On the Whittaker–Shannon sampling by means of Berrut's rational interpolant and its extension by Floater and Hormann

Len Bos* Stefano De Marchi†

Abstract

We discuss sampling (interpolation) by translates of sinc functions for data restricted to a finite interval. We indicate how the Floater–Hormann (cf. [8]) of the Berrut normalization (cf. [2]), in the case of equally spaced nodes, can be regarded as a sampling operator with improved approximation properties that remains numerically stable. We provide a compact formula for the denominator of the Floater–Hormann operator. Finally we use this compact formula to compute, for the case of the Berrut operator, the asymptotics of the associated quadrature weights.

AMS Subject Classification: 42A15, 41A05, 41A55, 41A20.

Keywords: truncated Whittaker–Shannon sampling, sinc function, rational interpolant, quadrature.

1 Introduction

The justly famous Whittaker–Shannon Sampling Theorem may be stated as follows.

Theorem 1. *Suppose that $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and that $\widehat{f}(\omega) = 0$ for $|\omega| \geq h/2$. Then*

$$f(x) = \sum_{k=-\infty}^{\infty} f(kh)\mathrm{sinc}\left(\frac{1}{h}(x - kh)\right). \tag{1}$$

Here

$$\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$

is the sinc function and we define the Fourier transform

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)dx.$$

In the case of f with domain restricted to some compact subinterval of $\mathbb{R}$, say to $[0, 1]$, the formula (1) of course no longer makes sense. However, taking $h = 1/n$, we may consider the partial

*University of Verona (Italy), leonardpeter.bos@univr.it
†University of Padua (Italy), demarchi@math.unipd.it

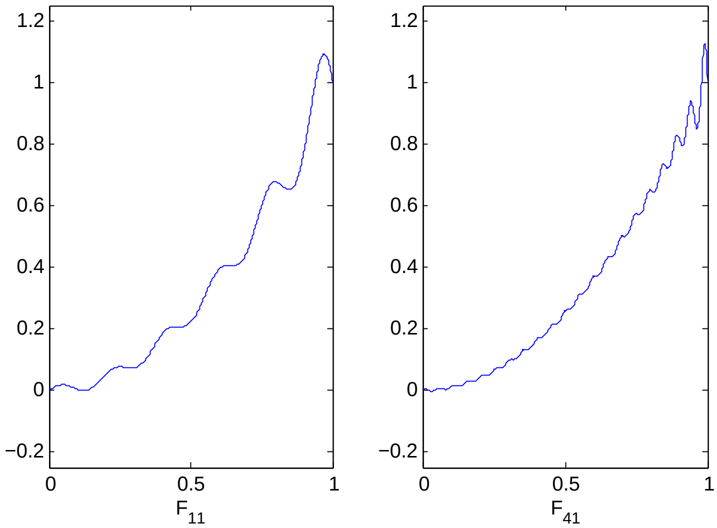

Figure 1: $F_{11}(x)$ and $F_{41}(x)$ for $f(x) = x^2$.

sum

$$f(x) \approx F_n(x) := \sum_{k=0}^{n} f(k/n)\text{sinc}\,(n(x - k/n))$$
$$= \sum_{k=0}^{n} f(x_k)\text{sinc}\,(n(x - x_k)) \tag{2}$$

where we have set $x_k := k/n$, $0 \le k \le n$. Although (2) no longer reproduces $f(x)$ for all $x \in [0,1]$, it is an *interpolant* in that

$$F_n(x_j) = f(x_j),\ \ 0 \le j \le n, \tag{3}$$

as easily follows from the cardinality property of the translated sinc functions, i.e.,

$$\text{sinc}(n(x_j - x_k)) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}.$$

This interpolant F_n was already studied by de la Vallée Poussin (cf. [7]) who showed that under some weak regularity conditions on $f(x)$,

$$\lim_{n\to\infty} F_n(x) = f(x),\ \ x \in [0,1],$$

with error essentially of $O(1/n)$. The reader interested in further details may find them in the excellent survey by Butzer and Stens (cf. [6]).

A decidedly negative property of the interpolant F_n of (2) is that it does not reproduce lines or even constants. Plots of F_{11} and F_{41} for $f(x) = x^2$ are given in Figure 1. Notice, in particular, the strong Gibbs phenomenon at the right end point $x = 1$.

