On the constrained mock-Chebyshev least-squares

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Abstract

The algebraic polynomial interpolation on uniformly distributed nodes is affected by the Runge phenomenon, also when the function to be interpolated is analytic. Among all techniques that have been proposed to defeat this phenomenon, there is the mock-Chebyshev interpolation which is an interpolation made on a subset of the given nodes which elements mimic *as well as possible* the Chebyshev-Lobatto points. In this work we use the simultaneous approximation theory to combine the previous technique with a polynomial regression in order to increase the accuracy of the approximation of an analytic function. We give indications on how to select the degree of the simultaneous regression so as to obtain polynomial approximant good in the uniform norm and provide a sufficient condition to improve, in that norm, the accuracy of the mock-Chebyshev interpolation with a simultaneous regression. Numerical results are provided.

Keywords: Runge phenomenon, Chebyshev-Lobatto nodes, mock-Chebyshev interpolation, simultaneous regression

1. Introduction

In many scientific disciplines when we want to study a phenomenon we can start in observing and recording what happens at regular instants of time. This provides a sample of information that we can use to give a more or less accurate approximation of the observed phenomenon. For this aim mathematical tools are needful. The first step is to imagine regular instants of time as a set of uniform distributed points and the sample of information as the evaluations of an unknown function. In this case a classical technique used to associate to the discrete set of experimental data a continuous approximation of the phenomenon is the algebraic polynomial interpolation. This technique has the well-known drawback that on uniformly distributed nodes might not converge even if the considered function is regular. A classical example is given by Runge's function

$$f(t) = \frac{1}{1 + 25t^2}, \; t \in [-1, 1]$$

on an equally spaced triangular array of nodes

$$x_{0,0}; \quad x_{0,1}, x_{1,1}; \quad x_{0,2}, x_{1,2}, x_{2,2}; \quad \ldots \quad ; \quad x_{0,n}, x_{1,n}, \ldots, x_{n,n}; \quad \ldots$$

where $x_{i,n} = -1 + \frac{2}{n}i$ for $i = 0, 1, \ldots, n, n \in \mathbb{N}_0$. In this case, the error made by interpolating f with polynomials has wild oscillations, a phenomenon known as *Runge Phenomenon*. Many techniques have been proposed to defeat this phenomenon; just to mention some of them, the least-squares fitting by polynomials [1], the barycentric rational interpolation [2, 3, 4], its extended version [5], the interpolation on subintervals [6]. A further technique exploited to cut down the Runge phenomenon is the so called mock-Chebyshev subset interpolation which takes advantages of the optimality of the interpolation processes on Chebyshev-Lobatto nodes [7]. The main goal of this paper consists in a

combination of this kind of interpolation with a regression aimed to improve the accuracy of the approximation of an analytic function; we will refer to this combination as *constrained mock-Chebyshev least-squares*.

The paper is structured as follows. In Section 2 we discuss some details on the mock-Chebyshev subset interpolation. The constrained mock-Chebyshev least-squares are introduced in the Section 3 and deeply investigated in Sections 4 and 5 in which we deal with the choice of the degree of the simultaneous regression and with an estimation for the error in the uniform norm, respectively. Section 6 is devoted to some numerical results. Last Section contains the algorithm.

2. Mock-Chebyshev subset interpolation

Let f be an analytic function with singularities close to the interval $[-1, 1]$ and suppose that its evaluations are known on $n+1$ equally spaced points of that interval. The idea that underlies the mock-Chebyshev subset interpolation is to interpolate f only on a proper subset consisting of $m + 1$ of the given nodes which "looks like" the Chebyshev-Lobatto grid of order $m + 1$. The result is that if we carefully choose m, the convergence of the interpolation process on such a subset of nodes for n which tends to infinity will be preserved (cf. [8]). Some notations: from here onwards we will indicate the equispaced grid of cardinality $n + 1$ with the symbol X_n, while the mock-Chebyshev subset of X_n of order $m + 1$ will be denoted by X'_m. To understand how to properly choose m (see e.g. [9]), let us remember that the $m + 1$ Chebyshev-Lobatto nodes are defined as

$$x_j^{CL} = -\cos\left(\frac{\pi}{m}j\right), \ j = 0, 1, \ldots, m.$$

Let us expand x_1^{CL} in Taylor series centered in zero

$$x_1^{CL} = -1 + \frac{\pi^2}{2m^2} + O\left(\frac{1}{m^4}\right) < -1 + \frac{\pi^2}{2m^2}. \tag{2.1}$$

Being $x_0^{CL} = -1$, the difference $x_1^{CL} - x_0^{CL}$ is a $O\left(\frac{1}{m^2}\right)$. In other words, this means that the $m + 1$ nodes of Chebyshev-Lobatto are distributed in $[-1, 1]$ with a density that is roughly quadratic in m. So for n proportional to m^2 or m proportional to $\sqrt{n}$, we can select among the given nodes a subset which mimic a sufficiently large Chebyshev-Lobatto grid. Let c be the constant of proportionality; a way to calculate it is to impose that the second node of the Chebyshev-Lobatto grid is as close as possible to the second node of the equispaced set X_n

$$-\cos\left(\frac{\pi}{m}\right) \simeq -1 + \frac{2}{n}.$$

This can be done in the following manner: by (2.1) we fix the largest integer m such that

$$-1 + \frac{1}{n} < -1 + \frac{\pi^2}{2m^2}$$

so for

$$m = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{n} \right\rfloor \tag{2.2}$$

for sure $-1 + \frac{2}{n}$ is the point of X_n closest to x_1^{CL} (for an example, see Figure 1). This choice of $c < \frac{\pi}{\sqrt{2}}$ avoids the fact that the endpoints -1 and 1 can be selected more than once.

For analytic functions the polynomial interpolation on Chebyshev nodes converges geometrically and stably. The mock-Chebyshev interpolation is a stable procedure but its rate of convergence is subgeometric. In [10] has been shown that on equispaced nodes no stable method can converge geometrically.
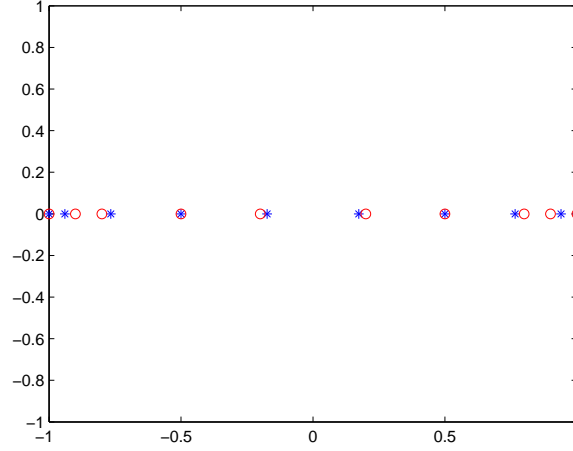
Figure 1: Plot of the Chebyshev-Lobatto nodes ($*$) and mock-Chebyshev nodes ($\circ$) for $n + 1 = 21, m = \frac{\pi}{\sqrt{2}}\sqrt{20} = 9$.

