Aldaz-Kounchev-Render operators on simplices

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Abstract

We introduce the Aldaz-Kounchev-Render operators on a multidimensional simplex. In the case of the unit simplex of $\mathbb{R}^m$ these operators preserve the functions $1, x_1^j, \ldots, x_m^j$, j a positive integer. The Voronovskaja formula, the behaviour with respect to the convex function, and the limit of iterates of the operators are investigated.

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1. Introduction

Starting from the classical Bernstein operators B_n defined on $C[0,1]$, several modifications have been considered with the aim to obtain operators preserving some prescribed functions. J.P. King [16] constructed linear positive operators which preserve the functions e_0 and e_2. Here and in what follows we use the notation $e_j(t) = t^j$, $j = 0, 1 \ldots$, $t \in [0,1]$.

For a fixed $j \in \mathbb{N}$ and $n \geq j$, Aldaz, Kounchev and Render [6] introduced a polynomial operator $B_{n,j} : C[0,1] \to C[0,1]$ preserving e_0 and e_j. The operator is a linear combination of the classical Bernstein basis functions $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ but using

$$t_{n,k,j} = \left(\prod_{l=0}^{j-1} \frac{k-l}{n-l} \right)^{1/j}$$

for the point evaluations of the function f, i.e.,

$$B_{n,j}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(t_{n,k,j}\right). \tag{1.1}$$

The properties of the Aldaz-Kounchev-Render operators were studied in several papers. The Voronovskaja formula conjectured in [10] was proved in [9]. For other properties the reader is referred to [12], [11], [4].

The Aldaz-Kounchev-Render operators on hypercube were introduced and studied in [1]. A conjecture concerning the Voronovskaja formula for these operators was formulated in the same paper and a partial solution was given in [2].

In this paper we introduce the Aldaz-Kounchev-Render operators $B_{m,j}$ on a simplex of $\mathbb{R}^m$. Their essential property is the preservation of the functions $1, x_1^j, \cdots, x_m^j$. Section 2 contains the corresponding definitions. In this section we use barycentric coordinates. For the next sections the cartesian coordinates are more convenient. A conjecture about the Voronovskaja type formula as well a partial solution are presented in Section 3. The behaviour of the operators in relation with convex functions is investigated in Section 4. Section 5 is devoted to the limit of iterates of Aldaz-Kounchev-Render operators.

Some basic definitions are recalled in what follows.

Let $\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_m \in \mathbb{R}^m$ be $m+1$ points in general position, i.e., the vectors $\overline{p}_k - \overline{p}_0$, $k = 1, \ldots, m$, are linearly independent. A point $\overline{p}$ in the affine hull of $\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_m$ can be uniquely represented as $\overline{p} = \sum_{k=0}^{m} u_k \overline{p}_k$, where $u_0 + u_1 + \cdots + u_m = 1$.

The components of $\overline{u} = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1}$ are called the *barycenter coordinates* of $\overline{p}$ with respect to $\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_m$. The points of the simplex spanned by $\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_m$ have nonnegative barycentric coordinates and are characterized by this property. Therefore, we can identify this simplex with

$$S_m := \{\overline{u} = (u_0, u_1, \ldots, u_m) \,|\, u_k \geq 0,\ u_0 + u_1 + \cdots + u_m = 1\}.$$

Let $U_m := \{\overline{x} = (x_1, \ldots, x_m) \,|\, x_k \geq 0,\ x_1 + \cdots + x_m \leq 1\}$ be the *unit simplex* in $\mathbb{R}^m$.

If $f : U_m \to \mathbb{R}$, it is quite useful to consider also f as a function on S_m, writing

$$f(u_0, u_1, \ldots, u_m) = f(x_1, \ldots, x_m) \tag{1.2}$$

with $u_1 = x_1, \ldots, u_m = x_m, u_0 = 1 - u_1 - \cdots - u_m$.

