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Abstract

In [2] we proved the stability, in L_2 and L_∞ norms, of kernel–based interpolation, essentially based on standard error estimates for radial basis functions. In this note we give a different proof based on a *sampling inequality* (cf. [14, Th. 2.6]). A few numerical examples, supporting the results, will be presented too.

Keywords: Radial basis functions, Lebesgue function, Lebesgue constant

1 Introduction

Let $s_{f,X}$ denote the recovery of a real–valued function $f : \Omega \to \mathbb{R}$ on some compact domain $\Omega \subseteq \mathbb{R}^d$ from its function values $f(x_j)$ on a scattered set $X = \{x_1, ..., x_N\} \subset \Omega \subseteq \mathbb{R}^d$. Independently on how the reconstruction is done we may assume that $s_{f,X}$ is a linear function of the data

$$s_{f,X} = \sum_{j=1}^{N} f(x_j) u_j \tag{1}$$

with certain continuous functions $u_j : \Omega \to \mathbb{R}$. To assert the stability of the recovery process $f \mapsto s_{f,X}$, we look for bounds of the form

$$\|s_{f,X}\|_{L_\infty(\Omega)} \le C(X) \|f\|_{\ell_\infty(X)} \tag{2}$$

which imply that the map, taking the data into the interpolant, is continuous in the $L_\infty(\Omega)$ and $\ell_\infty(X)$ norms. Of course, one can also use $L_2(\Omega)$ and $\ell_2(X)$ norms above.

An upper (naive) bound for the *stability constant* $C(X)$ is supplied by putting (1) into (2):

$$C(X) \ge \left\| \sum_{j=1}^{N} |u_j(x)| \right\|_{L_\infty(\Omega)} =: \Lambda_X.$$

This involves the *Lebesgue function*

$$\lambda_X(x) := \sum_{j=1}^{N} |u_j(x)| \, .$$

whose maximum value $\Lambda_X := \max_{x \in \Omega} \lambda_X(x)$ is called the associated *Lebesgue constant*.

It is a classical problem to derive upper bounds for the stability constant in (2) and for its upper bound, the Lebesgue constant Λ_X.

However, stability bounds for multivariate *kernel–based* recovery processes are missing. We shall derive them as follows. Given a positive definite kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$, the recovery of functions from function values $f(x_j)$ on the set $X = \{x_1, ..., x_N\} \subset \Omega \subseteq \mathbb{R}^d$ of N different data sites can be done via interpolants of the form

$$s_{f,X} := \sum_{j=1}^{N} \alpha_j \Phi(\cdot, x_j) \tag{3}$$

from the finite-dimensional space $V_X := \text{span}\,\{\Phi(\cdot, x) : x \in X\}$ of translates of the kernel, and satisfying the linear system

$$f(x_k) = \sum_{j=1}^{N} \alpha_j \Phi(x_k, x_j),\ 1 \le k \le N$$

based on the *kernel matrix* $A_{\Phi,X} := \Phi(x_k, x_j),\ 1 \le j, k \le N$. The case of *conditionally* positive definite kernels is similar, and we suppress details here.

The interpolant of (3), as in classical polynomial interpolation, can also be written in terms of *cardinal functions* $u_j \in V_X$ such that $u_j(x_k) = \delta_{j,k}$. Then, the interpolant (3) takes the usual *Lagrangian* form (1).

The reproduction quality of kernel–based methods is governed by the *fill distance* or *mesh norm*

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2 \tag{4}$$

describing the geometric relation of X to the domain Ω. In particular, the reproduction error is small if $h_{X,\Omega}$ is small. But unfortunately the kernel matrix $A_{\Phi,X}$ is badly conditioned if the data locations has small *separation distance*

$$q_X = \frac{1}{2} \min_{\substack{x_i, x_j \in X \\ x_i \neq x_j}} \|x_i - x_j\|\,. \tag{5}$$

Then the coefficients of the representation (3) get very large even if the data values $f(x_k)$ are small, and simple linear solvers will fail. However, users often report that the final function $s_{f,X}$ of (3) behaves nicely in spite of the large coefficients, and using stable solvers lead to useful results even in case of unreasonably large condition numbers. This means that the interpolant can be stably calculated in the sense of (2), while the coefficients in the basis supplied by the $\Phi(x, x_j)$ are unstable. This calls for the construction of new and more stable bases as shown in [9].

The fill distance (4) and the separation distance (5) are used for standard error and stability estimates for multivariate interpolants. For instace, in [2] we used them to derive the very first proofd of the results of the present paper.