3. Constrained mock-Chebyshev least-squares

In performing the mock-Chebyshev interpolation we know the evaluations of f on the whole set X_n, but actually we only use the information corresponding to the elements of X'_m. Indeed, in [9] the $n - m$ remaining nodes are definitively discarded and the corresponding evaluations are lost. Our idea is to use those nodes, whose set will be denoted by $X''_{n-m} = \left\{x''_{1,n-m}, x''_{2,n-m}, ..., x''_{n-m,n-m}\right\}$, $x''_{1,n-m} < x''_{2,n-m} < ... < x''_{n-m,n-m}$, to improve the accuracy of the approximation through a simultaneous regression. More precisely, let f be an analytic function on $[-1, 1]$ and let $\mathcal{P}^{r*} = \left\{P \in \mathcal{P}^r : P(x'_{i,m}) = f(x'_{i,m}), \ i = 0, 1, \ldots, m\right\}$ where $\mathcal{P}^r$ is the space of polynomials of degree $\leqslant r$ and $m < r \leqslant n$. We search the solution of the following *constrained least-squares problem* [11, 12, 13]

$$\min_{P \in \mathcal{P}^{r*}} \|f - P\|_2^2 \tag{3.1}$$

where $\|\cdot\|_2$ is the discrete 2-norm on X''_{n-m}.

Theorem 3.1. *The constrained least-squares problem (3.1) has a unique solution.*

Proof. Let us denote by $P_{X'}$ the interpolating polynomial for f on X'_m. It is not difficult to verify that a generic polynomial $P \in \mathcal{P}^{r*}$ is of the form $P(t) = P_{X'}(t) + Q(t)\omega_m(t)$ with $\omega_m(t) = \prod_{i=0}^{m}(t - x'_{i,m})$ and $Q(t)$ an arbitrary polynomial of degree $r - m - 1$. The problem (3.1) then becomes

$$\min_{Q \in \mathcal{P}^{r-m-1}} \|f - (P_{X'} + Q\omega_m)\|_2^2$$

$$= \min_{Q \in \mathcal{P}^{r-m-1}} \sum_{k=1}^{n-m} \left\{f\left(x''_{k,n-m}\right) - P_{X'}\left(x''_{k,n-m}\right) - Q\left(x''_{k,n-m}\right)\omega_m\left(x''_{k,n-m}\right)\right\}^2$$

$$= \min_{Q \in \mathcal{P}^{r-m-1}} \sum_{k=1}^{n-m} \left\{\frac{f\left(x''_{k,n-m}\right) - P_{X'}\left(x''_{k,n-m}\right)}{\omega_m\left(x''_{k,n-m}\right)} - Q\left(x''_{k,n-m}\right)\right\}^2 \omega_m^2\left(x''_{k,n-m}\right).$$

By introducing the following discrete weighted 2-norm

$$\|u\|_{2,\omega_m^2} = \left(\sum_{k=1}^{n-m} w_k u^2(x''_{k,n-m})\right)^{\frac{1}{2}}$$

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where $w_k = \omega_m^2(x''_{k,n-m})$ for $k = 1, \ldots, n - m$ and by defining $\hat{f}$ as

$$\hat{f}(t) := \frac{f(t) - P_{X'}(t)}{\omega_m(t)}, \quad t \in [-1, 1], \tag{3.2}$$

the problem (3.1) can be reduced to the following classical least-squares problem

$$\min_{Q \in \mathcal{P}^{r-m-1}} \left\| \hat{f} - Q \right\|^2_{2, \omega_m^2} \tag{3.3}$$

which has a unique solution. ■

By denoting with $\hat{Q}_{X''}(t)$ the solution of (3.3), the desired polynomial approximant is

$$\hat{P}_X(t) = P_{X'}(t) + \hat{Q}_{X''}(t)\omega_m(t). \tag{3.4}$$

To write $\hat{P}_X$ explicitly, let us introduce the discrete inner product associated to the norm $\|\cdot\|_{2,\omega_m^2}$

$$(u, v)_{\omega_m^2} = \sum_{k=1}^{n-m} w_k u(x''_{k,n-m}) v(x''_{k,n-m})$$

and let $\left\{ \pi_i(t, \omega_m^2) \right\}_{i=0}^{r-m-1}$ be a basis of $\mathcal{P}^{r-m-1}$ orthogonal with respect to the previous product. We can express $\hat{Q}_{X''}(t)$ with respect to that basis as

$$\hat{Q}_{X''}(t) = \sum_{i=0}^{r-m-1} q_i \pi_i(t), \; q_i = \frac{(\hat{f}, \pi_i)_{\omega_m^2}}{(\pi_i, \pi_i)_{\omega_m^2}}.$$

Then $\hat{P}_X(t)$ becomes explicitly

$$\hat{P}_X(t) = P_{X'}(t) + \left(\sum_{i=0}^{r-m-1} q_i \pi_i(t) \right) \prod_{i=0}^{m} (t - x'_{i,n}).$$

Theorem 3.2. *In the discrete 2-norm on X''_{n-m} the inequality*

$$\left\| f - \hat{P}_X \right\|_2 < \left\| f - P_{X'} \right\|_2$$

holds.

Proof. The choice of an orthogonal basis for $\mathcal{P}^{r-m-1}$ allows us to express the error $\hat{f} - \hat{Q}_{X''}$ in the $\|\cdot\|_{2,\omega_m^2}$ norm as follows:

$$\left\| \hat{f} - \hat{Q}_{X''} \right\|_{2,\omega_m^2} = \left\{ \left\| \hat{f} \right\|^2_{2,\omega_m^2} - \sum_{i=0}^{r-m-1} q_i^2 \left\| \pi_i \right\|^2_{2,\omega_m^2} \right\}^{\frac{1}{2}}, \; q_i = \frac{(\hat{f}, \pi_i)_{\omega_m^2}}{(\pi_i, \pi_i)_{\omega_m^2}}.$$

Therefore the error $f - \hat{P}_X$ in the 2-norm is

$$\left\| f - \hat{P}_X \right\|_2 = \left\{ \left\| f - P_{X'} \right\|_2^2 - \sum_{i=0}^{r-m-1} \tilde{q}_i^2 \left\| \pi_i \omega_m \right\|_2^2 \right\}^{\frac{1}{2}}, \; \tilde{q}_i = \frac{(f - P_{X'}, \pi_i \omega_m)}{(\pi_i \omega_m, \pi_i \omega_m)}.$$

■

In other words, the error made by using the constrained mock-Chebyshev least-squares method is, in the 2-norm, strictly smaller than the error produced when we restrict ourselves to the mock-Chebyshev subset interpolation.

4. The degree of simultaneous regression

As shown in the previous section we approximate the function f with a least-squares polynomial that satisfies interpolation conditions on a mock-Chebyshev subset of the given nodes. We have not specified yet how to choose the degree of the constructed approximant $\hat{P}_X$. When this degree increases up to the total number of nodes the approximation gets worse, since the combined approximant approaches the interpolating polynomial.