2. Bernstein operators and Aldaz-Kounchev-Render operators

Let $\overline{i} = (i_0, i_1, \ldots, i_m) \in \mathbb{N}_0^{m+1}$ be a multiindex with $|\overline{i}| = i_0 + i_1 + \cdots + i_m = n$. The $\overline{i}$-th Bernstein basic polynomial of degree n is defined by

$$b_{n,\overline{i}}(\overline{u}) := \frac{n!}{\overline{i}!} \overline{u}^{\overline{i}} = \frac{n!}{i_0! i_1! \ldots i_m!} u_0^{i_0} u_1^{i_1} \ldots u_m^{i_m},$$

where $\overline{u} = (u_0, u_1, \ldots, u_m) \in S_m$.

The n-th Bernstein polynomial associated with $f \in C(S_m)$ is defined as

$$B_n f(\overline{u}) := \sum_{|\overline{i}|=n} f\left(\frac{\overline{i}}{n}\right) b_{n,\overline{i}}(\overline{u}),$$

where $\dfrac{\overline{i}}{n} = \left(\dfrac{i_0}{n}, \dfrac{i_1}{n}, \dots, \dfrac{i_m}{n}\right) \in S_m$.

Let $j, n \in \mathbb{N}$, $j \leq n$, and $i \in \{0, 1, \dots, n\}$. Set

$$\gamma_{n,j,i} := \left(\frac{i(i-1)\dots(i-j+1)}{n(n-1)\dots(n-j+1)}\right)^{1/j}.$$

Then $\gamma_{n,j,i} \leq \dfrac{i}{n}$. If $\overline{i} \in \mathbb{N}_0^{m+1}$, $|\overline{i}| = n$, let $\overline{r}_{n,j,\overline{i}} := (r_{n,j,0}, r_{n,j,1}, \dots, r_{n,j,m})$, where $r_{n,j,l} := \gamma_{n,j,i_l}$, $l = 1, \dots, m$, and $r_{n,j,0} := 1 - r_{n,j,1} - \dots - r_{n,j,m}$.

The n-th Aldaz-Kounchev-Render polynomial associated with $f \in C(S_m)$ is defined by

$$B_{n,j} f(\overline{u}) := \sum_{|\overline{i}|=n} f\left(\overline{r}_{n,j,\overline{i}}\right) b_{n,\overline{i}}(\overline{u}).$$

Proposition 2.1. *(i) $B_{n,j}$ interpolates f at the vertices of S_m.*

(ii) $B_{n,j}\mathbf{1} = \mathbf{1}$.

(iii) Let $f \in C(S_m)$, $f(u_0, u_1, \dots, u_m) = u_l^j$ for a certain $l \in \{1, \dots, m\}$. Then $B_{n,j} f = f$.

Proof. (i) Indeed, let $\overline{u} \in S_m$ such that $u_l = 1$ for a certain $l \in \{0, 1, \dots, m\}$. Then $b_{n,\overline{i}}(\overline{u}) = 1$ iff $i_l = n$. It follows that $r_{n,j,l} = \gamma_{n,j,n} = 1$, and so $\overline{r}_{n,j,\overline{i}} = \overline{u}$. Summing-up, we see that $B_{n,j} f(\overline{u}) = f(\overline{u})$.

(iii) To simplify the notation we give the proof for $l = 1$. If $u_1 = 1$, we have $B_{n,j} f(\overline{u}) = \overline{u}$ according to (i). So let $0 \leq u_1 < 1$. Then

$$\begin{aligned}
B_{n,j} f(\overline{u}) &= \sum_{|\overline{i}|=n} r_{n,j,1}^j b_{n,\overline{i}}(\overline{u}) = \sum_{|\overline{i}|=n} \gamma_{n,j,i_1}^j b_{n,\overline{i}}(\overline{u}) \\
&= \sum_{|\overline{i}|=n} \frac{i_1(i_1-1)\dots(i_1-j+1)}{n(n-1)\dots(n-j+1)} \frac{n!}{i_0! i_1! \dots i_m!} u_0^{i_0} u_1^{i_1} \dots u_m^{i_m} \\
&= \sum_{i_1=0}^{n} \frac{i_1(i_1-1)\dots(i_1-j+1)}{n(n-1)\dots(n-j+1)} \frac{n!}{i_1!(n-i_1)!} u_1^{i_1}(1-u_1)^{n-i_1} \\
&\times \sum_{i_0+i_2+\dots+i_m=n-i_1} \frac{(n-i_1)!}{i_0! i_2! \dots i_m!} \frac{u_0^{i_0} u_2^{i_2} \dots u_m^{i_m}}{(1-u_1)^{n-i_1}}.
\end{aligned}$$

