{k}{n}, \frac{k+1}{n}\right]}{F\left[\frac{j}{n}, \frac{k+1}{n}\right]} \frac{1}{k+1-j},$$

and then this condition follows from Proposition 3.3 with $x = (k+1)/n$, $y = k/n$, and $z = j/n$. Condition (6) can be shown analogously by noticing that $x_k = \tilde{x}_{n-k}$, where $\tilde{X}_n = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n\}$ are the interpolation nodes generated by the regular distribution function $\tilde{F}(t) = 1 - F(1-t)$. For condition (7), we write, similar as above,

$$\frac{x_{k+1} - x_k}{x_k - x_{k-1}} = \frac{F\left[\frac{k}{n}, \frac{k+1}{n}\right]}{F\left[\frac{k-1}{n}, \frac{k}{n}\right]},$$

and conclude from Proposition 3.4 with $x = k/n$ and $s = 1/n$ that this condition holds for all $n \ge 1/\varepsilon$, where ε is the constant introduced in Proposition 3.4. But as the number of remaining cases for $n < 1/\varepsilon$ is finite, we can enlarge the constant R from Proposition 3.4 so that it also covers these cases, and then condition (7) holds for all $n \in \mathbb{N}$. □

4 Examples

4.1 Regular distributions

The simplest example of a regular distribution function is the function $F_1(t) = t$, which generates equidistant nodes (see Figure 1). For these nodes, it has already been shown by Bos et al. [2] that the Lebesgue constant for Berrut's rational interpolant grows logarithmically, and their upper bound is even tighter than our general upper bound in Theorem 2.2. However, the results from the previous sections allow us to handle more general node distributions.

For example, Bos et al. [2] report numerical tests which show that the Lebesgue constant grows logarithmically with n for logarithmically distributed interpolation nodes. As these nodes are in fact generated by the distribution function $F_2(t) = \ln(1 + t(e-1))$, which clearly is regular because $F_2'(t)$ is strictly positive, Theorems 2.2 and 3.5 now provide a proof of this behaviour (see Figure 2).

As a second example, consider the function $F_3(t) = (1-\cos(\pi t))/2$, which is a regular distribution function by Definition 3.2, as its first derivative has $l = 2$ zeros at $t_1 = 0$ and $t_2 = 1$ with multiplicities $r_1 = r_2 = 1$. This function generates the extrema of the Chebyshev polynomials (of the first kind), mapped to $[0, 1]$,

$$x_k = (1 - \cos(k\pi/n))/2, \qquad k = 0, \ldots, n,$$

which are also known as *Chebyshev–Gauss–Lobatto nodes* or *Clenshaw–Curtis nodes* (see Figure 3). By Theorem 3.5, these nodes are well-spaced, and so the Lebesgue constant (1) grows logarithmically with n, due to

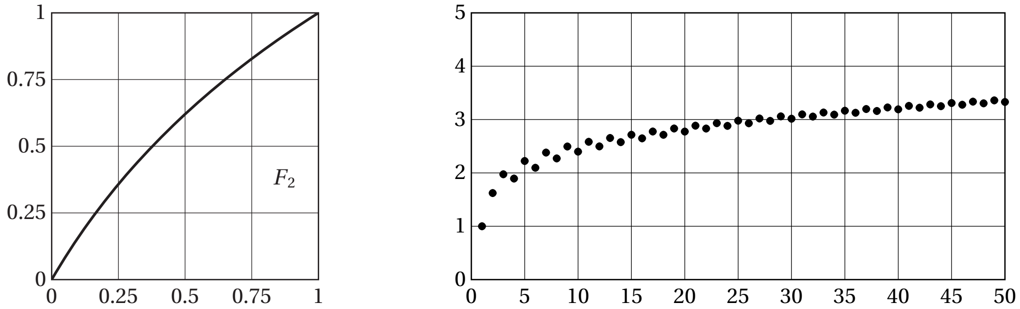

Figure 2: Lebesgue constant for Berrut's rational interpolant at $n+1$ interpolation nodes, generated by the regular distribution function $F_2(t) = \ln(1 + t(e-1))$ for $1 \le n \le 50$.