In order to alleviate the poor approximation quality of F_n Berrut (cf. [2]) has suggested normalizing the formula (2) for F_n to obtain

$$B_n(x) := \frac{\displaystyle\sum_{k=0}^{n} f(x_k)\mathrm{sinc}\,(n(x-x_k))}{\displaystyle\sum_{k=0}^{n} \mathrm{sinc}\,(n(x-x_k))}. \tag{4}$$

As is easily seen, B_n remains an interpolant of f at the nodes x_k, $k = 0, \cdots, n$ but has the advantage of reproducing constants, i.e., if $f(x) = 1$ then $B_n(x) = 1$. It turns out that the formula for B_n can be simplified. Specifically, just notice that

$$\begin{aligned}\mathrm{sinc}(n(x-x_k) &= \mathrm{sinc}(n(x-k/n)) \\ &= \frac{\sin(n\pi x - k\pi)}{n\pi(x-x_k)} \\ &= (-1)^k \frac{\sin(n\pi x)}{n\pi(x-x_k)}.\end{aligned}$$

Hence,

$$\begin{aligned}B_n(x) &= \frac{\sum_{k=0}^{n} f(x_k)\mathrm{sinc}\,(n(x-x_k))}{\sum_{k=0}^{n} \mathrm{sinc}\,(n(x-x_k))} \\ &= \frac{\sin(n\pi x)\sum_{k=0}^{n}(-1)^k \frac{f(x_k)}{x-x_k}}{\sin(n\pi x)\sum_{k=0}^{n}(-1)^k \frac{1}{x-x_k}} \\ &= \frac{\sum_{k=0}^{n}(-1)^k f(x_k)/(x-x_k)}{\sum_{k=0}^{n}(-1)^k/(x-x_k)}.\end{aligned} \tag{5}$$

This latter formula remains an interpolant for *any* set of distinct nodes x_k and is, in general, the barycentric rational interpolant introduced by Berrut in [1].

As already noted, B_n reproduces constants. If we take $f(x) = x$ then we may calculate

$$\begin{aligned}B_n(x) &= \frac{\sum_{k=0}^{n}(-1)^k x_k/(x-x_k)}{\sum_{k=0}^{n}(-1)^k/(x-x_k)} \\ &= \frac{\sum_{k=0}^{n}(-1)^k((x_k-x)+x)/(x-x_k)}{\sum_{k=0}^{n}(-1)^k/(x-x_k)} \\ &= \frac{\sum_{k=0}^{n}(-1)^{k+1} + x\sum_{k=0}^{n}(-1)^k/(x-x_k)}{\sum_{k=0}^{n}(-1)^k/(x-x_k)} \\ &= x\end{aligned}$$

for n *odd*, as in this case $\sum_{k=0}^{n}(-1)^{k+1} = 0$. In other words, for n odd, B_n also reproduces linears, resulting in improved approximation properties, as is already evident by comparing Figure 1 with Figure 2 where we show $B_n(x)$ for $f(x) = x^2$ and $n = 11$ and $n = 41$.

Besides being an improved approximant, B_n is also numerically stable as its associated Lebesgue constant has $O(\log(n))$ growth, as was recently shown in [4].

Floater and Hormann in [8], have suggested an even greater improvement to the approximation power of the Berrut interpolant (also for general nodes x_k) by introducing a family of weights

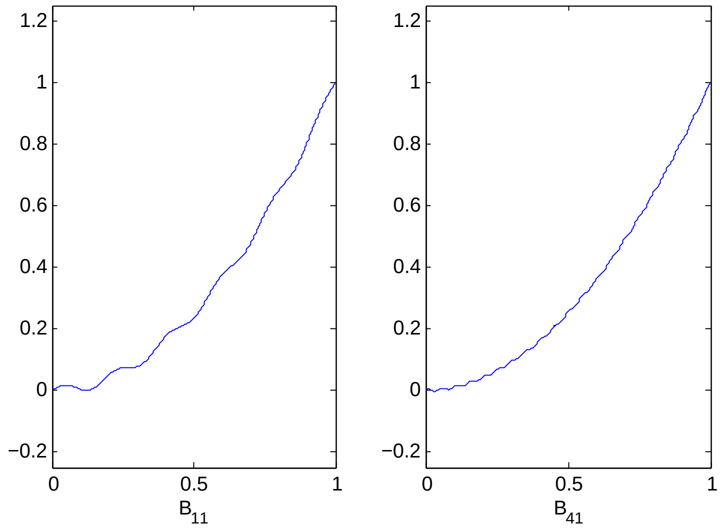

Figure 2: $B_{11}(x)$ and $B_{41}(x)$ for $f(x) = x^2$.

$\beta_k = \beta_k^{(d)}$, depending on a degree $d \geq 0$, into the formula for B_n, resulting in

$$FH_n(x) := \frac{\sum_{k=0}^{n} (-1)^k \beta_k f(x_k)/(x - x_k)}{\sum_{k=0}^{n} (-1)^k \beta_k/(x - x_k)}. \tag{6}$$

In general, the weights $\beta_k^{(d)}$ are specifically chosen so that FH_n reproduces polynomials of degree at most d and how to do this is perhaps the main result of their paper. However, in the case of equally spaced nodes, $x_k = k/n$, this is rather easy to explain, and we offer a simplified proof of this special case of [8], culminating in Proposition 4 below.