The second sum is equal to $\dfrac{1}{(1-u_1)^{n-i_1}}(u_0 + u_2 + \dots + u_m)^{n-i_1} = 1$. The first sum is equal to u_1^j according to [6, Proposition 11]. So we have

$$B_{n,j} f(\overline{u}) = u_1^j = f(\overline{u}).$$

$\square$

3. Voronovskaja type formulas

In studying the approximation properties of a sequence of positive linear operators, Voronovskaja formula is an essential tool. In this section we present results in this sense by using cartesian coordinates. In this framework the Bernstein operators and Aldaz-Kounchev-Render operators are described as

$$B_n f(\overline{x}) = \sum_{i_1+\cdots+i_m\le n} p_{n,i_1,\ldots,i_m}(\overline{x}) f\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right), \tag{3.1}$$

$$B_{n,j} f(\overline{x}) = \sum_{i_1+\cdots+i_m\le n} p_{n,i_1,\ldots,i_m}(\overline{x}) f\left(\gamma_{n,j,i_1},\ldots,\gamma_{n,j,i_m}\right), \tag{3.2}$$

where

$$p_{n,i_1,\ldots,i_m}(\overline{x}) = \frac{n!}{i_1!\ldots i_m!(n-i_1-\ldots-i_m)!} x_1^{i_1}\ldots x_m^{i_m}(1-x_1-\ldots-x_m)^{n-i_1-\cdots-i_m},$$

$\overline{x} = (x_1,\ldots,x_m) \in U_m$, $f \in C(U_m)$.

Let $C^2(U_m)$ be the set of the functions $f \in C(U_m)$ having on $int(U_m)$ continuous partial derivatives of order ≤ 2, which can be continuously extended to U_m.

Conjecture 3.1. *Let $f \in C^2(U_m)$. Then, for $x \in int(U_m)$,*

$$\lim_{n\to\infty} n(B_{n,j}f(x) - f(x))$$
$$= \frac{1}{2}\left[\sum_{i=1}^m x_i(1-x_i) f''_{x_i^2}(x) - 2\sum_{1\le i<k\le m} x_i x_k f''_{x_i x_k}(x) - (j-1)\sum_{l=1}^m (1-x_l) f'_{x_l}(x)\right].$$

Let $D_{m,j} := \left\{\overline{y} = (y_1,\ldots,y_m) \mid y_k \ge 0,\ y_1^j + \cdots + y_m^j \le 1\right\}$. Note that $U_m = D_{m,1} \subset D_{m,j}$.

In what follows we will give a result related to Conjecture 3.1. In order to simplify the notation we consider the case $m = 2$, $j = 2$.

Theorem 3.1. *Let $f \in C(D_{2,2})$ such that the function*

$$(u,v) \in U_2 \to f(\sqrt{u},\sqrt{v})$$

is in $C^2(U_2)$. Then, for $(x,y) \in int(U_2)$,

$$\begin{aligned}\lim_{n\to\infty} & n(B_{n,2}f(x,y) - f(x,y)) \\ &= \frac{x(1-x)}{2} f''_{x^2}(x,y) - xy f''_{xy}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y) \\ &- \frac{1-x}{2} f'_x(x,y) - \frac{1-y}{2} f'_y(x,y).\end{aligned} \tag{3.3}$$

Proof. A key ingredient is the identity

$$\frac{i(i-1)}{n(n-1)} - \left(\frac{i}{n}\right)^2 = \frac{1}{n-1}\frac{i}{n}\left(\frac{i}{n}-1\right).$$