2 The main results

Later, we shall consider arbitrary sets with different cardinalities, but with uniformity constants bounded above by a fixed number. Note that $h_{X,\Omega}$ and

q_X play an important role in finding good points for radial basis function interpolation, as recently studied in [7, 1, 3].

To generate interpolants, we allow conditionally positive definite translation-invariant kernels

$$\Phi(x, y) = K(x - y) \text{ for all } x, y \in \mathbb{R}^d,\ K\ :\ \mathbb{R}^d \to \mathbb{R}$$

which are reproducing in their "native" Hilbert space $\mathcal{N}$ which we assume to be norm–equivalent to some Sobolev space $W_2^\tau(\Omega)$ with $\tau > d/2$. The kernel will then have a Fourier transform satisfying

$$0 < c(1 + \|\omega\|_2^2)^{-\tau} \leq \hat{K}(\omega) \leq C(1 + \|\omega\|_2^2)^{-\tau} \tag{6}$$

at infinity. This includes Poisson radial functions [5, 6], Sobolev/Matern kernels, and Wendland's compactly supported kernels (cf. e.g. [12]). It is well-known that under the above assumptions the interpolation problem is uniquely solvable, and the space V_X is a subspace of Sobolev space $W_2^\tau(\Omega)$.

In the following, the constants are dependent on the space dimension, the domain, and the kernel, and the assertions hold for all sets X of scattered data locations with sufficiently small fill distance $h_{X,\Omega}$.

Our central result is

Theorem 1 *The classical Lebesgue constant for interpolation with Φ on $N = |X|$ data locations $X = \{x_1, \ldots, x_N\}$ in a bounded domain $\Omega \subseteq \mathbb{R}^d$ satisfying an outer cone condition has a bound of the form*

$$\Lambda_X \leq C\sqrt{N} \left(\frac{h_{X,\Omega}}{q_X}\right)^{\tau - d/2}.$$

For quasi-uniform sets with bounded uniformity γ, this simplifies to

$$\Lambda_X \leq C\sqrt{N}.$$

Each single cardinal function is bounded by

$$\|u_j\|_{L_\infty(\Omega)} \leq C \left(\frac{h_{X,\Omega}}{q_X}\right)^{\tau - d/2},$$

which in the quasi-uniform case simplifies to

$$\|u_j\|_{L_\infty(\Omega)} \leq C.$$

There also is an L_2 analog of this:

Theorem 2 *Under the above assumptions, the cardinal functions have a bound*

$$\|u_j\|_{L_2(\Omega)} \leq C \left(\frac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2} h_{X,\Omega}^{d/2}$$

and for quasi-uniform data locations they behave like

$$\|u_j\|_{L_2(\Omega)} \leq C h_{X,\Omega}^{d/2}.$$

But the Lebesgue constants are only upper bounds for the stability constant in function space. In fact, we can do better:

Theorem 3 *Interpolation on sufficiently many quasi–uniformly distributed data is stable in the sense of*

$$\|s_{f,X}\|_{L_\infty(\Omega)} \leq C \left(\|f\|_{\ell_\infty(X)} + \|f\|_{\ell_2(X)}\right)$$

and

$$\|s_{f,X}\|_{L_2(\Omega)} \leq C h^{d/2} \|f\|_{\ell_2(X)}$$

with a constant C independent of X.

Note that the right-hand side of the final inequality is a properly scaled discrete version of the L_2 norm.

We shall prove these results in the next section. However, it turns out there that the assumption (6) is crucial, and we were not able to extend the results to kernels with infinite smoothness. The final chapter will provide examples showing that similar results are not possible for kernels with infinite smoothness.

2.1 Proofs

Our most important tool for the proof of Theorem 1 is a *sampling inequality* (cf. [14, Th. 2.6]). For any a bounded Lipschitz domain Ω with an inner cone condition, and for Sobolev space $W_2^\tau(\Omega) \subset \mathbb{R}^d$ with $\tau > d/2$ there are positive constants C and h_0 such that

$$\|u\|_{L_\infty(\Omega)} \leq C \left(h_{X,\Omega}^{\tau-d/2} \|u\|_{W_2^\tau(\Omega)} + \|u\|_{\ell_\infty(X)}\right) \tag{7}$$

holds for all $u \in W_2^\tau(\Omega)$ and all finite subsets $X \subset \Omega$ with $h_{X,\Omega} \leq h_0$. This is independent of kernels.