Firstly, in the specific case of equally spaced nodes their formula for the $\beta_k^{(d)}$ reduces to

$$\beta_k^{(d)} := \begin{cases} \sum_{j=0}^{k} \binom{d}{k} & 0 \leq k \leq d \\ 2^d & d \leq k \leq n-d \\ \beta_{n-k} & n-d \leq k \leq n \end{cases} \tag{7}$$

where $n \geq 2d$, by assumption. Note that for $d = 0$, $FH_n(x)$ reduces to $B_n(x)$.

Lemma 2. *Assume that $n \geq 2d$. Then the generating function of the $\beta_k^{(d)}$ is given by*

$$\begin{aligned} \beta^{(n,d)}(t) := \sum_{k=0}^{n} \beta_k^{(d)} t^k &= (1+t)^d \left(\frac{1 - t^{n+1-d}}{1-t} \right) \\ &= (1+t)^d \left\{ 1 + t + t^t + \cdots + t^{n-d} \right\}. \end{aligned}$$

Proof. Write

$$(1+t)^d \left\{1+t+t^t+\cdots+t^{n-d}\right\} = \left(\sum_{k=0}^{\infty} a_k t^k\right)\left(\sum_{k=0}^{\infty} b_k t^k\right)$$

where

$$a_k = \begin{cases} \binom{d}{k} & 0 \le k \le d \\ & \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_k = \begin{cases} 1 & 0 \le k \le n-d \\ & \\ 0 & \text{otherwise} \end{cases} .$$

But

$$\left(\sum_{k=0}^{\infty} a_k t^k\right)\left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{k=0}^{\infty} t^k \left\{\sum_{j=0}^{k} a_j b_{k-j}\right\},$$

and, for $0 \le k \le d$,

$$\sum_{j=0}^{k} a_j b_{k-j} = \sum_{j=0}^{k} \binom{d}{j} \times 1 = \beta_k^{(d)}$$

while for $d < k \le n-d$,

$$\begin{aligned}
\sum_{j=0}^{k} a_j b_{k-j} &= \sum_{j=0}^{d} a_j b_{k-j} \quad (\text{since } a_j = 0 \text{ for } j > d) \\
&= \sum_{j=0}^{d} \binom{d}{j} \times 1 \\
&= 2^d \\
&= \beta_k^{(d)}
\end{aligned}$$

and for $n-d < k \le n$,

$$\begin{aligned}
\sum_{j=0}^{k} a_j b_{k-j} &= \sum_{j=0}^{k} b_j a_{k-j} \\
&= \sum_{j=0}^{n-d} b_j a_{k-j} \quad (\text{since } b_j = 0 \text{ for } j > n-d) \\
&= \sum_{j=0}^{n-d} 1 \times a_{k-j} \\
&= \sum_{j=k-d}^{n-d} a_{k-j} \quad (\text{since } a_{k-j} = 0 \text{ for } j < k-d,\ i.e.\ k-j > d) \\
&= \sum_{j=k-d}^{n-d} \binom{d}{k-j} \\
&= \sum_{j=0}^{n-k} \binom{d}{d-j} \quad (\text{letting } j' = d+j-k) \\
&= \sum_{j=0}^{n-k} \binom{d}{j} \\
&= \beta_k^{(d)}.
\end{aligned}$$

□

These $\beta_k^{(d)}$ have the property that their alternating moments up to order $d-1$ are 0.

Lemma 3. *For $n \ge 2d$ and $0 \le j \le d-1$, the alternating moments*

$$\sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k^j = 0.$$

Proof. From the generating function we have

$$\beta^{(n,d)}(-t) = \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} t^k = (1-t)^d \{1 - t + t^2 - t^3 + \cdots + (-1)^{n-d} t^{n-d}\}$$

and hence each of the derivatives

$$\left.\left(\frac{d^j}{dt^j} \beta^{(n,d)}(-t)\right)\right|_{t=1} = 0, \quad 0 \le j \le d-1.$$

In other words,

$$0 = \left.\left(\sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k(k-1)\cdots(k-j+1) t^{k-j}\right)\right|_{t=1}, \quad 0 \le j \le d-1.$$

From $j = 0$ we have

$$0 = \sum_{k=0}^{n} (-1)^k \beta_k^{(d)}.$$

From $j = 1$ we have

$$0 = \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k.$$

From $j = 2$ we have

$$\begin{aligned}
0 &= \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k(k-1) \\
&= \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} (k^2 - k) \\
&= \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k^2 - \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k \\
&= \sum_{k=0}^{n} (-1)^k \beta_k^{(d)} k^2
\end{aligned}$$

by the $j = 1$ case. Continuing in this way we have that all the (alternating) moments up to order $d - 1$ are zero. □