Theorem 4.1. *Let r be the degree of $\hat{P}_X$ and let us denote by P_X the interpolating polynomial of f on X_n. If $r = n$ then*

$$\hat{P}_X \equiv P_X.$$

Proof. Recalling that

$$\hat{P}_X(t) = P_{X'}(t) + \hat{Q}_{X''}(t)\omega_m(t)$$

if $\hat{P}_X$ is an n degree polynomial, the regression polynomial $\hat{Q}_{X''}$ must be a $n - m - 1$ degree polynomial. Since the least-squares set X''_{n-m} has cardinality $n - m$, $\hat{Q}_{X''}$ is the interpolating polynomial for $\hat{f}$ on X''_{n-m} that is

$$\hat{Q}_{X''}(x''_{k,n-m}) = \hat{f}(x''_{k,n-m}), \; k = 1, \ldots, n - m.$$

From the previous relation it follows that

$$
\begin{aligned}
\hat{P}_X(x''_{k,n-m}) &= P_{X'}(x''_{k,n-m}) + \hat{Q}_{X''}(x''_{k,n-m})\omega_m(x''_{k,n-m}) \\
&= P_{X'}(x''_{k,n-m}) + \hat{f}(x''_{k,n-m})\omega_m(x''_{k,n-m}) \\
&= P_{X'}(x''_{k,n-m}) + \frac{f(x''_{k,n-m}) - P_{X'}(x''_{k,n-m})}{\omega_m(x''_{k,n-m})}\omega_m(x''_{k,n-m}) \\
&= f(x''_{k,n-m})
\end{aligned}
$$

that is $\hat{P}_X$ interpolates f on X''_{n-m}. However, by construction $\hat{P}_X$ interpolates also f on X'_m then it coincides with the interpolating polynomial for f on X_n by the uniqueness of the interpolating polynomial of degree n on X_n. $\blacksquare$

By taking into account this result, let us come back to the choice of a proper degree for $\hat{P}_X$. Clearly, it depends on the degree of the simultaneous regression polynomial, namely of the polynomial $\hat{Q}_{X''}$. In order to determine a degree for $\hat{Q}_{X''}$ which gives, in the uniform norm, better accuracy of the constrained mock-Chebyshev least-squares with respect to the mock-Chebyshev interpolation we use a result presented by L. Reichel in [14]. This result implies that for an equispaced set of q (internal) nodes of $[-1, 1]$

$$z_k = -1 + \frac{2k - 1}{q}, \; k = 1, \ldots, q, \tag{4.1}$$

the degree p of the least-squares polynomial should be selected so that there is a subset of cardinality $p + 1$ of the equispaced set which is close, in the mock-Chebyshev sense, to the $p + 1$ Chebyshev grid. Actually, the result presented in [14] is more general since it deals with the least-squares approximation of a function on a Jordan curve in the complex plane. To explain the outlines of Reichel's idea we use his notation. Let Γ be a Jordan curve or Jordan arc in the complex plane and let Ω the open set bounded by Γ. If Γ is a Jordan arc then Ω is void. Let $\{z_{k,q}\}_{k=1}^q$ be a set of q distinct nodes on Γ. For a given function φ on Γ, let $L_{p,q}\varphi$ denote the least-squares polynomial of degree $\leqslant p$ with respect to the semi-norm

$$\|\varphi\| := (\varphi, \varphi)^{\frac{1}{2}}$$

defined through the inner product

$$(\varphi, \psi) := \sum_{k=1}^q \varphi(z_{k,q})\overline{\psi(z_{k,q})}.$$

Moreover, let $I_p\varphi$ be the interpolating polynomial of φ at $p + 1$ distinct points $\{w_{k,p}\}_{k=0}^p$ on Γ. We write $I_p \prec L_{p,q}$ if $\{w_{k,p}\}_{k=0}^p \subset \{z_{k,q}\}_{k=1}^q$. We equip the domain and the range of $L_{p,q}$ and I_p with the uniform norm on Γ

$$\|\varphi\|_\Gamma = \sup_{z \in \Gamma} |\varphi(z)|$$

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and we denote the induced operator norm with the symbol $\|\cdot\|$. Finally, we define

$$E_p(\varphi) := \inf_{Q_p \in \mathcal{P}^p} \|\varphi - Q_p\|_\Gamma$$

The following theorem [14, Theorem 2.1] bounds the norm of the least-squares projection $L_{p,q}$ in terms of the norm of the interpolation projection I_p.

Theorem 4.2. *Let $L_{p,q}$ and I_p be defined on the set of continuous function on $\Gamma \cup \Omega$ and analytic in Ω. Then*

$$\|L_{p,q}\| \leqslant \|I_p\| \left(1 + \sqrt{q} \sup_{\|\varphi\|_\Gamma = 1} E_p(\varphi)\right), \ \forall I_p \prec L_{p,q}, \ \forall q \geqslant p. \tag{4.2}$$

By means of examples it has been shown that also when p is fixed the $\sqrt{q}$ growth of the right-hand side of (4.2) can be achieved. This suggests to make further assumptions on the distribution of the interpolation nodes and on the smoothness of the function. Generally, we will assume that p is an increasing function of q. Using a Jackson's theorem [15, p. 147] the following corollary [14, Corollary 2.1] shows that additional smoothness of the function to be approximated decreases the growth of $\|L_{p,q}\|$ with $q, p(q)$.

Corollary 4.3. *Let $\Gamma = [-1, 1]$ and let $F_{d,k,\Gamma} := \left\{ \varphi : \varphi \in C^k[-1, 1], \left\|\frac{d^k \varphi}{dz^k}\right\|_\Gamma \leqslant d \right\}$ be the domain of $L_{p,q}$. Then for some constant D depending on the constant d and on the integer k*

$$\|L_{p,q}\| \leqslant \|I_p\| \left(1 + D\sqrt{q}(p + 1)^{-k}\right), \ \forall I_p \prec L_{p,q}.$$

The next step is to determine a bound for $\min_{I_p \prec L_{p,q}} \|I_p\|$. We do not discuss in detail the estimates calculated for $\|I_p\|$ in [14] but only mention that a useful bound for $\min_{I_p \prec L_{p,q}} \|I_p\|$ is obtained when the interpolation points are Fejér points or points close to Fejér points. Let us recall that for a generic curve Γ the Fejér points are defined as the image on Γ of equispaced nodes onto the unit circle through a particular conformal mapping [14]. In particular, if $\Gamma = [-1, 1]$ the Chebyshev points are Fejér points [14, Example 3.1]. The estimates obtained for $\|I_p\|$ in [14] suggest the following least-squares approximation method:

Criterion 4.4. *Let $\Gamma = [-1, 1]$. Given a function $\varphi \in F_{d,k,\Gamma}$ and q least-squares nodes $\{z_{k,q}\}_{k=1}^q$ on Γ, choose the degree of the approximating polynomial $L_{p,q}\varphi$ as the greatest p such that $p + 1$ points are close to $p + 1$ Fejér points.*

When the q nodes are equispaced like in (4.1) this means that the degree p of the least-squares approximant should be selected so that there are $p + 1$ points among the equispaced ones which are close to the $p + 1$ Chebyshev nodes. In other words, p should be selected in the mock-Chebyshev sense.

In the case of simultaneous regression the least-squares nodes are those of X''_{n-m} and therefore they are not equally spaced. However, when the cardinality of X_n is sufficiently large we can approximate an equispaced grid with width $\geqslant 2h, h = \frac{2}{n}$ using nodes belonging to X''_{n-m}. In fact, the maximum distance between two consecutive nodes of X''_{n-m} is at most $2h$. To be aware of it, let us observe that the interval $I = \left[x''_{1,n-m}, x''_{n-m,n-m}\right]$ according to the mock-Chebyshev extraction is properly contained in $[-1, 1]$ and symmetric with respect to the origin. Because of the choice of m the first and the second node of X'_m are equal to $x_{0,n}$ and $x_{1,n}$, respectively, i.e. $X'_m = \{x_{0,n}, x_{1,n}, \dots\}$. Moreover, we have

Lemma 4.5. *The first three nodes of X_n belong to X'_m, i.e.*

$$X'_m = \{x_{0,n}, x_{1,n}, x_{2,n}, \dots\}.$$

Proof. To prove that $x_{2,n}$ together with $x_{0,n}, x_{1,n}$ has been taken during the mock-Chebyshev extraction, we need to expand in Taylor series the difference between the second and the third Chebyshev-Lobatto node

$$x_2^{CL} - x_1^{CL} = -\cos\left(\frac{2\pi}{m}\right) + \cos\left(\frac{\pi}{m}\right) = -2\sin\left(\frac{3\pi}{2m}\right)\sin\left(-\frac{\pi}{2m}\right) = 2\frac{\pi}{2m}\frac{3\pi}{2m} + O\left(\frac{\pi^4}{m^4}\right) < 2\frac{\pi}{2m}\frac{3\pi}{2m}.$$