We can apply the sampling inequality in two ways:

$$
\begin{aligned}
\|s_{f,X}\|_{L_\infty(\Omega)} &\leq C\left(h_{X,\Omega}^{\tau-d/2}\|s_{f,X}\|_{W_2^\tau(\Omega)} + \|s_{f,X}\|_{\ell_\infty(X)}\right)\\
&\leq C\left(h_{X,\Omega}^{\tau-d/2}\|s_{f,X}\|_{W_2^\tau(\Omega)} + \|f\|_{\ell_\infty(X)}\right),\\
&\leq C\left(h_{X,\Omega}^{\tau-d/2}\|s_{f,X}\|_{\mathcal{N}} + \|f\|_{\ell_\infty(X)}\right),\\
\|u_j\|_{L_\infty(\Omega)} &\leq C\left(h_{X,\Omega}^{\tau-d/2}\|u_j\|_{W_2^\tau(\Omega)} + \|u_j\|_{\ell_\infty(X)}\right)\\
&\leq C\left(h_{X,\Omega}^{\tau-d/2}\|u_j\|_{W_2^\tau(\Omega)} + 1\right)\\
&\leq C\left(h_{X,\Omega}^{\tau-d/2}\|u_j\|_{\mathcal{N}} + 1\right)
\end{aligned}
$$

since we know that the space V_X is contained in $W_2^\tau(\Omega)$. To get a bound on the norm in native space, we need bounds of the form

$$\|s\|_{\mathcal{N}} \leq C(X,\Omega,\Phi)\|s\|_{\ell_\infty(X)}$$

for arbitrary elements $s \in V_X$. Such bounds are available from [12], but we repeat the basic notation here. Let Φ satisfy (6). Then [12] has

$$\|s\|^2_{W_2^\tau(\Omega)} \leq C q_X^{-2\tau+d}\|s\|^2_{\ell_2(X)} \leq CNq_X^{-2\tau+d}\|s\|^2_{\ell_\infty(X)} \text{ for all } s \in V_X$$

with a different generic constant. If we apply this to u_j, we get

$$\|u_j\|_{L_\infty(\Omega)} \leq C\left(\left(\frac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2} + 1\right),$$

while application to $s_{f,X}$ yields

$$
\begin{aligned}
\|s_{f,X}\|_{L_\infty(\Omega)} &\leq C\left(\left(\tfrac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2}\|f\|_{\ell_2(X)} + \|f\|_{\ell_\infty(X)}\right)\\
&\leq C\left(\sqrt{N}\left(\tfrac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2} + 1\right)\|f\|_{\ell_\infty(X)}.
\end{aligned}
$$

Then the assertions of Theorem 1 and the first part of Theorem 3 follow.□

For the L_2 case covered by Theorem 2, we take the sampling inequality

$$\|f\|_{L_2(\Omega)} \leq C\left(h_{X,\Omega}^{\tau}\|f\|_{W_2^\tau(\Omega)} + \|f\|_{\ell_2(X)}h_{X,\Omega}^{d/2}\right), \quad \text{for all } f \in W_2^\tau(\Omega) \tag{8}$$

of [8, Thm. 3.5]. We can apply the sampling inequality as

$$
\begin{aligned}
\|s_{f,X}\|_{L_2(\Omega)} &\leq C\left(h_{X,\Omega}^{\tau}\|s_{f,X}\|_{W_2^\tau(\Omega)} + \|s_{f,X}\|_{\ell_2(X)}h_{X,\Omega}^{d/2}\right)\\
&\leq C\left(h_{X,\Omega}^{\tau}\|s_{f,X}\|_{W_2^\tau(\Omega)} + \|f\|_{\ell_2(X)}h_{X,\Omega}^{d/2}\right),\\
&\leq C\left(\tfrac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2}\|f\|_{\ell_2(X)}h_{X,\Omega}^{d/2},\\
\|u_j\|_{L_2(\Omega)} &\leq C\left(h_{X,\Omega}^{\tau}\|u_j\|_{W_2^\tau(\Omega)} + \|u_j\|_{\ell_2(X)}h_{X,\Omega}^{d/2}\right)\\
&\leq C\left(h_{X,\Omega}^{\tau-d/2}\|u_j\|_{W_2^\tau(\Omega)} + 1\right)h_{X,\Omega}^{d/2}\\
&\leq C\left(\left(\tfrac{h_{X,\Omega}}{q_X}\right)^{\tau-d/2} + 1\right)h_{X,\Omega}^{d/2}.
\end{aligned}
$$