Let $g \in C^2(U_2)$ be defined by $g(u,v) := f(\sqrt{u},\sqrt{v})$, $(u,v) \in U_2$. Then

$$\lim_{n\to\infty} n\left[B_{n,2}(g(s^2,t^2);(x,y)) - B_n(g(s^2,t^2);(x,y))\right]$$

$$= \lim n \sum_{i+k\le n} p_{n,i,k}(x,y)\left[g\left(\frac{i(i-1)}{n(n-1)},\frac{k(k-1)}{n(n-1)}\right) - g\left(\left(\frac{i}{n}\right)^2,\left(\frac{k}{n}\right)^2\right)\right]$$

$$= L_1 + L_2,$$

where

$$\begin{aligned}
L_1 &= \lim_{n\to\infty} n \sum_{i+k\le n} p_{n,i,k}(x,y)\left\{\frac{1}{n-1}\frac{i}{n}\left(\frac{i}{n}-1\right)g'_x\left(\left(\frac{i}{n}\right)^2,\left(\frac{k}{n}\right)^2\right)\right.\\
&\quad \left. + \frac{1}{n-1}\frac{k}{n}\left(\frac{k}{n}-1\right)g'_y\left(\left(\frac{i}{n}\right)^2,\left(\frac{k}{n}\right)^2\right)\right\}\\
&= \lim_{n\to\infty} B_n\left(s(s-1)g'_x(s^2,t^2) + t(t-1)g'_y(s^2,t^2)\,;\,(x,y)\right)\\
&= (x^2-x)g'_x(x^2,y^2) + (y^2-y)g'_y(x^2,y^2),\\
L_2 &= \lim_{n\to\infty} \frac{n}{2(n-1)^2}\sum_{i+k\le n} p_{n,i,k}(x,y)\left\{\left(\frac{i}{n}\left(\frac{i}{n}-1\right)\right)^2 g''_{x^2}(\xi,\eta)\right.\\
&\quad \left. +2\frac{i}{n}\left(\frac{i}{n}-1\right)\frac{k}{n}\left(\frac{k}{n}-1\right)g''_{xy}(\xi,\eta) + \left(\frac{k}{n}\left(\frac{k}{n}-1\right)\right)^2 g''_{y^2}(\xi,\eta)\right\}
\end{aligned}$$

for suitable (ξ,η) furnished by Taylor's formula.

The absolute value of the above sum is dominated by

$$\max\{\|g''_{x^2}\|_\infty, \|g''_{xy}\|_\infty, \|g''_{y^2}\|_\infty\}\times$$

$$\sum_{i+k\le n} p_{n,i,k}(x,y)\left[\left(\frac{i}{n}\left(\frac{i}{n}-1\right)\right)^2 + 2\frac{i}{n}\left(\frac{i}{n}-1\right)\frac{k}{n}\left(\frac{k}{n}-1\right) + \left(\frac{k}{n}\left(\frac{k}{n}-1\right)\right)^2\right],$$

which tends to

$$\max\{\|g''_{x^2}\|_\infty, \|g''_{xy}\|_\infty, \|g''_{y^2}\|_\infty\}\left[(x(x-1))^2 + 2xy(x-1)(y-1) + (y(y-1))^2\right],$$

and so $L_2 = 0$. We conclude that

$$\begin{aligned}
&\lim_{n\to\infty} n\left[B_{n,2}(g(s^2,t^2);(x,y)) - B_n(g(s^2,t^2);(x,y))\right]\\
&= (x^2-x)g'_x(x^2,y^2) + (y^2-y)g'_y(x^2,y^2).
\end{aligned} \tag{3.4}$$

Now $g(s^2,t^2) = f(s,t)$, $f'_x(x,y) = 2xg'_x(x^2,y^2)$, $f'_y(x,y) = 2yg'_y(x^2,y^2)$, and (3.4) shows that

$$\lim_{n\to\infty} n\left[B_{n,2}f(x,y) - B_n f(x,y)\right]$$
$$= \frac{1}{2}(x-1)f'_x(x,y) + \frac{1}{2}(y-1)f'_y(x,y),\ (x,y) \in int(U_2). \tag{3.5}$$

On the other hand, it is well known that

$$\lim_{n\to\infty} n(B_n f(x,y) - f(x,y)) \tag{3.6}$$
$$= \frac{x(1-x)}{2} f''_{x^2}(x,y) - xyf''_{xy}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y),\ (x,y) \in int(U_2). \tag{3.7}$$

From (3.5) and (3.7) we get (3.3), and the proof is finished. □

4. Aldaz-Kounchev-Render operators and convex functions

The behaviour of positive linear operators with respect to the convex functions is an important topic of study. This section is devoted to such problems in the framework of Bernstein operators and Aldaz-Kounchev-Render operators on the simplex.