Recalling that m is given by (2.2) the previous difference can be rounded up by $\frac{3}{n}$ and the thesis follows (see Figure 2). ∎
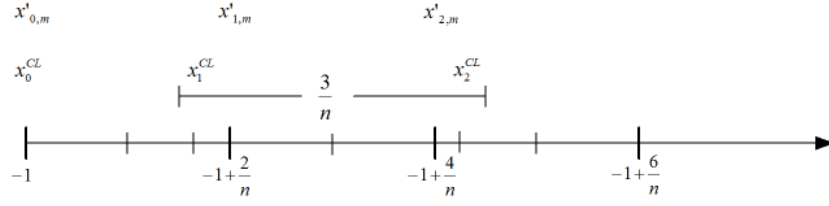
Figure 2: Proof of Lemma 4.5

Lemma 4.6. *For $n > 7$, $x_{3,n}$ does not belong to X'_m, i.e.*

$$x_{3,n} \in X''_{n-m}, \quad n > 7.$$

Proof. Let us expand x_3^{CL} in Taylor series

$$x_3^{CL} = -\cos\left(\frac{3\pi}{m}\right) = -1 + \frac{9\pi^2}{2m^2} - \frac{81\pi^4}{24m^4} + O\left(\left(\frac{3\pi}{m}\right)^6\right) > -1 + \frac{9}{n} - \frac{27}{2n^2}$$

and check for which values of $n \in \mathbb{N}$ the following inequality holds

$$-1 + \frac{9}{n} - \frac{27}{2n^2} > -1 + \frac{7}{n}.$$

We obtain that

$$n > \frac{27}{4}$$

and therefore $\left|x_3^{CL} + 1 - \frac{6}{n}\right| > \left|x_3^{CL} + 1 - \frac{8}{n}\right|$. ∎

Proposition 4.7. *For sufficiently large n the following inequality*

$$\max_{2 \leqslant i \leqslant n-m} \left|x''_{i-1,n-m} - x''_{i,n-m}\right| \leqslant 2h$$

holds.

Proof. The thesis is equivalent to the fact that among the nodes of X'_m belonging to $I = \left[-1 + \frac{6}{n}, 1 - \frac{6}{n}\right]$ there are not two consecutive nodes of X_n. By Lemma 4.5 and Lemma 4.6 the nodes of the $m + 1$ Chebyshev-Lobatto grid which are contained in I are

$$x_j^{CL} = -\cos\left(\frac{\pi}{m}j\right), \quad j = 3, \ldots, m - 3. \tag{4.3}$$

It is well-known that the nodes (4.3) are more dense near the endpoints of I and less near its center, therefore it is sufficient to verify that the distance between x_3^{CL} and x_4^{CL} is greater than $2h$. Let us expand in Taylor series $x_4^{CL} - x_3^{CL}$

$$
\begin{aligned}
x_4^{CL} - x_3^{CL} &= -\cos\left(\frac{4\pi}{m}\right) + \cos\left(\frac{3\pi}{m}\right) = -2\sin\left(\frac{7\pi}{2m}\right)\sin\left(-\frac{\pi}{2m}\right) \\
&= 2\left(\frac{7\pi}{2m} - \left(\frac{7\pi}{2m}\right)^3\frac{1}{6} + O\left(\left(\frac{7\pi}{2m}\right)^5\right)\right)\left(\frac{\pi}{2m} - \left(\frac{\pi}{2m}\right)^3\frac{1}{6} + O\left(\left(\frac{7\pi}{2m}\right)^5\right)\right) \\
&= \frac{7\pi^2}{2m^2} - \frac{175\pi^4}{24m^4} + O\left(\frac{\pi^6}{m^6}\right)
\end{aligned}
$$

round downward by

$$\frac{7}{n} - \frac{175}{6n^2} < x_4^{CL} - x_3^{CL}$$

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and impose that

$$\frac{4}{n} < \frac{7}{n} - \frac{175}{6n^2}.$$

From the previous inequality it follows that

$$n > \frac{175}{18} \simeq 9.72$$

and the thesis holds. ∎

At this point we can apply the results presented in [14] to the simultaneous regression. Taking into account that the grid (4.1) is equispaced with width $\frac{2}{q}$ in $\left[-1 + \frac{1}{q}, 1 - \frac{1}{q}\right]$, we note that, for n sufficiently large, we can approximate such a grid with $q = \frac{n}{6} = \frac{1}{3h}$ and nodes coming from X''_{n-m}. We denote this grid with $\tilde{X}''_{n-m}$. The choice for the degree of the simultaneous regression which gives good approximation in the uniform norm is therefore

$$p = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{q} \right\rfloor = \left\lfloor \frac{\pi}{\sqrt{2}} \sqrt{\frac{n}{6}} \right\rfloor. \tag{4.4}$$

Let us observe that since the degree of the mock-Chebyshev interpolation and the degree of the regression are chosen in the same way, we can obtain the previous result applying to X''_{n-m} the idea explained in [9], that is imposing that

$$-\cos\left(\frac{\pi}{p}\right) \simeq -1 + \frac{6}{n}.$$

It is a straightforward calculus to prove that p will be like in (4.4).

5. Uniform norm estimation

We have determined the degree p as in (4.4) for the polynomial $\hat{Q}_{X''}$ which, according to Reichel's theory, gives good approximation in the uniform norm. Now, we want to calculate an estimation for the norm error $E_{\hat{P}_X}(f) = \|f - \hat{P}_X\|_\infty$ in the uniform norm. Let $\hat{P}_X : C[-1,1] \to \mathcal{P}^{r*}$ the projection operator which associates to a continuous function in $[-1,1]$ its constrained mock-Chebyshev polynomial and let $\hat{Q}_{X''} : C[-1,1] \to \mathcal{P}^{r-m-1}$ the projection operator which associates to a continuous function in $[-1,1]$ its least-squares polynomial in the norm $\|\cdot\|_{2,\omega_m^2}$.

As in the proof of Theorem 4.2 and Corollary 4.3 we get an estimate for the operator norm $\|\hat{Q}_{X''}\|$.

Theorem 5.1. *Let $\varphi \in C[-1,1]$ and $I_p\varphi$ be the interpolating polynomial of φ on the $p + 1$ mock-Chebyshev subset $X'''_p = \left\{x'''_{k,p}\right\}_{k=0}^p$ of $\tilde{X}''_n$. Then*

$$\left\|\hat{Q}_{X''}\right\| \leqslant \|I_p\| \left(1 + \frac{\left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\min_{j=0,\dots,p} \sqrt{\tilde{w}_j}} \sup_{\|\varphi\|_\infty = 1} E_p(\varphi)\right)$$

Proof. Let $Q_p^*\varphi$ be the polynomial of degree $\leqslant p$ such that $E_p(\varphi) = \left\|\varphi - Q_p^*\varphi\right\|_\infty$. By (3.3)

$$\left\|\hat{Q}_{X''}\varphi - \varphi\right\|_{2,\omega_m^2} \leqslant \left\|Q_p^*\varphi - \varphi\right\|_{2,\omega_m^2}.$$