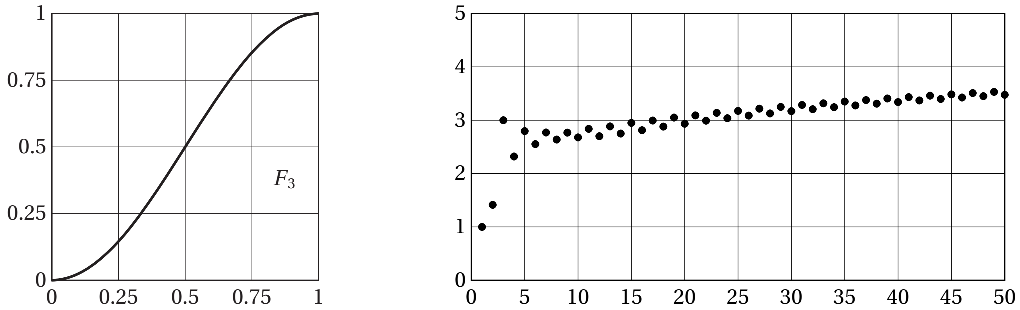

Figure 3: Lebesgue constant for Berrut's rational interpolant at $n+1$ Chebyshev–Gauss–Lobatto nodes, generated by the regular distribution function $F_3(t) = (1 - \cos(\pi t))/2$ for $1 \le n \le 50$.

Theorem 2.2. Following the proofs of Propositions 3.3 and 3.4, using $\delta = 1/2$ and $\varepsilon = 1/4$, the constants in Definition 3.2 turn out to be $C = 2\pi$ and $R = 9\pi/2$ for this node distribution, leading to the upper bound

$$\Lambda(X_n) \le (9\pi/2 + 1)(1 + 4\pi \ln(n)),$$

according to (8). Admittedly, this bound is not very tight, and it requires further investigations to derive better upper bounds.

4.2 Non-regular distributions

While the regularity condition for distribution functions in Definition 3.2 is clearly sufficient for the associated nodes to be well-spaced and the growth of the Lebesgue constant to be logarithmic (see Theorems 2.2 and 3.5), we believe that this condition is also necessary. To this end, let us consider the distribution function

$$F_4(t) = \begin{cases} 0, & t = 0, \\ \exp(1 - 1/t), & 0 < t \le 1. \end{cases} \tag{11}$$

This function is C^∞-flat at the origin and hence not regular in our sense, because its derivative has a zero with infinite multiplicity at $t = 0$. For these nodes, the numerical estimates in Figure 4 indicate that the Lebesgue constant for Berrut's rational interpolant grows exponentially with n.

However, the line between exponential and logarithmic growth seems to be very thin, as a second example demonstrates. The distribution function

$$F_5(t) = \frac{1}{2} \begin{cases} 1 - \exp(1 + 1/(2t-1)), & 0 \le t < 1/2, \\ 1, & t = 1/2, \\ 1 + \exp(1 - 1/(2t-1)), & 1/2 < t \le 1. \end{cases} \tag{12}$$

is not regular, because its derivative has a zero with infinite multiplicity at $t = 1/2$, and we observe in Figure 5 that the corresponding Lebesgue constant grows exponentially with n. But this is true only for odd n, while for even n the growth seems to be only logarithmic.

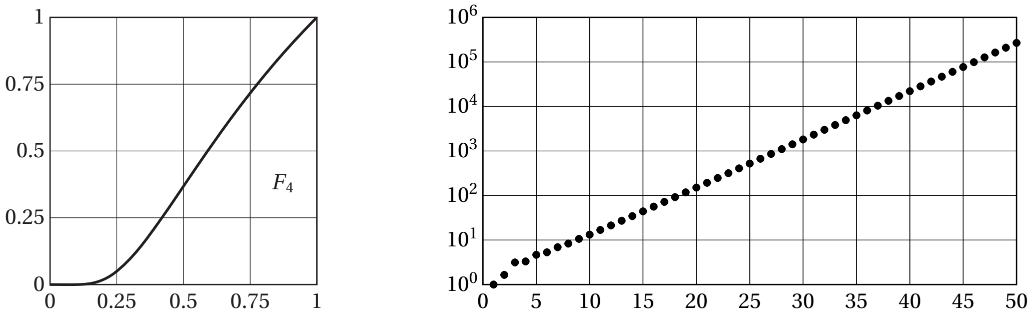

Figure 4: Lebesgue constant for Berrut's rational interpolant at $n+1$ interpolation nodes, generated by the non-regular distribution function F_4 in (11) for $1 \le n \le 50$. Note that the ordinate is plotted in logarithmic scale, so that the linear growth observed in the plot corresponds to an exponential growth of the values themselves.