Proposition 4. *(Floater and Hormann [8]) Consider the Floater-Hormann inerpolant FH_n with weights $\beta_k^{(d)}$ given by* (7)*, $n \geq 2d$, and equally spaced nodes $x_k = k/n$. Then if $f(x)$ is a polynomial of degree at most d,*

$$FH_n(x) = f(x).$$

Proof. For simplicity's sake we will write β_k for $\beta_k^{(d)}$. It is also somewhat convenient to write the Floater-Hormann formula (6) as

$$FH_n(x) = \sum_{k=0}^{n} f(x_k) b_k(x)$$

where

$$b_k(x) := \left\{ (-1)^k \beta_k / (x - x_k) \right\} / \sum_{j=0}^{n} (-1)^j \beta_j / (x - x_j). \tag{8}$$

Clearly,

$$\sum_{k=0}^{n} b_k(x) = 1 \tag{9}$$

(a property that is true for any weights and any nodes).

We will prove the Proposition for $f(x) = x^j$, $0 \leq j \leq d$, by induction on j. The $j = 0$ case follows directly from (9). Hence suppose that the claim holds for x^i, $i \leq j - 1$; we will show that it then also holds for x^j, $1 \le j \le d$. Note that for $1 \le j \le d$,

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\beta_k(x-x_k)^j/(x-x_k) &= \sum_{k=0}^{n}(-1)^k\beta_k(x-x_k)^{j-1} \\
&= \sum_{k=0}^{n}(-1)^k\beta_k(x-k/n)^{j-1} \\
&= \sum_{k=0}^{n}(-1)^k\beta_k\left\{\sum_{i=0}^{j-1}(-1)^i(k^i/n^i)x^{j-1-i}\right\} \\
&= \sum_{i=0}^{j-1}\left\{\binom{j-1}{i}(-1)^i x^{j-1-i}n^{-i}\left(\sum_{k=0}^{n}(-1)^k\beta_k k^i\right)\right\} \\
&= 0
\end{aligned}$$

by Lemma 3. Consequently we have also that

$$\sum_{k=0}^{n} b_k(x)(x-x_k)^j = 0$$

for $1 \le j \le d$. Thus

$$\begin{aligned}
0 &= \sum_{k=0}^{n} b_k(x)(x-x_k)^j \\
&= \sum_{k=0}^{n} b_k(x)\left\{\sum_{i=0}^{j}\binom{j}{i}x^{j-i}(-1)^i x_k^i\right\} \\
&= \sum_{i=0}^{j}\left\{(-1)^i\binom{j}{i}x^{j-i}\sum_{k=0}^{n}x_k^i b_k(x)\right\}.
\end{aligned}$$

But by the induction assumptiom, for $i < j$,

$$\sum_{k=0}^{n} x_k^i b_k(x) = x^i$$

and so we have

$$\begin{aligned}
0 &= \left(\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}x^{j-i}x^i\right) + (-1)^j\binom{j}{j}x^{j-j}\sum_{k=0}^{n}x_k^j b_k(x) \\
&= x^j\left((1-1)^j - (-1)^j\binom{j}{j}\right) + (-1)^j\sum_{k=0}^{n}x_k^j b_k(x) \\
&= (-1)^{j+1}x^j + (-1)^j\sum_{k=0}^{n}x_k^j b_k(x).
\end{aligned}$$

It follows that

$$\sum_{k=0}^{n} x_k^j b_k(x) = x^j$$

and we are done. □

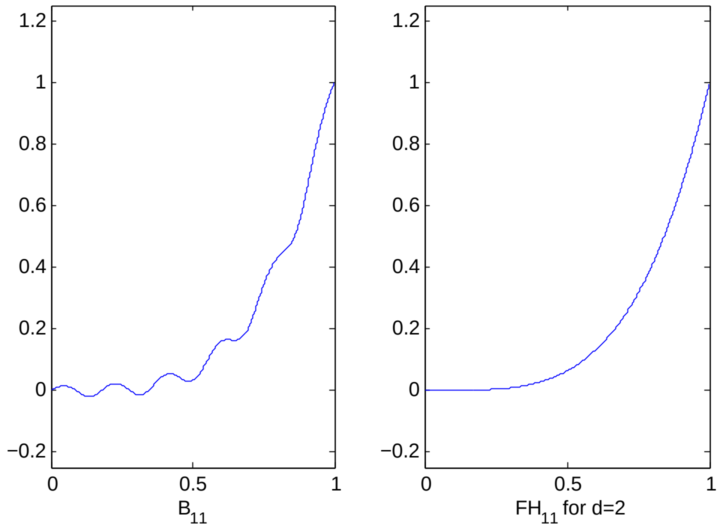

Figure 3: $B_{11}(x)$ and FH_{11} (with $d = 2$) for $f(x) = x^4$.

Besides having improved approximation properties, the Floater-Hormann remains numerically stable as its associated Lebesgue constant is also of logarithmic growth in n, as is shown in the recent paper [5].