On the other hand,

$$\begin{aligned}
\left\|Q_p^*\varphi - \varphi\right\|_{2,\omega_m^2} &= \left(\sum_{k=1}^{n-m} w_k \left(Q_p^*(x''_{k,n-m}) - \varphi(x''_{k,n-m})\right)^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}} \left\|Q_p^*\varphi - \varphi\right\|_\infty \\
&= \left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}} E_p(\varphi).
\end{aligned} \tag{5.1}$$

Let $l_k(t)$ $k = 0, \ldots, p$ be the elementary Lagrangian polynomials associated with X_p''', that is

$$I_p\varphi(t) = \sum_{j=0}^{p} \varphi(x_{j,p}''')l_j(t).$$

Let us express $\hat{Q}_{X''}\varphi$ in the same basis as

$$\hat{Q}_{X''}\varphi(t) = \sum_{j=0}^{p} \alpha_j l_j(t)$$

for some coefficients α_j. From (5.1) it follows that

$$\sqrt{\widetilde{w}_j}\left|\alpha_j - \varphi(x_{j,p}''')\right| \leqslant \left\|\hat{Q}_{X''}\varphi - \varphi\right\|_{2,\omega_m^2} \leqslant \left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}} E_p(\varphi), \; j = 0, \ldots, p.$$

where $\widetilde{w}_j$, $j = 0, \ldots, p$ are the positive weights corresponding to the nodes $\left\{x_{k,p}'''\right\}_{k=0}^{p}$ and then

$$\left|\alpha_j - \varphi(x_{j,p}''')\right| \leqslant \frac{\left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\sqrt{\widetilde{w}_j}} E_p(\varphi).$$

Substituting the previous relation into

$$\left|\hat{Q}_{X''}\varphi(t)\right| \leqslant \sum_{j=0}^{p}\left|\alpha_j - \varphi(x_{j,p}''')\right| |l_j(t)| + \sum_{j=0}^{p}\left|\varphi(x_{j,p}''')\right| |l_j(t)|$$

we obtain

$$\left\|\hat{Q}_{X''}\right\| = \sup_{\|\varphi\|_\infty = 1}\left\|\hat{Q}_{X''}\varphi\right\|_\infty \leqslant \|I_p\| \frac{\left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\min_{j=0,\ldots,p} \sqrt{\widetilde{w}_j}} \sup_{\|\varphi\|_\infty = 1} E_p(\varphi) + \|I_p\|$$

which proves the theorem. ∎

Recall that, fixed $\Gamma = [-1, 1]$, according to [14], for each $k \in \mathbb{N}$ and $d > 0$ we set

$$F_{d,k,\Gamma} := \left\{\varphi : \varphi \in C^k[-1, 1], \; \left\|\frac{d^k\varphi}{dz^k}\right\|_\Gamma \leqslant d\right\}.$$

Corollary 5.2. *If $\hat{Q}_{X''}$ has domain $F_{d,k,\Gamma}$ there exists a constant D depending on d and on the integer k such that*

$$\left\|\hat{Q}_{X''}\right\| \leqslant \|I_p\| \left(1 + D\frac{\left(\sum_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\min_{j=0,\ldots,p} \sqrt{\widetilde{w}_j}}(p + 1)^{-k}\right) \tag{5.2}$$

Proof. From a Jackson's theorem [15, p. 147] for $\varphi \in F_{d,k,\Gamma}$ it follows

$$E_p(\varphi) \leqslant D(p + 1)^{-k}$$

where D is a constant depending on d and on the integer k. ∎

With these results in mind we can provide an estimate in the uniform norm for the error of the constrained mock-Chebyshev least-squares.

Theorem 5.3. *Let* $f \in F_{d,p,\Gamma}$. *Then*

$$E_{\hat{P}_X}(f) \leqslant \left(1 + \|I_p\| \left(1 + D \frac{\left(\sum\limits_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\min\limits_{j=0,\ldots,p} \sqrt{\widetilde{w}_j}}(p+1)^{-p}\right)\right) E_p(\hat{f}) \|\omega_m\|_\infty \tag{5.3}$$

Proof. Let us start from the following relations

$$\begin{aligned} E_{\hat{P}_X}(f) &= \left\| f - P_{X'}f - \hat{Q}_{X''}\hat{f}\omega_m \right\|_\infty \\ &= \left\| \frac{f - P_{X'}f}{\omega_m}\omega_m - \hat{Q}_{X''}\left(\frac{f - P_{X'}f}{\omega_m}\right)\omega_m \right\|_\infty \\ &\leqslant E_{\hat{Q}_{X''}}\left(\frac{f - P_{X'}f}{\omega_m}\right)\|\omega_m\|_\infty . \end{aligned}$$

where $E_{\hat{Q}_{X''}}\left(\frac{f-P_{X'}f}{\omega_m}\right)$ is the uniform norm error made in approximating $\hat{f}$ with its least-squares polynomial in the norm $\|\cdot\|_{2,\omega_m^2}$. Since $\hat{Q}_{X''}$ is a projection operator which reproduces the polynomials the following inequality holds

$$E_{\hat{Q}_{X''}}\left(\frac{f - P_{X'}f}{\omega_m}\right) \leqslant \left(1 + \|\hat{Q}_{X''}\|\right) E_p(\hat{f})$$

where $E_p(\hat{f}) = \min\limits_{Q \in \mathcal{P}^p} \|\hat{f} - Q\|_\infty$. Therefore

$$E_{\hat{P}_X}(f) \leqslant \left(1 + \|\hat{Q}_{X''}\|\right) E_p(\hat{f}) \|\omega_m\|_\infty$$

which applying Corollary 5.2 to f gives the thesis. ∎

Theorem 5.3 gives a sufficient condition to improve in the uniform norm the accuracy of the mock-Chebyshev interpolation through the constrained mock-Chebyshev least-squares.

Corollary 5.4. *Let* $f \in C^{m+1}[-1,1]$. *If*

$$\left(1 + \|I_p\| \left(1 + D \frac{\left(\sum\limits_{k=1}^{n-m} w_k\right)^{\frac{1}{2}}}{\min\limits_{j=0,\ldots,p} \sqrt{\widetilde{w}_j}}(p+1)^{-p}\right)\right) E_p(\hat{f}) < \frac{\|f^{(m+1)}\|}{(m+1)!}$$

then

$$E_{\hat{P}_X}(f) < E_{P_{X'}}(f)$$

where $E_{P_{X'}}(f) = \|f - P_{X'}\|_\infty$.

Proof. Let us recall that the error in the Lagrange interpolation can be bounded as follows

$$E_{P_{X'}}(f) \leqslant \frac{\|f^{(m+1)}\|}{(m+1)!} \|\omega_m\|_\infty .$$

From Theorem 5.3 we get the thesis. ∎

Finally, the following corollary shows that the operator $\hat{P}_X$ reproduces polynomials of degree $\leqslant m + p$.