Proposition 4.1. *Let $f \in C(D_{m,j})$ such that $g : U_m \to \mathbb{R}$, $g(t_1,\dots,t_m) := f\left(\sqrt[j]{t_1},\dots,\sqrt[j]{t_m}\right)$ is a convex function. Then*

$$B_{n,j}f(\overline{x}) \geq f(\overline{x}),\ \overline{x} \in U_m.$$

Proof. Due to the continuity, it suffices to prove this for $x \in int(U_m)$. So, let $x \in int(U_m)$ be fixed. Set $s_k := x_k^j,\ k = 1,\dots,m$. Then $(s_1,\dots,s_m) \in int(U_m)$ and consequently there exist $a_0, a_1, \dots, a_m \in \mathbb{R}$ such that

$$g(t_1,\dots,t_m) \geq a_0 + a_1t_1 + \cdots + a_mt_m,\ \overline{t} \in U_m,$$
$$g(s_1,\dots,s_m) = a_0 + a_1s_1 + \cdots + a_ms_m.$$

It follows that

$$f\left(\sqrt[j]{t_1},\dots,\sqrt[j]{t_m}\right) \geq a_0 + a_1t_1 + \cdots + a_mt_m,\ \overline{t} \in U_m,$$
$$f\left(\sqrt[j]{s_1},\dots,\sqrt[j]{s_m}\right) = a_0 + a_1s_1 + \cdots + a_ms_m.$$

Setting $t_k = y_k^j$, $k = 1,\dots,m$, we get

$$f(y_1,\dots,y_m) \geq a_0 + a_1y_1^j + \cdots + a_my_m^j,\ \overline{y} \in D_{m,j}$$
$$f(x_1,\dots,x_m) = a_0 + a_1x_1^j + \cdots + a_mx_m^j.$$

Now $B_{n,j}f(y_1,\dots,y_m) \geq a_0 + a_1y_1^j + \cdots + a_my_m^j$, and in particular

$$B_{n,j}f(x_1,\dots,x_m) \geq a_0 + a_1x_1^j + \cdots + a_mx_m^j = f(x_1,\dots,x_m).$$

□

To present the next result, which involves the Voronovskaja operator, let $m = 2$, $j \geq 1$,

$$U_2 = \{(s,t) \,|\, s \geq 0,\, t \geq 0,\, s+t \leq 1\},$$
$$D_{2,j} = \left\{(x,y) \,|\, x \geq 0,\, y \geq 0,\, x^j + y^j \leq 1\right\}.$$

Let $g \in C^2(U_2)$, $f \in C^2(D_{2,j})$, $f(x,y) = g(x^j, y^j)$. Suppose that g is convex. Define, for $(x,y) \in D_{2,j}$,

$$\begin{aligned} Vf(x,y) &= x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y) - 2xyf''_{xy}(x,y) \\ &\quad - (j-1)(1-x)f'_x(x,y) - (j-1)(1-y)f'_y(x,y). \end{aligned}$$

Remark that $\dfrac{1}{2}V$ is the Voronovskaja operator for the sequence $(B_{n,j})$, when $m = 2$.

Set $s = x^j$, $t = y^j$. Then $f(x,y) = g(s,t) = g(x^j, y^j)$. We have

$$f'_x(x,y) = jx^{j-1}g'_s(x^j,y^j),\;\; f''_{xy}(x,y) = j^2x^{j-1}y^{j-1}g''_{st}(x^j,y^j),$$
$$f''_{x^2}(x,y) = j^2x^{2j-2}g''_{s^2}(x^j,y^j) + j(j-1)x^{j-2}g'_s(x^j,y^j).$$

Therefore,

$$Vf(x,y) = x^{2j-1}(1-x)g''_{s^2}(x^j,y^j) + y^{2j-1}(1-y)g''_{t^2}(x^j,y^j) - 2x^jy^jg''_{st}(x^j,y^j).$$