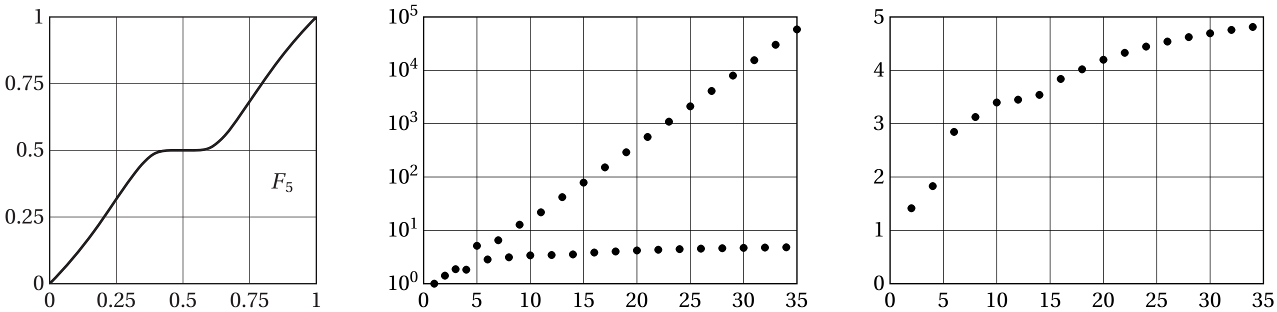

Figure 5: Lebesgue constant for Berrut's rational interpolant at $n+1$ interpolation nodes, generated by the non-regular distribution function F_5 in (12) for $1 \le n \le 35$. As in Figure 4, the growth is exponential, but only for odd n, while the values seem to grow logarithmically for even n.

4.3 Extended Chebyshev nodes

Besides nodes that are generated by regular distribution functions, we can also verify directly if a given family of nodes is well-spaced. For example, the Chebyshev nodes, mapped to $[0,1]$, which are also known as *extended Chebyshev nodes* [4] and given by

$$x_k = \frac{1}{2}\left[1 - \cos\left(\frac{2k+1}{2n+2}\pi\right) \Big/ \cos\left(\frac{1}{2n+2}\pi\right)\right], \qquad k = 0, \dots, n, \tag{13}$$

are *not* generated by a distribution function, but they are still well-spaced, as the following propositions confirm. Therefore, it follows from Theorem 2.2 that the Lebesgue constant for Berrut's rational interpolant at these nodes grows logarithmically (see Figure 6), as already observed numerically by Bos et al. [2].

Proposition 4.1. *For the extended Chebyshev nodes in* (13)*,*

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} \le \frac{\pi^2/2}{k+1-j}, \qquad j = 0, \dots, k, \quad k = 0, \dots, n-1,$$

for any $n \in \mathbb{N}$. That is, condition (5) *of Definition 2.1 holds with $C = \pi^2/2$.*

Proof. First note that

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} = \frac{\cos\left(\frac{2k+1}{2n+2}\pi\right) - \cos\left(\frac{2k+3}{2n+2}\pi\right)}{\cos\left(\frac{2j+1}{2n+2}\pi\right) - \cos\left(\frac{2k+3}{2n+2}\pi\right)} = \frac{\sin\left(\frac{2k+2}{2n+2}\pi\right)\sin\left(\frac{1}{2n+2}\pi\right)}{\sin\left(\frac{k+j+2}{2n+2}\pi\right)\sin\left(\frac{k-j+1}{2n+2}\pi\right)}.$$