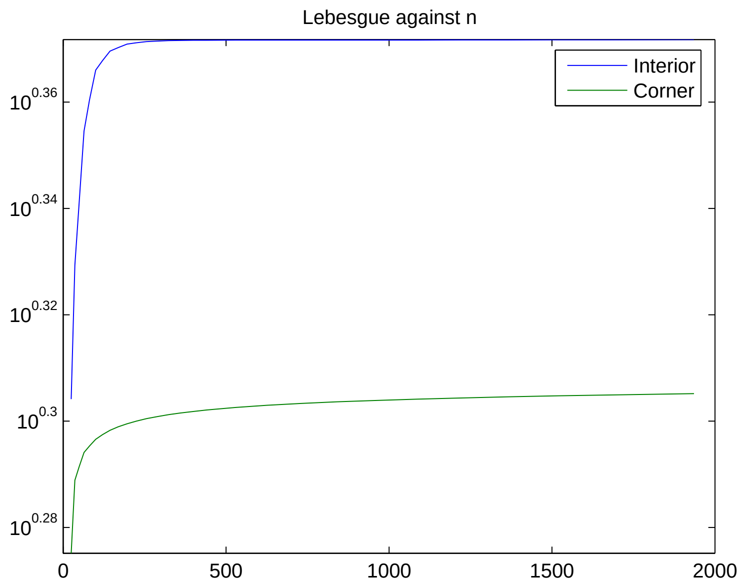

Figure 1: Lebesgue constants for the Sobolev/Matern kernel

This proves Theorem 2 and the second part of Theorem 3. $\square$

3 Examples

We present two series of examples on uniform grids on $[-1,1]^2$ and increasing numbers N of data locations.

Figure 1 shows the values Λ_X of the Lebesgue constants for the Sobolev/Matern kernel $(r/c)^\nu K_\nu(r/c)$ for $\nu = 1.5$ at scale $c = 20$. In this and other examples for kernels with finite smoothness, one can see that our bounds on the Lebesgue constants are valid, but the experimental Lebesgue constants seem to be uniformly bounded. In all cases, the maximum of the Lebesgue function is attained in the interior of the domain.

Things are different for infinitely smooth kernels. Figure 2 shows the behavior for the Gaussian. The maximum of the Lebesgue function is attained near the corners for large scales, while the behavior in the interior is as stable as for kernels with limited smoothness. The Lebesgue constants do not seem to be uniformly bounded.

A second series of examples was run on 225 regular points in $[-1,1]^2$ for different kernels at different scales using a parameter c as $\Phi_c(x) = \Phi(x/c)$.

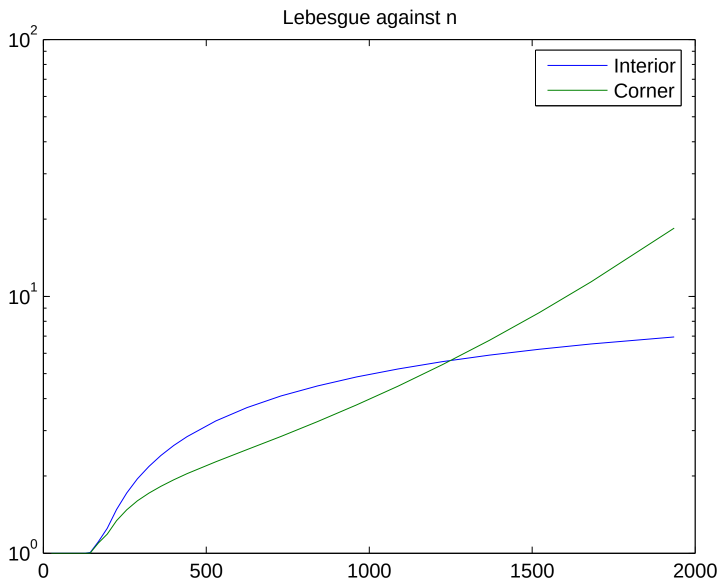

Figure 2: Lebesgue constants for the Gauss kernel

Figures 3 to 5 show how the scaling of the Gaussian kernel influences the shape of the associated Lagrange basis functions. The limit for large scales is called the *flat limit* [4] which is a Lagrange basis function of the de Boor/Ron polynomial interpolation [11]. It cannot be expected that such Lagrange basis functions are uniformly bounded.