Theorem 4.1. *Suppose that the function f verifies the above hypothesis. If $(x,y) \in U_2$, then $Vf(x,y) \geq 0$.*

Proof. Let $x = s^{1/j}$, $y = t^{1/j}$, so that

$$Vf(x,y) = s^{2-1/j}(1-s^{1/j})g''_{s^2}(s,t) + t^{2-1/j}(1-t^{1/j})g''_{t^2}(s,t) - 2stg''_{st}(s,t).$$

Since g is convex, we have $g''_{s^2} \geq 0$, $g''_{t^2} \geq 0$, $g''_{s^2}g''_{t^2} \geq {g''_{st}}^2$, for $(s,t) \in U_2$. Now

$$Vf(x,y) \geq 2\sqrt{(st)^{2-1/j}(1-s^{1/j})(1-t^{1/j})}|g''_{st}(s,t)| - 2stg''_{st}(s,t).$$

It remains to prove that

$$\sqrt{(st)^{2-1/j}(1-s^{1/j})(1-t^{1/j})} \geq st.$$

This reduces to $1 - s^{1/j} - t^{1/j} \geq 0$, i.e., $x + y \leq 1$, which is true because $(x,y) \in U_2$. □

Let $f \in C(D_{m,j})$, $g \in C(U_m)$, $g(t_1,\ldots,t_m) = f(\sqrt[j]{t_1},\ldots,\sqrt[j]{t_m})$. We write $\overline{y} \leq \overline{x}$ if $y_k \leq x_k$, $k = 1,\ldots,m$.

Theorem 4.2. *If g is convex and $f(\overline{y}) \leq f(\overline{x})$ whenever $\overline{y}, \overline{x} \in U_m$, $\overline{y} \leq \overline{x}$, then*

$$B_nf(\overline{x}) \geq B_{n,j}f(\overline{x}) \geq f(\overline{x}),\; \overline{x} \in U_m. \tag{4.1}$$

Proof. In (3.1) and (3.2) we have $\gamma_{n,j,i_k} \leq \frac{i_k}{n}$, $k = 1, \ldots, m$. Thus

$$f\left(\frac{i_1}{n}, \ldots, \frac{i_m}{n}\right) \geq f(\gamma_{n,j,i_1}, \ldots, \gamma_{n,j,i_m}) \tag{4.2}$$

and this proves the first inequality in (4.1). For the second inequality see Proposition 4.1. $\square$

Example 4.1. *The function $f(\overline{x}) = \exp(x_1^j + \cdots + x_m^j)$, $\overline{x} \in U_m$, satisfies the hypothesis of Theorem 4.2. For it the approximation provided by $B_{n,j}$ is better than that provided by B_n.*

For $n = 10$ and $j = 2$ Figures 1 and 2 illustrate the inequalities $f \leq B_{10,2}f \leq B_{10}f$.

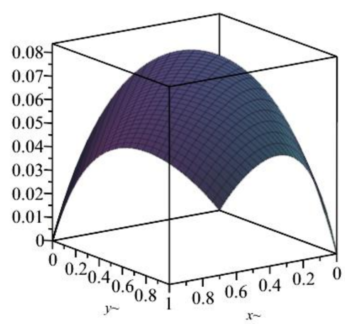

Figure 1: Graph of $B_{10}f - B_{10,2}f$

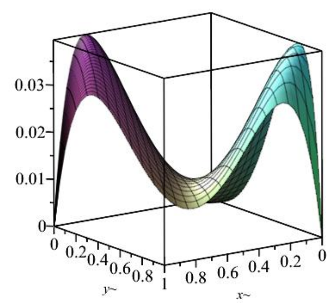

Figure 2: Graph of $B_{10,2}f - f$

Theorem 4.3. *Let $f \in C(U_m)$ be convex and $f(\overline{y}) \geq f(\overline{x})$, whenever $\overline{y}, \overline{x} \in U_m$, $\overline{y} \leq \overline{x}$. Then*

$$B_{n,j}f(\overline{x}) \geq B_n f(\overline{x}) \geq f(\overline{x}), \ \overline{x} \in U_m. \tag{4.3}$$

Proof. The first inequality follows from (4.2). Since Bernstein operators preserve the affine functions and f is convex, the second inequality is also valid. $\square$

Example 4.2. *The function $f(\overline{x}) = \exp(-x_1 - \cdots - x_m)$, $\overline{x} \in U_m$, satisfies the hypothesis of Theorem 4.3. For it the approximation provided by B_n is better than that provided by $B_{n,j}$.*

For $n = 2$ and $j = 2$ Figures 3 and 4 illustrate the inequalities $f \leq B_2 f \leq B_{2,2}f$.