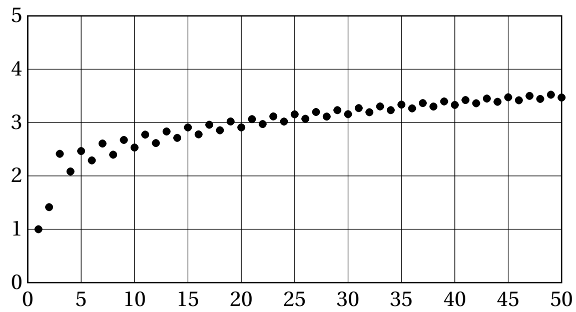

Figure 6: Lebesgue constant for Berrut's rational interpolant at $n+1$ extended Chebyshev nodes for $1 \le n \le 50$.

Moreover, for the considered ranges of j and k, we always have $k-j+1 \le n+1$, hence $\frac{k-j+1}{2n+2}\pi \le \frac{\pi}{2}$. Now, if in addition $\frac{k+j+2}{2n+2}\pi \le \frac{\pi}{2}$, then using the inequality $2\varphi/\pi \le \sin(\varphi) \le \varphi$ for $\varphi \in [0, \pi/2]$, we have

$$\frac{x_{k+1}-x_k}{x_{k+1}-x_j} \le \frac{\left(\frac{2k+2}{2n+2}\pi\right)\left(\frac{1}{2n+2}\pi\right)}{\frac{2}{\pi}\left(\frac{k+j+2}{2n+2}\pi\right)\frac{2}{\pi}\left(\frac{k-j+1}{2n+2}\pi\right)} = \frac{\pi^2}{4}\frac{2k+2}{k+j+2}\frac{1}{k-j+1} \le \frac{\pi^2}{2}\frac{1}{k-j+1}.$$

On the other hand, if $\frac{k+j+2}{2n+2}\pi > \frac{\pi}{2}$, then $\frac{2k+2}{2n+2}\pi > \frac{k+j+2}{2n+2}\pi > \frac{\pi}{2}$, and so

$$\frac{\sin\left(\frac{2k+2}{2n+2}\pi\right)}{\sin\left(\frac{k+j+2}{2n+2}\pi\right)} \le 1,$$

because $\sin(\varphi)$ is decreasing on $[\pi/2, \pi]$. Therefore, in this case,

$$\frac{x_{k+1}-x_k}{x_{k+1}-x_j} \le \frac{\sin\left(\frac{1}{2n+2}\pi\right)}{\sin\left(\frac{k-j+1}{2n+2}\pi\right)} \le \frac{\frac{1}{2n+2}\pi}{\frac{2}{\pi}\frac{k-j+1}{2n+2}\pi} = \frac{\pi}{2}\frac{1}{k-j+1}.$$

□

Note again that condition (6) in Definition 2.1 is entirely symmetric, hence we leave out the details.

Proposition 4.2. *For the extended Chebyshev nodes in* (13),

$$\frac{1}{2} \le \frac{x_{k+1}-x_k}{x_k-x_{k-1}} \le 2, \qquad k = 1, \dots, n-1,$$

for any $n \ge 2$. That is, condition (7) *of Definition* (2.1) *holds with $R = 2$.*

Proof. We first observe that

$$\frac{x_{k+1}-x_k}{x_k-x_{k-1}} = \frac{\cos\left(\frac{2k+1}{2n+2}\pi\right)-\cos\left(\frac{2k+3}{2n+2}\pi\right)}{\cos\left(\frac{2k-1}{2n+2}\pi\right)-\cos\left(\frac{2k+1}{2n+2}\pi\right)} = \frac{2\sin\left(\frac{2k+2}{2n+2}\pi\right)\sin\left(\frac{1}{2n+2}\pi\right)}{2\sin\left(\frac{2k}{2n+2}\pi\right)\sin\left(\frac{1}{2n+2}\pi\right)} = \frac{\sin\left(\frac{k+1}{n+1}\pi\right)}{\sin\left(\frac{k}{n+1}\pi\right)},$$

and further, with $\theta = \frac{k\pi}{n+1} \in \left[\frac{\pi}{n+1}, \pi - \frac{2\pi}{n+1}\right]$,

$$\frac{x_{k+1}-x_k}{x_k-x_{k-1}} = \frac{\sin\left(\theta + \frac{\pi}{n+1}\right)}{\sin(\theta)} = \cos\left(\frac{\pi}{n+1}\right) + \cot(\theta)\sin\left(\frac{\pi}{n+1}\right).$$