We would like to emphasize that the Floater-Hormann interpolant for equally spaced nodes may also be regarded as a simple improvement on the sampling operator. Indeed we may write

$$FH_n(x) = \frac{\sum_{k=0}^{n} \beta_k^{(d)} f(x_k)\operatorname{sinc}(n(x-x_k))}{\sum_{k=0}^{n} \beta_k^{(d)} \operatorname{sinc}(n(x-x_k))}.$$

The weights $\beta_k^{(d)}$ are constant except for the first d and last d and hence (for small d) are only a small modification of the normalized sampling operator $B_n(x)$, but FH_n reproduces polynomials of degree d and yet enjoys a Lebesgue constant of minimal growth.

In Figure 3 we show $B_{11}(x)$ for $f(x) = x^4$ together with $FH_{11}(x)$ for $d = 2$. We hope to have convinced the reader that the interpolation operators FH_n are indeed worthy of further study, and provide an interesting improvement to (truncated) sampling.

2 A Compact Formula for the Basis Functions $b_k(x)$

We first consider the case $d = 0$ and equally spaced points (in which case $FH_n(x) = B_n(x)$) for which (8) reduces to

$$b_k(x) = \frac{(-1)^k/(x-x_k)}{\sum_{j=0}^{n}(-1)^j/(x-x_j)}. \tag{10}$$

We will give a compact formula for $b_k(x)$ in terms of the so-called *Bateman G-function*, which we now introduce.

Let $\psi(x)$ denote the classical digamma function, i.e.,

$$\psi(x) := \frac{d}{dx}\log(\Gamma(x)), \quad x \neq 0, -1, -2, \cdots$$

where $\Gamma(x)$ is the Gamma function. Bateman's G-function is defined as

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right). \tag{11}$$

Some important equations that we will make use of are (equation numbers refer to *An Atlas of Functions*, [10]):

$$G(x) = 2\int_0^\infty \frac{e^{-xt}}{1+e^{-t}}dt, \ x > 0, \tag{44:13:3}$$

$$G(1-x) = 2\pi\csc(\pi x) - G(x), \tag{44:13:5}$$

$$G(x) = 2\sum_{j=0}^\infty \frac{(-1)^j}{x+j}, \ x \neq 0, -1, -2, \ldots, \tag{44:14:5}$$

$$\sum_{j=0}^n \frac{(-1)^j}{jb+c} = \frac{1}{2b}\left\{G\left(\frac{c}{b}\right) + (-1)^n G\left(\frac{c}{b}+n+1\right)\right\}, \ c \neq 0, -b, -2b, -3b, \ldots. \tag{44:14:4}$$

Consider now the denominator of (10),

$$\begin{aligned} D_n(x) &:= \sum_{j=0}^n \frac{(-1)^j}{x-x_j} \\ &= n\sum_{j=0}^n \frac{(-1)^j}{nx-j} \\ &= \frac{n}{2(-1)}\left\{G\left(\frac{nx}{-1}\right) + (-1)^n G\left(\frac{nx}{-1}+n+1\right)\right\} \end{aligned}$$

by (44:14:4) with $b = -1$ and $c = nx$. Simplifying, we obtain

$$D_n(x) = -\frac{n}{2}\left\{G(-nx) + (-1)^n G(-nx+n+1)\right\}$$

which is valid for $nx \neq 0, 1, 2, \cdots$, i.e., for $x \neq 0, 1/n, 2/n, \cdots, 1$.

Now, by the reflection formula (44:13:5) above, it follows that

$$\begin{aligned} G(-nx) &= G(1-(nx+1)) \\ &= 2\pi\csc(\pi(nx+1)) - G(nx+1) \\ &= -2\pi\csc(n\pi x) - G(nx+1). \end{aligned}$$

Hence,

$$\begin{aligned} D_n(x) &= -\frac{n}{2}\left\{-2\pi\csc(n\pi x) - G(nx+1) + (-1)^n G(n(1-x)+1)\right\} \\ &= \frac{n}{2}\left\{2\pi\csc(n\pi x) + G(nx+1) + (-1)^{n+1} G(n(1-x)+1)\right\} \end{aligned}$$