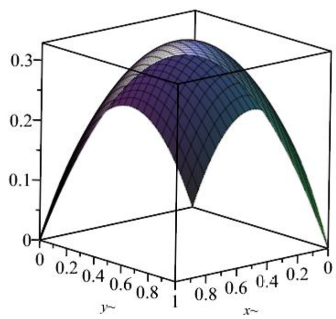

Figure 3: Graph of $B_{2,2}f - B_2 f$

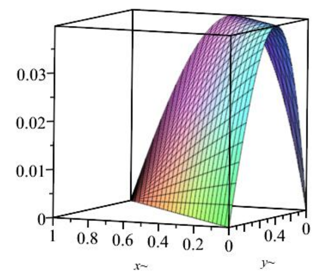

Figure 4: Graph of $B_2 f - f$

5. Limit of iterates of Aldaz-Kounchev-Render operators

The iterates of a Markov operator (i.e., a positive linear operator preserving the constant functions) are investigated in many papers. There are several methods for determining the limits of such iterates, see, e.g., [5], [7], [8], [13], [14], [15], [17], [18], [19] and the references therein.

Let $\overline{v}_0 = (0, 0. \dots, 0)$, $\overline{v}_1 = (1, 0, \dots, 0)$,..., $\overline{v}_m = (0, \dots, 0, 1)$ be the vertices of the simplex U_m.

Theorem 5.1. *For each $f \in C(U_m)$, one has*

$$\lim_{p\to\infty} B^p_{n,j} f(\overline{x}) = f(\overline{v}_0)(1 - x_1^j - \cdots - x_m^j) + f(\overline{v}_1)x_1^j + \cdots + f(\overline{v}_m)x_m^j, \quad (5.1)$$

uniformly for $\overline{x} \in U_m$.

First proof. Given $a_0, \dots, a_m \in \mathbb{R}$, let

$$X := \{f \in C(U_m) \mid f(\overline{v}_k) = a_k,\ k = 0, 1, \dots, m\}.$$

Endowed with the metric $d(f, g) := \|f - g\|_\infty$, $C(U_m)$ is a complete metric space and X is a closed linear subspace, so that (X, d) is also a complete metric space.

Let $f \in X$ and $k \in \{0, 1, \dots, m\}$. Then $B_{n,j} f(\overline{v}_k) = f(\overline{v}_k) = a_k$, and so $B_{n,j} f \in X$. We can consider the operator $B_{n,j} : X \to X$.

Let $\overline{x} \in U_m$. By the power means inequality we have

$$\left(\frac{x_1^n + \cdots + x_m^n + (1 - x_1 - \cdots - x_m)^n}{m+1}\right)^{1/n} \geq \frac{x_1 + \cdots + x_m + (1 - x_1 - \cdots - x_m)}{m+1} = \frac{1}{m+1},$$

which leads to

$$x_1^n + \cdots + x_m^n + (1 - x_1 - \cdots - x_m)^n \geq \frac{1}{(m+1)^{n-1}}. \quad (5.2)$$

If $f, g \in X$, then

$$\begin{aligned}|B_{n,j}f(\overline{x}) - B_{n,j}g(\overline{x})| &= \left|\sum_{|\overline{i}|=n} \left(f\left(\overline{r}_{n,j,\overline{i}}\right) - g\left(\overline{r}_{n,j,\overline{i}}\right)\right) b_{n,\overline{i}}(\overline{x})\right| \\ &\le \sum \left|f\left(\overline{r}_{n,j,\overline{i}}\right) - g\left(\overline{r}_{n,j,\overline{i}}\right)\right| b_{n,\overline{i}}(\overline{x}), \end{aligned} \tag{5.3}$$

where the last sum runs over all $\overline{i}$ with $|\overline{i}| = n$, except for the case where $\overline{r}_{n,j,\overline{i}}$ is a vertex of U_m, when f and g coincide. Now we have from (5.2) and (5.3),

$$\begin{aligned}|B_{n,j}f(\overline{x}) - B_{n,j}g(\overline{x})| &\le \|f-g\|_\infty \left(1 - x_1^n - \cdots - x_m^n - (1 - x_1 - \cdots - x_m)^n\right) \\ &\le \left(1 - \frac{1}{(m+1)^{n-1}}\right) \|f-g\|_\infty. \end{aligned} \tag{5.4}$$