As $\cot(\varphi)$ is strictly decreasing on $(0,\pi)$, we conclude $\cot(\theta) \le \cot\left(\frac{\pi}{n+1}\right)$, hence

$$\frac{x_{k+1}-x_k}{x_k-x_{k-1}} \le \cos\left(\frac{\pi}{n+1}\right) + \cot\left(\frac{\pi}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right) = 2\cos\left(\frac{\pi}{n+1}\right) \le 2.$$

Similarly,

$$\frac{x_{k+1}-x_k}{x_k-x_{k-1}} \ge \cos\left(\frac{\pi}{n+1}\right) + \cot\left(\pi - \frac{2\pi}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right) = \frac{1}{2}\sec\left(\frac{\pi}{n+1}\right) \ge \frac{1}{2}$$

where we used the trigonometric identity $\cos(\varphi) + \cot(\pi - 2\varphi)\sin(\varphi) = \sec(\varphi)/2$ with $\varphi = \pi/(n+1)$. □

By (8), we therefore get the upper bound

$$\Lambda(X_n) \le 3 + 3\pi^2 \ln(n)$$

on the Lebesgue constant for Berrut's rational interpolant at extended Chebyshev nodes.

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A Appendix

Lemma A.1. *Let $u, v, r \in \mathbb{R}$ with $u < v$ and $r \geq 0$. Then,*

$$\frac{v^r}{r+1} \leq \frac{1}{v-u}\int_u^v |t|^r\,dt \leq v^r, \qquad \text{if} \quad 0 \leq u < v, \tag{14}$$

and

$$\frac{v^r}{2(r+1)} \leq \frac{1}{v-u}\int_u^v |t|^r\,dt \leq \frac{v^r}{r+1}, \qquad \text{if} \quad u < 0 < v, \quad |u| \leq |v|\,. \tag{15}$$

Proof. Using the assumption $0 \leq u < v$ in (14), the two bounds can be established by noticing that

$$\frac{1}{v-u}\int_u^v |t|^r\,dt = \frac{1}{v-u}\int_u^v t^r dt \leq \frac{1}{v-u}\int_u^v v^r dt = v^r$$

and

$$\frac{1}{v-u}\int_u^v |t|^r\,dt = \frac{1}{v-u}\int_u^v t^r dt = \frac{1}{r+1}\frac{v^{r+1}-u^{r+1}}{v-u} \geq \frac{1}{r+1}\frac{v^{r+1}-v^r u}{v-u} = \frac{v^r}{r+1}.$$

Similarly, the upper bound in (15) follows from the given conditions on u and v, because

$$\frac{1}{v-u}\int_u^v |t|^r\,dt = \frac{1}{v-u}\left(\int_u^0 (-t)^r dt + \int_0^v t^r dt\right) = \frac{1}{r+1}\frac{(-u)^{r+1}+v^{r+1}}{v-u} \leq \frac{1}{r+1}\frac{(-u)v^r+v^{r+1}}{v-u} = \frac{v^r}{r+1}.$$

The lower bound in (15) is equivalent to

$$\frac{(-u)^{r+1}+v^{r+1}}{v-u} \geq \frac{v^r}{2} \qquad \Longleftrightarrow \qquad 2(-u)^{r+1}+v^{r+1} \geq (-u)v^r,$$

which is true, since $v \geq (-u) > 0$. □

Lemma A.2. *Let $a, b, c, r \in \mathbb{R}$ with $a \leq b < c$ and $r \geq 0$. Then,*

$$\left(\frac{1}{c-b}\int_b^c |t|^r\,dt\right)\Big/\left(\frac{1}{c-a}\int_a^c |t|^r\,dt\right) \leq 2(r+1).$$

Proof. For our convenience, let

$$B := \frac{1}{c-b}\int_b^c |t|^r\,dt \qquad \text{and} \qquad A := \frac{1}{c-a}\int_a^c |t|^r\,dt.$$