and

$$
\begin{aligned}
b_k(x) &= \frac{(-1)^k/(x-x_k)}{D_n(x)} \\
&= \frac{\frac{2}{n}(-1)^k/(x-x_k)}{2\pi\csc(n\pi x)+G(nx+1)+(-1)^{n+1}G(n(1-x)+1)} \\
&= \frac{2(-1)^k/(n(x-x_k))}{2\pi\csc(n\pi x)+G(nx+1)+(-1)^{n+1}G(n(1-x)+1)} \\
&= \frac{2(-1)^k\frac{\sin(n\pi x)}{n(x-x_k)}}{\sin(n\pi x)\{2\pi\csc(n\pi x)+G(nx+1)+(-1)^{n+1}G(n(1-x)+1)\}} \\
&= \frac{2\pi(-1)^k\frac{\sin(n\pi x)}{n\pi(x-x_k)}}{\sin(n\pi x)\{2\pi\csc(n\pi x)+G(nx+1)+(-1)^{n+1}G(n(1-x)+1)\}} \\
&= \frac{2\pi\frac{\sin(n\pi(x-x_k))}{n\pi(x-x_k)}}{\sin(n\pi x)\{2\pi\csc(n\pi x)+G(nx+1)+(-1)^{n+1}G(n(1-x)+1)\}} \\
&= \frac{2\pi\mathrm{sinc}(n(x-x_k))}{2\pi+\sin(n\pi x)G(nx+1)+(-1)^{n+1}\sin(n\pi x)G(n(1-x)+1)} \\
&= \frac{2\pi\mathrm{sinc}(n(x-x_k))}{2\pi+\sin(n\pi x)G(nx+1)+\sin(n\pi(1-x))G(n(1-x)+1)}.
\end{aligned}
$$

Simplifying slightly, we arrive at our formula

$$
b_k(x) = \frac{\mathrm{sinc}(n(x-x_k))}{1+\frac{1}{2\pi}\{\sin(n\pi x)G(nx+1)+\sin(n\pi(1-x))G(n(1-x)+1)\}}. \tag{12}
$$

Note that since $G(x)$ is defined in terms of the digamma function (see (11)), there are fast, accurate algorithms for its evaluation. For example, one may use Matlab's `psi` function. Note also that by (44:13:3), $G(x)$ is strictly decreasing for $x > 0$. Hence, for $0 \le x \le 1$, both $G(nx+1) \le G(1)$ and $G(n(1-x)+1) \le G(1)$, where the value of $G(1)$ is known to have the value

$$
G(1) = \log(4) = 1.3863\cdots.
$$

It follows that the denominator in (12) is, on $[0,1]$, bounded below by

$$
1 - \frac{1}{2\pi}2\log(4) = 0.5587\cdots. \tag{13}
$$

The formula for $d > 0$ is a slight modification of (12). Notice that in the formula for b_k, (8), we may divide the numerator and denominator by 2^d, or in other words, use the normalize weights

$$
\hat{\beta}_k^{(d)} := \begin{cases} 2^{-d}\sum_{j=0}^{k}\binom{d}{k} & 0 \le k \le d \\ 1 & d \le k \le n-d \\ \hat{\beta}_{n-k} & n-d \le k \le n \end{cases} \tag{14}
$$

which are equal to 1 except for the first and last d weights. Hence the formula for the denominator is the same except for a modification to the first and last d terms.

Proposition 5. *Suppose that $n \geq 2d$. Then*

$$b_k(x) = \frac{\hat{\beta}_k^{(d)} \operatorname{sinc}(n(x - x_k))}{A_d(x) + \left[1 + \frac{1}{2\pi}\{\sin(n\pi x)G(nx+1) + \sin(n\pi(1-x))G(n(1-x)+1)\}\right] + B_d(x)}$$

where

$$A_d(x) := \sum_{j=0}^{d-1} (\hat{\beta}_j^{(d)} - 1) \operatorname{sinc}(n(x - x_j)),$$

$$B_d(x) := \sum_{j=n-d+1}^{n} (\hat{\beta}_j^{(d)} - 1) \operatorname{sinc}(n(x - x_j)).$$

Proof. The proof is elementary bookkeeping and we suppress the details. □

3 Quadrature Based on FH_n

The interpolation formula FH_n leads naturally to an interpolatory quadrature formula of the type

$$\int_0^1 f(x)dx \approx \sum_{k=0} w_k f(x_k)$$

where the weights w_k are given by

$$w_k := \int_0^1 b_k(x)dx. \tag{15}$$

The study of these quadrature formulas is still at a preliminary stage. Indeed, the recent paper [9] studies the approximation order of the quadrature formula and gives a numerical study of the weights for $d > 0$.

In particular, for stability reasons it is interesting to know if the weights w_k are positive or not. We can use the compact formula given in Proposition 5 to show that, at least for $d = 0$, the weights are (essentially) asymptotic to $1/n$.