Corollary 5.5. *If* $f = p_r$ *with* $p_r \in \mathcal{P}^{m+p}$, *then*

$$\hat{P}_X f = f$$

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Proof. If $f = p_r$ with $r \leqslant m$

$$\hat{f}(t) = \frac{p_r(t) - P_{X'} p_r(t)}{\omega_m(t)} = \frac{p_r^{(m+1)}(\xi_t)}{(m+1)!} \equiv 0.$$

If $f = p_r$ with $m < r \leqslant m + p$

$$\hat{f}(t) = \frac{p_r(t) - P_{X'} p_r(t)}{\omega_m(t)}$$

is a polynomial of degree $r - (m + 1)$. In both cases $E_p(\hat{f}) = 0$ and the right-hand side of (5.3) is zero. ∎

6. Numerical results

We finally carried out a series of numerical tests to compare, in the uniform norm, the approximation of the constrained mock-Chebyshev least-squares and the mock-Chebyshev interpolation. A first set of test functions includes the following ones (the first three functions were already considered in [16]):

$$f_1(t) = \sqrt{|t|},$$

$$f_2(t) = \frac{1}{1 + 25t^2},$$

$$f_3(t) = \frac{10^{-15}}{10^{-15} + 25t^2}, \qquad t \in [-1, 1].$$

$$f_4(t) = t\,|t|,$$

The function f_1 is Hölder continuous with exponent $1/2$, the function f_3 is a modification of f_2 obtained by introducing the exponential 10^{-15} in order to squash f_2 on x and y axes, the function f_4 is of class C^1. The errors are computed as the maximum absolute value of the difference between the approximant and the exact function at 10001 equispaced points in $[-1, 1]$. Let us rename with p the degree of the simultaneous regression polynomial $\hat{Q}_{X''}$. In Table 1 p ranges from $p = 28$ to $p = 100$. We denote with p^* the degree of the simultaneous regression which, according to the theory explained above, gives good approximation in the uniform norm. Table 1 allows to compare the two errors of interest in the case of $n + 1 = 1001$ equispaced interpolation nodes. At the top of the table, in green, is highlighted the error $E_{\hat{P}_X}(f_i)$ in correspondence of the degree p^*. In red is highlighted the minimum possible error $E_{\hat{P}_X}(f_i)$ in the range $[1, n - m - 1]$. At the bottom, in blue, is represented the error $E_{P_{X'}}(f_i)$. As we can see, the constrained mock-Chebyshev least-squares improve the accuracy of the approximation of the mock-Chebyshev interpolation. We note that in correspondence of the degree p^* we obtain an improvement of the accuracy of approximation. More in detail, for f_1 there is an interval for p in which the approximation obtained with our method is better than the one coming from the mock-Chebyshev interpolation. In this case the improvement involves only the coefficients. When the function to be approximated is the Runge function, our approximation is everywhere more accurate for p ranging from 1 to 100. In particular, there is a range for p in which we get 2 digits of precision more than the mock-Chebyshev interpolation and p^* lies in this range. For f_3 our approximation is, up to a certain value, better but almost the same of the approximation obtained with the mock-Chebyshev interpolation and then gets little worse. In the case of f_4 there is an interval for p in which we get 1 digits of precision more than the mock-Chebyshev interpolation.

We have done further tests using the Runge function and the following ones:

$$f_5(t) = \frac{1}{t^2 - (1 + 0.5)},$$

$$f_6(t) = \frac{1}{t^4 + \left(\frac{\sqrt{26}}{5} - 1\right)t^2 + \left(\frac{13}{50}\right)^2}, \quad t \in [-1, 1],$$

$$f_7(t) = \frac{1}{t^4 + \left(\frac{2}{50}\right)^2},$$

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p	$E_{\hat{P}_X}(f_1)$	$E_{\hat{P}_X}(f_2)$	$E_{\hat{P}_X}(f_3)$	$E_{\hat{P}_X}(f_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	7.9726586e − 002	9.7493857e − 009	9.9994994e − 001	5.4308526e − 005
29	7.8915085e − 002	8.5899644e − 009	9.9994769e − 001	5.4308526e − 005
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
33	7.7268588e − 002	6.2480174e − 009	9.9994276e − 001	4.8879070e − 005
34	7.7268642e − 002	6.2483426e − 009	9.9994277e − 001	4.6554802e − 005
35	7.7593676e − 002	7.6886833e − 009	9.9994378e − 001	4.6554852e − 005
36	7.7593662e − 002	7.6886787e − 009	9.9994377e − 001	4.8513243e − 005
37	7.6667437e − 002	5.8468658e − 009	9.9994084e − 001	4.8512907e − 005
38	7.6667394e − 002	5.8470333e − 009	9.9994083e − 001	5.0626752e − 005
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
47	7.5926645e − 002	7.2563305e − 009	9.9993836e − 001	7.8662677e − 005
48	7.5926555e − 002	7.2566879e − 009	9.9993834e − 001	8.3106886e − 005
49	7.6081471e − 002	8.0118418e − 009	9.9993892e − 001	8.3106580e − 005
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
59	9.8058844e − 002	9.7826094e − 009	9.9993832e − 001	1.2059132e − 004
60	9.8061139e − 002	9.7829010e − 009	9.9993831e − 001	1.2342356e − 004
61	1.0514604e − 001	1.1889342e − 008	9.9993920e − 001	1.2342492e − 004
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
99	3.5158374e − 001	2.9978376e − 008	3.0304570e + 000	3.8993185e − 004
100	3.5157737e − 001	2.9977317e − 008	3.0304057e + 000	4.0022643e − 004

$E_{P_{X'}}(f_1)$	$E_{P_{X'}}(f_2)$	$E_{P_{X'}}(f_3)$	$E_{P_{X'}}(f_4)$
8.7569583e − 002	8.9863528e − 007	9.9996656e − 001	1.5095571e − 004

Table 1: Comparison between $E_{\hat{P}_X}(f_i)$ and $E_{P_{X'}}(f_i)$ for $n = 1000$. In this case $m = 70$, $p^* = 28$.

which, as the Runge function, are analytic in the interval $[-1, 1]$. The function f_5 has poles at $\pm \sqrt{1 + 0.5}$, while the function f_6 has poles at $\frac{1}{5} \pm i\frac{1}{10}$ and $-\frac{1}{5} \pm i\frac{1}{10}$ and the function f_7 has poles at $\frac{1}{5\sqrt{2}} \pm i\frac{1}{5\sqrt{2}}$ and $-\frac{1}{5\sqrt{2}} \pm i\frac{1}{5\sqrt{2}}$. Figure 3 compares the errors for f_2. The error in the constrained mock-Chebyshev least-squares is, for every $30 \leqslant n \leqslant 3530$, smaller than the error in the mock-Chebyshev interpolation. The number $n = 3530$ is due to the fact that the constrained mock-Chebyshev least-squares method reaches order 10^{-15} on $n + 1 = 3531$ equispaced nodes. The accuracy of the mock-Chebyshev interpolation on the same set of nodes is of order 10^{-12}. Figure 4 shows how the errors vary for the function f_5 when $20 \leqslant n \leqslant 292$. Also in this case the approximation provided by the constrained mock-Chebyshev least-squares is more accurate than the one provided by the mock-Chebyshev interpolation and again when the accuracy of the former is of order 10^{-15} the accuracy of the latter is of order 10^{-11}. Figure 5 shows the errors behaviour for the function f_6 when $40 \leqslant n \leqslant 924$ and the results are similar than in the previous cases. Finally, Figure 6 compares the errors for f_7. In this case, the maximum order of precision that can be reached by the constrained mock-Chebyshev method is 10^{-12}.

The remaining part of the present Section is devoted to the comparison of the constrained mock-Chebyshev method with some Radial Basis Functions, Hermite Function interpolation (cf. [17]) and Floater-Hormann barycentric interpolation. A difference between these techniques and the constrained mock-Chebyshev least-squares is the structure of the approximation. Indeed, only the constrained mock-Chebyshev least-squares is based on polynomials, while the other approximants belong to other classes of functions.