Therefore,

$$d(B_{n,j}f, B_{n,j}g) \le \left(1 - \frac{1}{(m+1)^{n-1}}\right) d(f,g),$$

which shows that $B_{n,j} : X \to X$ is a contraction on the complete metric space X. According to Banach's fixed point theorem, $B_{n,j}$ has a unique fixed point $\varphi \in X$ and for each $f \in X$,

$$\lim_{p\to\infty} B_{n,j}^p f = \varphi.$$

It is easy to see that the unique fixed point of $B_{n,j}$ is in fact the function

$$\varphi(\overline{x}) = a_0(1 - x_1^j - \cdots - x_m^j) + a_1 x_1^j + \cdots + a_m x_m^j, \ \overline{x} \in U_m.$$

It follows that for each $f \in X$,

$$\lim_{p\to\infty} B_{n,j}^p f(\overline{x}) = f(\overline{v}_0)(1 - x_1^j - \cdots - x_m^j) + f(\overline{v}_1)x_1^j + \cdots + f(\overline{v}_m)x_m^j,$$

uniformly for $\overline{x} \in U_m$. Since in the definition of X, $a_0, \dots, a_m$ were arbitrary, we conclude that (5.1) holds for each $f \in C(U_m)$, and the first proof is finished.

Second proof. We use notation and results from [5]. Let $S = B_{n,j}$. The associated matrix M has the form

$$M = \left(\begin{array}{c|c} I_{\nu\times\nu} & O_{\nu\times(s-\nu+1)} \\ \hline R_{(s-\nu+1)\times\nu} & Q_{(s-\nu+1)\times(s-\nu+1)} \end{array}\right), \tag{5.5}$$

with $\nu = m+1$. According to [5, Theorem 2.1], there exists

$$Tf := \lim_{p\to\infty} B_{n,j}^p f, \ f \in C(U_m). \tag{5.6}$$

Corollary 2.1 from [5] tells us that

$$Tf = \sum_{k=0}^m f(\overline{v}_k)\varphi_k, \tag{5.7}$$

where φ_k are fixed points of $B_{n,j}$, $\varphi_k \geq 0$, $\varphi_0 + \cdots + \varphi_m = \mathbf{1}$.

Let $g_0(\overline{x}) = 1 - x_1^j - \cdots - x_m^j$, $g_1(\overline{x}) = x_1^j, \ldots, g_m(\overline{x}) = x_m^j$.

These are fixed points of $B_{n,j}$, and Corollary 2.1 from [5] shows that

$$g_i = a_{i0}\varphi_0 + a_{i1}\varphi_1 + \cdots + a_{im}\varphi_m, \; i = 0, \ldots, n,$$

for some $a_{ik} \in \mathbb{R}$. Now $Tg_i(\overline{v}_l) = g_i(\overline{v}_l) = \delta_{il}$ (since Tg_i interpolates g_i on the extreme points) and also

$$Tg_i(\overline{v}_l) = \sum_{k=0}^{m} g_i(\overline{v}_k)\varphi_k(\overline{v}_l) = \varphi_i(\overline{v}_l).$$

It follows that $\varphi_i(\overline{v}_l) = \delta_{il}$. Now

$$\delta_{il} = g_i(\overline{v}_l) = a_{i0}\varphi_0(\overline{v}_l) + \cdots + a_{im}\varphi_m(\overline{v}_l) = a_{il},$$

i.e., $a_{il} = \delta_{il}$ and so $g_i = \varphi_i$. Combined with (5.6) and (5.7), this leads to (5.1) and this concludes the second proof.

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