In the $d = 0$ case we have b_k given by (12), which we may write in the form

$$b_k(x) = \frac{\operatorname{sinc}(n(x - x_k))}{D_n(x)}$$

where

$$D_n(x) := 1 + \frac{1}{2\pi}\{\sin(n\pi x)G(nx+1) + \sin(n\pi(1-x))G(n(1-x)+1)\}.$$

Proposition 6. *Suppose that $d = 0$ and r is such that that $k/n \to r \in (0,1)$, with both $k, n \to \infty$. Then*

$$\lim_{k/n \to r} n\, w_k = \lim_{n\to\infty} n \int_0^1 \operatorname{sinc}(n(x - x_k))dx = 1.$$

Proof. We calculate

$$
\begin{aligned}
&n\int_0^1 \operatorname{sinc}(n(x-x_k)) - b_k(x)\,dx \\
&= n\int_0^1 \operatorname{sinc}(n(x-x_k))\left\{1-\frac{1}{D_n(x)}\right\}dx \\
&= \frac{n}{2\pi}\int_0^1 \operatorname{sinc}(n(x-x_k))\frac{\sin(n\pi x)G(nx+1)+\sin(n\pi(1-x))G(n(1-x)+1)}{D_n(x)}dx \\
&= \frac{n}{2\pi}\left\{\int_0^1 \frac{\operatorname{sinc}(n(x-x_k))\sin(n\pi x)G(nx+1)}{D_n(x)}dx+\int_0^1 \frac{\operatorname{sinc}(n(x-x_k))\sin(n\pi(1-x))G(n(1-x)+1)}{D_n(x)}dx\right\}.
\end{aligned}
\tag{16}
$$

Consider first,

$$n\int_0^1 \frac{\operatorname{sinc}(n(x-x_k))\sin(n\pi x)G(nx+1)}{D_n(x)}dx.$$

Letting $x=t/n$, we obtain

$$
\begin{aligned}
&\int_0^n \frac{\operatorname{sinc}(t-k)\sin(\pi t)G(t+1)}{D_n(t/n)}dt \\
&= \left(\int_0^{k-1}+\int_{k-1}^{k+1}+\int_{k+1}^{n}\right)\frac{\operatorname{sinc}(t-k\sin(\pi t)G(t+1)}{D_n(t/n)}dt \\
&= A+B+C, \quad \text{say.}
\end{aligned}
$$

Using the fact that $D_n(x)\geq 1-\ln(4)/\pi$ (see (13)) we may bound $|A|$ by

$$
\begin{aligned}
|A| &= \left|\int_0^{k-1}\frac{\operatorname{sinc}(t-k)\sin(\pi t)G(t+1)}{D_n(t/n)}dt\right| \\
&\leq \frac{1}{1-\log(4)/\pi}\int_0^{k-1}|\operatorname{sinc}(t-k)|\cdot|\sin(\pi t)|\cdot G(t+1)dt \\
&\leq \frac{1}{1-\log(4)/\pi}\int_0^{k-1}|\operatorname{sinc}(t-k)|\cdot G(t+1)dt.
\end{aligned}
$$

Now, by (44:13:3),

$$G(x)=2\int_0^\infty \frac{e^{-xt}}{1+e^{-t}}dt\leq\int_0^\infty e^{-xt}dt=\frac{2}{x}.$$

Further,

$$|\operatorname{sinc}(x)|\leq\frac{1}{\pi|x|}.$$

Hence,

$$
\begin{aligned}
|A| &\leq \frac{2}{\pi-\log(4)}\int_0^{k-1}\frac{1}{(k-t)(t+1)}dt \\
&= \frac{2}{\pi-\log(4)}\frac{1}{k+1}\log\left(\frac{t+1}{k-t}\right)\Bigg|_{t=0}^{t=k-1} \\
&= \frac{4}{\pi-\log(2)}\frac{\log(k)}{k+1}.
\end{aligned}
\tag{17}
$$

Similarly, using the fact that $|\text{sinc}(x)| \leq 1$, we have for $|B|$

$$\begin{aligned}
|B| &= \left| \int_{k-1}^{k+1} \frac{\text{sinc}(t-k)\sin(\pi t)G(t+1)}{D_n(t/n)} dt \right| \\
&\leq \frac{1}{1-\log(4)/\pi} \int_{k-1}^{k+1} G(t+1)dt \\
&\leq \frac{1}{1-\log(4)/\pi} \int_{k-1}^{k+1} \frac{2}{t+1} dt \\
&= \frac{2\pi}{\pi - \log(4)} \log\left(\frac{k+2}{k}\right). \qquad (18)
\end{aligned}$$

Finally, for $|C|$ we have

$$\begin{aligned}
|C| &= \left| \int_{k+1}^{n} \frac{\text{sinc}(t-k)\sin(\pi t)G(t+1)}{D_n(t/n)} dt \right| \\
&\leq \frac{1}{1-\log(4)/\pi} \int_{k+1}^{n} |\text{sinc}(t-k)| \cdot |\sin(\pi t)| \cdot G(t+1)dt \\
&\leq \frac{2}{\pi - \log(4)} \int_{k+1}^{n} \frac{1}{(t-k)(t+1)} dt \\
&\leq \frac{2}{\pi - \log(4)} \int_{k+1}^{\infty} \frac{1}{(t-k)(t+1)} dt \\
&= \frac{2}{\pi - \log(4)} \frac{1}{k+1} \log\left(\frac{t-k}{t+1}\right)\Bigg|_{t=k+1}^{t=\infty} \\
&= \frac{2}{\pi - \log(4)} \frac{\log(k+1)}{k+1}. \qquad (19)
\end{aligned}$$