Constrained mock-Chebyshev method vs RBF interpolation

Given n points $\xi_1, \ldots \xi_n$ in $[-1, 1]$ (called centers) and the corresponding values f_i of a given function f on them, an RBF interpolant for f takes the form
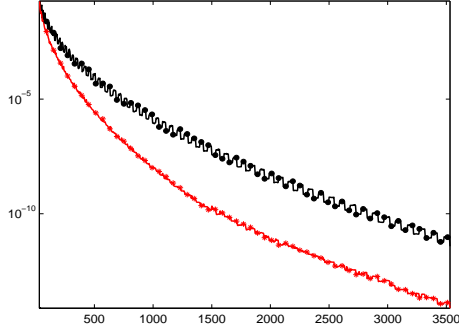
$$S(t) = \sum_{i=1}^{n} \lambda_i \phi(|t - \xi_i|)$$

Figure 3: Comparison between $E_{\hat{P}_X}(f_2)$ ($*$) (lower curve) and $E_{P_{X'}}(f_2)$ ($\bullet$) (upper curve) for $30 \leqslant n \leqslant 3530$. When $n = 3530$, $\deg(\hat{P}_X f_2) = m + p^* = 131 + 53$.
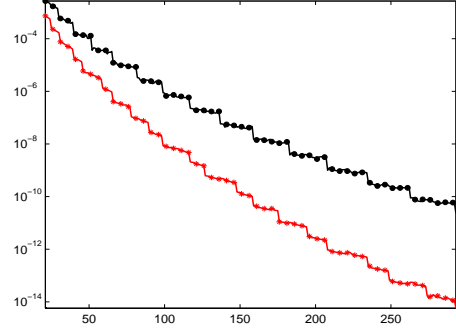

Figure 4: Comparison between $E_{\hat{P}_X}(f_5)$ ($*$) (lower curve) and $E_{P_{X'}}(f_5)$ ($\bullet$) (upper curve) for $20 \leqslant n \leqslant 292$. When $n = 292$, $\deg(\hat{P}_X f_5) = m + p^* = 37 + 15$.
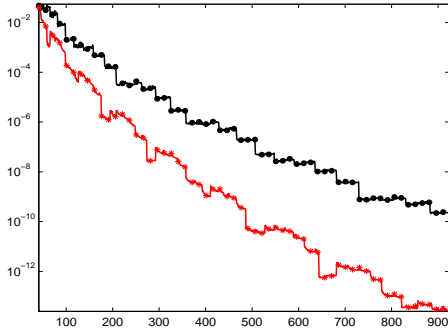

Figure 5: Comparison between $E_{\hat{P}_X}(f_6)$ ($*$) (lower curve) and $E_{P_{X'}}(f_6)$ ($\bullet$) (upper curve) for $40 \leqslant n \leqslant 924$. When $n = 923$, $\deg(\hat{P}_X f_6) = m + p^* = 67 + 27$.
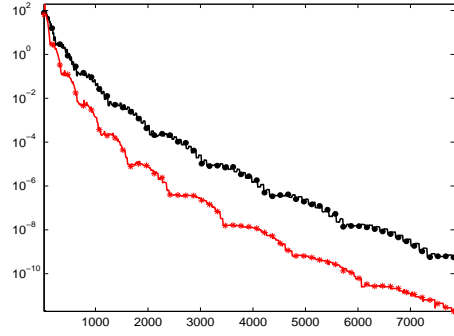
Figure 6: Comparison between $E_{\hat{P}_X}(f_7)$ ($*$) (lower curve) and $E_{P_{X'}}(f_7)$ ($\bullet$) (upper curve) for $20 \leqslant n \leqslant 7843$. When $n = 7843$, $\deg(\hat{P}_X f_7) = m + p^* = 196 + 80$.

where $\phi(r)$ is a function defined for $r \geqslant 0$. The λ_i are determined, as usual, by imposing the interpolation conditions $S(\xi_j) = f_j$, $j = 1, ..., n$. Popular choices for $\phi(r)$ are

- $\phi(r) = |r|^{2m+1}$, Monomials (MN),

- $\phi(r) = (1 - r)_+^4 (1 + 4r)$, Wendland (W2),

- $\phi(r) = \frac{1}{\sqrt{1+(\varepsilon r)^2}}$, Inverse Multiquadric (IMQ),

- $\phi(r) = \exp(-(\varepsilon r)^2)$, Gaussian (G),

ε is known as *shape parameter* since as $\varepsilon \to 0$ RBFs become flatter, while $\varepsilon \to \infty$ makes the RBFs spiky. The first two are parameter-free and piecewise smooth, while inverse multiquadrics and gaussians are infinitely smooth and depend on ε. Although we will numerically compare the constrained mock-Chebyshev method with the RBF interpolants associated to every choice of ϕ listed above, from a theoretical point of view we focus our attention on the Gaussian RBFs (GRBFs). In [18] it has been proved that, when $\varepsilon \to 0$, smooth RBF interpolants converges on the polynomial interpolants on the same nodes. This means that, in such a flat limit case, as the polynomial interpolation also the RBF approximation on uniform grids suffers of the Runge phenomenon. Furthermore, in [19] the author showed that the GRBFs on equally spaced nodes and fixed parameter diverge when interpolating functions that have poles in the Runge region of polynomial interpolation. A way to avoid the Runge phenomenon when interpolating with GRBF is

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to vary the shape parameter with n. Indeed, as suggested in [20], if we define $\alpha = \varepsilon\frac{2}{n}$, for $\alpha = O\left(\frac{1}{\sqrt[4]{n}}\right)$ the Runge phenomenon disappears. Such a choice has a drawback since, as $n \to \infty$, the condition number of the interpolation matrix increases exponentially. Hence, the GRBFs can defeat the Runge Phenomenon just as the constrained mock-Chebyshev least-squares, but being ill-conditioned they can be used only on few nodes. Ill-conditioning, mainly due to the basis of translates, can be reduced significantly by using stable bases, as discussed in [21].
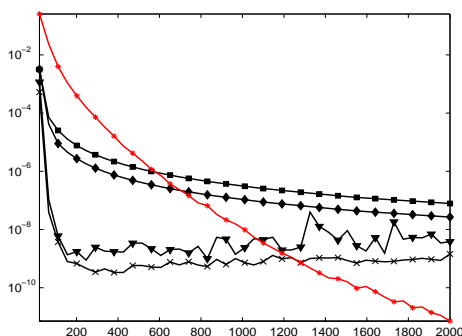
Figure 7: Comparison between $E_{\hat{P}_X}(f_2)$ ($*$) and the errors obtained in approximating f_2 with (from top to bottom) W2 ($\blacksquare$), MN ($\blacklozenge$), G ($\blacktriangledown$), and IMQ ($\times$) RBF interpolants for $20 \leqslant n \leqslant 2000$.
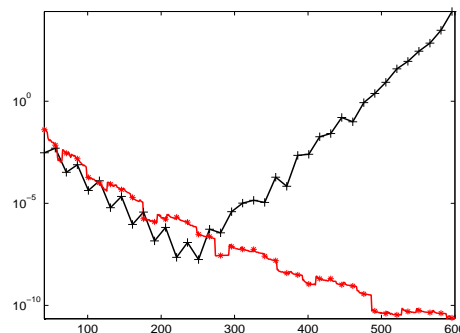
Figure 8: Comparison between $E_{\hat{P}_X}(f_6)$ ($*$) and the error obtained in approximating f_6 with the Hermite function interpolant ($+$) for $40 \leqslant n \leqslant 600$.
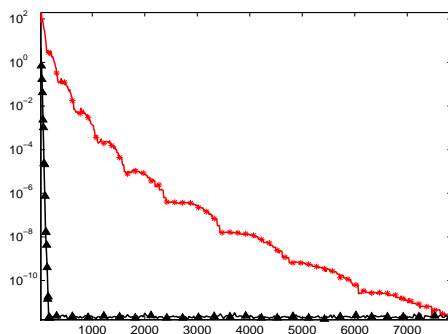

Figure 9: Comparison between $E_{\hat{P}_X}(f_7)$ ($*$) (upper curve), and the error in the Floater-Hormann barycentric interpolation ($\blacktriangle$) (lower curve) for $20 \leqslant n \leqslant 7843$.