By Lemma A.1, we then have

$$B \leq \begin{cases} c^r, & 0 \leq b < c, \\ c^r/(r+1), & b < 0 < c, \quad |b| \leq |c|\,, \\ (-b)^r/(r+1), & b < 0 < c, \quad |b| \geq |c|\,, \\ (-b)^r, & b < c \leq 0 \end{cases} \qquad \text{and} \qquad A \geq \begin{cases} c^r/(r+1), & 0 \leq a < c, \\ c^r/(2(r+1)), & a < 0 < c, \quad |a| \leq |c|\,, \\ (-a)^r/(2(r+1)), & a < 0 < c, \quad |a| \geq |c|\,, \\ (-a)^r/(r+1), & a < c \leq 0, \end{cases}$$

where the first two cases in both inequalities follow directly from (14) and (15), respectively. The other two cases can be derived similarly by noticing the apparent symmetry of A and B with respect to changing the signs of a, b, and c. Combining all cases, we find

$$\frac{B}{A} \leq \begin{cases} \frac{c^r}{c^r/(r+1)} = r+1, & 0 \leq a \leq b < c, \\ \frac{c^r}{c^r/(2(r+1))} = 2(r+1), & a < 0 \leq b < c, \quad |a| \leq |c|\,, \\ \frac{c^r}{(-a)^r/(2(r+1))} \leq \frac{c^r}{c^r/(2(r+1))} = 2(r+1), & a < 0 \leq b < c, \quad |a| \geq |c|\,, \\ \frac{c^r/(r+1)}{c^r/(2(r+1))} = 2, & a \leq b < 0 < c, \quad |b| \leq |c|\,, \quad |a| \leq |c|\,, \\ \frac{c^r/(r+1)}{(-a)^r/(2(r+1))} \leq \frac{c^r/(r+1)}{c^r/(2(r+1))} = 2, & a \leq b < 0 < c, \quad |b| \leq |c|\,, \quad |a| \geq |c|\,, \\ \frac{(-b)^r/(r+1)}{(-a)^r/(2(r+1))} \leq \frac{(-b)^r/(r+1)}{(-b)^r/(2(r+1))} = 2, & a \leq b < 0 < c, \quad |b| \geq |c|\,, \\ \frac{(-b)^r}{(-a)^r/(r+1)} \leq \frac{(-b)^r}{(-b)^r/(r+1)} = r+1, & a \leq b < c \leq 0, \end{cases}$$

where we used the fact that the conditions of the last two cases imply $|a| \geq |b|$, and also $|a| \geq |c|$ in the second-last case. Note that this covers all possible cases of a, b, c with $a \leq b < c$, as assumed in the statement of the lemma. ∎

Lemma A.3. *Let $u, v, s \in \mathbb{R}$ with $u \geq 0$, $v \geq 0$, and $s \geq 1$. Then,*

$$(u + 2v)^s - u^s \leq 2^s((u + v)^s - u^s).$$

Proof. Let us fix $v \geq 0$ and consider the function $f(u) = (u + 2v)^s - u^s$. It follows from the assumption $s - 1 \geq 0$ that $f'(u) = s((u + 2v)^{s-1} - u^{s-1}) \geq 0$ for $u \geq 0$. Hence, $f(u)$ is increasing, and so

$$(u + 2v)^s - u^s = f(u) \leq f(2u) = (2u + 2v)^s - (2u)^s = 2^s((u + v)^s - u^s).$$

∎

Lemma A.4. *Let $a, b, r \in \mathbb{R}$ with $b > 0$, and $r \geq 0$. Then,*

$$3^{-(r+1)} \leq \int_a^{a+b} |t|^r \, dt \Big/ \int_{a-b}^{a} |t|^r \, dt \leq 3^{r+1}. \tag{16}$$

Proof. Substituting t with $-s$ in both integral gives

$$\int_a^{a+b} |t|^r \, dt = \int_{(-a)}^{-(a+b)} |-s|^r \, (-ds) = \int_{(-a)-b}^{(-a)} |s|^r \, ds$$

and

$$\int_{a-b}^{a} |t|^r \, dt = \int_{-(a-b)}^{(-a)} |-s|^r \, (-ds) = \int_{(-a)}^{(-a)+b} |s|^r \, ds,$$

and as the bounds in (16) are symmetric with respect to multiplicative inversion, we can assume without loss of generality that $a \geq 0$.