Next consider (2π times) the second integral of the right side of (16), i.e.,

$$n \int_0^1 \frac{\text{sinc}(n(x-x_k))\sin(n\pi(1-x))G(n(1-x)+1)}{D_n(x)} dx.$$

Let $x = 1 - x'$ to get

$$\begin{aligned}
& n \int_0^1 \frac{\text{sinc}(n(1-x-x_k))\sin(n\pi x)G(nx+1)}{D_n(1-x)} dx \\
&= \int_0^1 \frac{\text{sinc}(n(x-(1-x_k)))\sin(n\pi x)G(nx+1)}{D_n(x)} dx \quad (\text{as } D_n(1-x) = D_n(x)) \\
&= \int_0^1 \frac{\text{sinc}(n(x-x_{n-k}))\sin(n\pi x)G(nx+1)}{D_n(x)} dx \quad (\text{as } D_n(1-x) = D_n(x)) \\
&= \int_0^1 \frac{\text{sinc}(n(x-x_{k'}))\sin(n\pi x)G(nx+1)}{D_n(x)} dx \quad (\text{as } D_n(1-x) = D_n(x))
\end{aligned}$$

with $k' := n - k$. This is the same as the first integral, but with k replaced by $k' = n - k$. (Note that as $k/n \to r \in (0,1)$, $(n-k)/k \to 1 - r \in (0,1)$. In particular $k' \to \infty$ as well.) Hence the bounds developed for the first integral also hold for the second. It follows from (16), (17), (18) and (19) that

$$
\begin{aligned}
\left| n\int_0^1 \operatorname{sinc}(n(x-x_k)) - b_k(x)\,dx \right| &\le \frac{1}{2\pi}\frac{4}{\pi-\log(2)}\left\{\frac{\log(k)}{k+1}+\frac{\log(k')}{k'+1}\right\}\\
&\quad+\frac{1}{2\pi}\frac{2\pi}{\pi-\log(4)}\left\{\log\left(\frac{k+2}{k}\right)+\log\left(\frac{k'+2}{k'}\right)\right\}\\
&\quad+\frac{1}{2\pi}\frac{2}{\pi-\log(4)}\left\{\frac{\log(k+1)}{k+1}+\frac{\log(k'+1)}{k'+1}\right\}
\end{aligned}
$$

which tends to 0 as $k, n \to \infty$ with $k/n \to r \in (0, 1)$.

We are almost finished. Note now that

$$
\begin{aligned}
n\int_0^1 \operatorname{sinc}(n(x-x_k))\,dx &= \int_0^n \operatorname{sinc}(x-k)\,dx \quad (\text{letting } x' = nx)\\
&= \int_{-k}^{n-k} \operatorname{sinc}(x)\,dx \quad (\text{letting } x' = x-k)\\
&\to \int_{-\infty}^{\infty} \operatorname{sinc}(x)\,dx \quad (\text{since } \ k, n, n-k \to \infty)\\
&= 1
\end{aligned}
$$

as is well known. □

Acknowledgments

This work has been done with support of the *60% funds, year 2011* of the University of Padua, as well as those of the University of Verona.

References

[1] J.-P. Berrut. Rational functions for guaranteed and experimentally well-conditioned global interpolation. *Comput. Math. Appl.*, 15(1):1–16, 1988.

[2] J.-P. Berrut. Barycentric Formulae for Cardinal (SINC–)Interpolants. *Numer. Math.*, 54, 703–718 (1989).

[3] J.-P. Berrut and H. D. Mittelmann. Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. *Comput. Math. Appl.*, 33(6):77–86, 1997.

[4] L. Bos, K. Hormann and S. De Marchi. On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes. *J. Comput. Appl. Math.*, 236:504-510, 2011.

[5] L. Bos, K. Hormann, S. De Marchi and G. Klein. On the Lebesgue constant of barycentric rational interpolant at equidistant nodes. to appear on Numer. Math.

[6] P. L. Butzer and R. L. Stens Sampling theory of not necessarily band-limited functions: a historical overview. *SIAM Review*, Vol. 34, No. 1, 40–53, 1992.

[7] Ch.-J. de la Vallée Poussin. Sur la convergence des formules d'interpolation entre ordonées équidistantes. *Bull. Cl. Sci. Acad. Roy. Belg.*, 4, 319–410, 1908.

[8] M. S. Floater and K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. *Numer. Math.*, 107(2):315–331, 2007.

[9] G. Klein and J.-P. Berrut. Linear barycentric rational quadrature, to appear in BIT Num. Math., http://dx.doi.org/10.1007/s10543-011-0357-x

[10] K. Oldham, J. Myland and J. Spanier. An Atlas of Functions, 2nd edition. Springer, 2008.