Figure 7 shows that, in approximating the Runge function f_2, the constrained mock-Chebyshev least-squares are, for initial values of n, less accurate than the RBFs interpolants, while, as n increases, they become more accurate. To have an idea of the discrepancy, while the constrained mock-Chebyshev least-squares reach order 10^{-15} (see Figure 3), the order of the RBFs interpolants for large n ranges from 10^{-7} to 10^{-9}. In performing this numerical test, for every fixed n, we have determined the shape parameter of IMQ and GRBFs using the so called *Trial & Error* technique which consists in varying ε into a fixed (discrete) range and choosing the "optimal" parameter as the one that produces the minimum error. Unfortunately this method requires a lot of CPU time for finding the "optimal" shape parameter. Other techniques are also available, as those described in [22, Ch. 17], but for our purposes the Trial & Error is a good way to estimate the optimal ϵ.

Constrained mock-Chebyshev method vs Hermite function interpolation

For a given function f the Hermite function interpolant on n points $\xi_1, \ldots \xi_n$ in $[-1, 1]$ can be expressed in the first

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barycentric form as

$$H(t) = \Omega(t) \sum_{j=1}^{n} \frac{\mu_j}{t - \xi_j} f(\xi_j), \quad \Omega(t) = \exp(-(n-1)/2 \log(4)\gamma^2 t^2) \prod_{i=1}^{n} (t - \xi_j), \quad \mu_j = \left(\frac{d\Omega}{dt}(\xi_j) \right)^{-1}$$

where γ is a free parameter (optimal choices are 1 or slightly smaller). As stated in [17], the computational cost of the previous formula is $O(n^2)$ which means that the Hermite function interpolation is cheaper than the GRBF interpolation. Furthermore, in the same paper the authors give numerical evidence that the Hermite function interpolation is substantially more accurate than the GRBF interpolation. However, as RBFs, also this kind of interpolation is strongly ill-conditioned and therefore its use must be limited to a maximum of about 250 interpolation points. Figure 8 shows how the ill-conditioning limits to 10^{-8} the best attainable accuracy in approximating f_6 with the Hermite interpolant, while the constrained mock-Chebyshev least-squares are very close to machine precision (see Figure 5).

Constrained mock-Chebyshev method vs Floater-Hormann interpolation

A Floater-Hormann interpolant is a rational global approximant obtained blending local interpolating polynomials. More precisely, given $n + 1$ distinct points $-1 = x_0 < x_2 < \ldots < x_n = 1$ and fixed an integer d such that $0 \leqslant d \leqslant n$, a Floater-Hormann barycentric interpolant for f can be written as

$$R(t) = \sum_{i=0}^{n-d} v_i(t) p_i(t) \Big/ \sum_{i=0}^{n-d} v_i(t)$$

where $p_i(t)$ is the polynomial of degree at most d which interpolates f in $x_i, \ldots, x_{i+d}$, $i = 0, \ldots, n - d$, while

$$v_i(t) = \frac{(-1)^i}{(t - x_i) \ldots (t - x_{i+d})}.$$

This is a stable technique as confirmed by the study of the Lebesgue constant in [23]. Looking at Figure 9, it is evident that, in approximanting f_7, the Floater-Hormann interpolant reaches 10^{-12} on few nodes, but then stabilizes without gaining anymore precision. Such a limit seems to be related to the smoothness of the function and to the location of its poles within the Runge region. The error in the Floater-Hormann barycentric interpolation has been calculated using the Chebfun algorithms which for each value of n choose the "best" blending parameter [24].

From previous comparisons we can conclude that the constrained mock-Chebyshev least-squares are a competitive *polynomial strategy* for defeat the Runge phenomenon. In this context, we can affirm that this method currently provides the best we can expect from polynomials.

7. Algorithm

Let us recall that, fixed p as in (4.4), the polynomial $\hat{P}_X$ is given by

$$\hat{P}_X(t) = P_{X'}(t) + \hat{Q}_{X''}(t)\omega_m(t)$$

where the polynomial $\hat{Q}_{X''}$ is the solution of the following least-squares problem

$$\min_{Q \in \mathcal{P}^p} \| f - P_{X'} - Q\omega_m \|_2^2.$$

We can express the previous minimum problem in matrix-form as follows

$$\min_{c \in \mathbb{R}^{p+1}} \| Ac - b \|_2^2 \tag{7.1}$$

where $A = \left[\omega_m(x''_{i,n-m}) \times (x''_{i,n-m})^{j-1} \right]_{\substack{i=1,\ldots,n-m \\ j=1,\ldots,p+1}}$ is a real $(n - m) \times (p + 1)$ matrix, $c = [c_1, \ldots, c_{p+1}]^T$ is the vector of coefficients of $\hat{Q}_{X''}$ and $b = \left[P_{X'}(x''_{1,n-m}) - f(x''_{1,n-m}), \ldots, P_{X'}(x''_{n-m,n-m}) - f(x''_{n-m,n-m}) \right]^T$. Thus, the polynomial $\hat{P}_X$ can be computed using the following algorithm:

Algorithm 1 Constrained mock-Chebyshev least-squares

Input: X_n, the set of $n + 1$ equispaced nodes in $[-1, 1]$ and the evaluations of f at X_n

    1. Determine the subset X'_m of X_n whose elements are the nearest to the $m + 1$ Chebyshev-Lobatto nodes and its complement X''_{n-m};

    2. Compute the polynomial $P_{X'}$ of degree m which interpolates f on X'_m;

    3. Compute the polynomial ω_m;

    4. Form the matrix A;

    5. Solve $\min_{c \in \mathbb{R}^{p+1}} \|Ac - b\|_2^2$;

  Output: $\hat{P}_X = P_{X'} + \hat{Q}_{X''}\omega_m$.

For the sake of better readability, in Algorithm 1 we have not specified that, when we deal with the computation of a polynomial (cf. Steps 2-3), we refer to its evaluations on a given array. To improve the performance of this algorithm we implemented Step 2 using the barycentric formula (cf. [25]). Such a formula is stable (cf. [26]) and its computational cost is $O(m^2) = O(n)$. The evaluations of $\hat{Q}_{X''}$ and ω_m are performed using the Horner algorithm. Let us observe that Step 5 is the most expensive one. Since A has full rank, if we solve (7.1) with the Householder QR factorization (which is a stable method) we need $2(n-m)(p+1)^2 - 2(p+1)^3/3$ flops (cf. [27]). Recalling that both m and p are proportional to $\sqrt{n}$, solving (7.1) requires $O(n^2)$ flops. Thus, the cost of the constrained mock-Chebyshev least-squares is $O(n^2)$.

8. Conclusion and perspective

In this work, we have combined the mock-Chebyshev interpolation with a simultaneous regression, to defeat the Runge Phenomenon for analytic functions with singularities close to the interval $[-1, 1]$. We have determined a degree for the simultaneous regression and a sufficient condition under which for such a degree the error of the constrained mock-Chebyshev method is, in the uniform norm, less than the error of the mock-Chebyshev interpolation. The proposed examples confirms that, in the uniform norm, the constrained mock-Chebyshev least-squares has better accuracy than the mock-Chebyshev interpolation. It might be interesting to extend this idea to the multivariate case on domains whose optimal distribution of nodes is known (cf. [28]).

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