We first consider the case $a \geq b$, so that $0 \leq a - b < a < a + b$ and

$$\int_a^{a+b} |t|^r \, dt \Big/ \int_{a-b}^{a} |t|^r \, dt = \int_a^{a+b} t^r dt \Big/ \int_{a-b}^{a} t^r dt = \frac{(a + b)^{r+1} - a^{r+1}}{a^{r+1} - (a - b)^{r+1}}.$$

The upper bound in (16) then follows from Lemma A.3 with $u = a - b$, $v = b$, and $s = r + 1$, because

$$\begin{aligned}
& (a + b)^{r+1} - (a - b)^{r+1} \leq 2^{r+1}(a^{r+1} - (a - b)^{r+1}) \\
\iff\quad & (a + b)^{r+1} \leq 2^{r+1}a^{r+1} - 2^{r+1}(a - b)^{r+1} + (a - b)^{r+1} \\
\iff\quad & (a + b)^{r+1} - a^{r+1} \leq (2^{r+1} - 1)(a^{r+1} - (a - b)^{r+1}) \\
\implies\quad & \frac{(a + b)^{r+1} - a^{r+1}}{a^{r+1} - (a - b)^{r+1}} \leq 2^{r+1} - 1 \leq 3^{r+1}.
\end{aligned}$$

For the lower bound, notice that the convexity of $f(u) = u^{r+1}$ for $u \geq 0$ and $r \geq 0$ implies

$$\begin{aligned}
& \frac{1}{2}(a - b)^{r+1} + \frac{1}{2}(a + b)^{r+1} \geq \left(\frac{1}{2}(a - b) + \frac{1}{2}(a + b)\right)^{r+1} = a^{r+1} \\
\iff\quad & (a + b)^{r+1} - a^{r+1} \geq a^{r+1} - (a - b)^{r+1} \\
\implies\quad & \frac{(a + b)^{r+1} - a^{r+1}}{a^{r+1} - (a - b)^{r+1}} \geq 1 \geq 3^{-(r+1)}.
\end{aligned}$$

In the second case with $0 \leq a < b$ we have

$$\int_a^{a+b} |t|^r \, dt \Big/ \int_{a-b}^{a} |t|^r \, dt = \int_a^{a+b} t^r dt \Big/ \left(\int_{a-b}^{0} (-t)^r dt + \int_0^a t^r dt \right) = \frac{(a + b)^{r+1} - a^{r+1}}{a^{r+1} + (b - a)^{r+1}}.$$

The upper bound in (16) then follows by substituting $b - a$ with c, where $c > 0$, and considering

$$
\begin{aligned}
(2a+c)^{r+1} \leq \begin{cases} (3c)^{r+1}, & a \leq c \\ (3a)^{r+1}, & a \geq c \end{cases} \quad &\Longrightarrow \quad (2a+c)^{r+1} \leq (3a)^{r+1} + (3c)^{r+1} + a^{r+1} \\
&\Longleftrightarrow \quad (2a+c)^{r+1} - a^{r+1} \leq 3^{r+1}(a^{r+1} + c^{r+1}) \\
&\Longleftrightarrow \quad \frac{(2a+c)^{r+1} - a^{r+1}}{a^{r+1} + c^{r+1}} \leq 3^{r+1}.
\end{aligned}
$$

To establish the lower bound, notice that

$$
(a+b)^{r+1} = (2a + (b-a))^{r+1} \geq (2a)^{r+1} + (b-a)^{r+1} = 2^r 2a^{r+1} + (b-a)^{r+1} \geq 2a^{r+1} + (b-a)^{r+1},
$$

hence

$$
(a+b)^{r+1} - a^{r+1} \geq a^{r+1} + (b-a)^{r+1} \quad \Longrightarrow \quad \frac{(a+b)^{r+1} - a^{r+1}}{a^{r+1} + (b-a)^{r+1}} \geq 1 \geq 3^{-(r+1)}.
$$

□