In contrast to this, Figure 6 shows the corresponding Lagrange basis function for the Sobolev/Matern kernel at scale 320. The scales were such that the conditions of the kernel matrices were unfeasible for the double scale. Figure 7 shows the Lebesgue function in the situation of Figure 5, while Figure 8 shows the Sobolev/Matern case in the situation of Figure 6.

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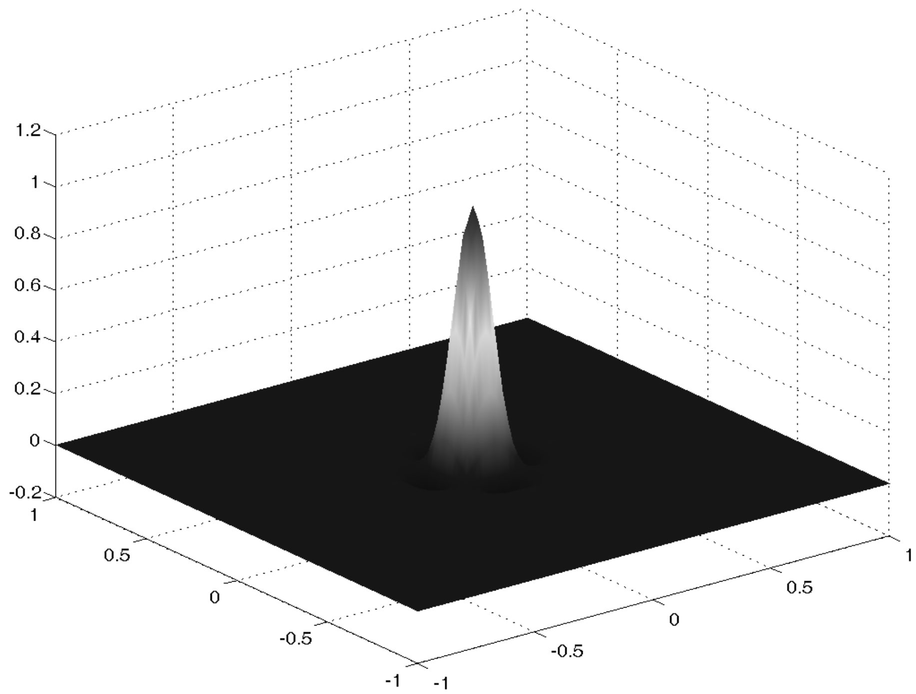

Figure 3: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.1

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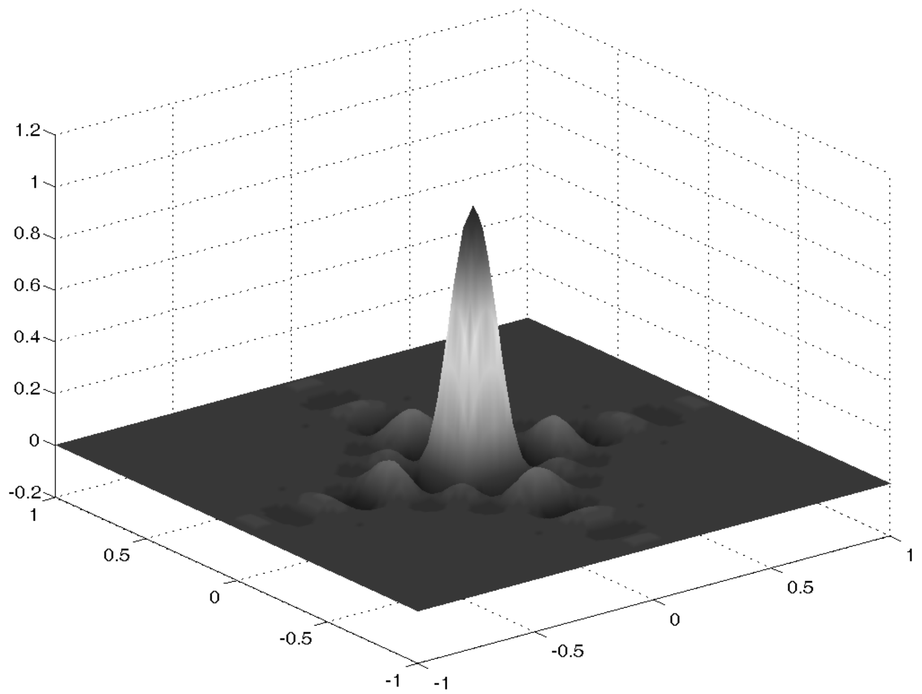

Figure 4: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.2

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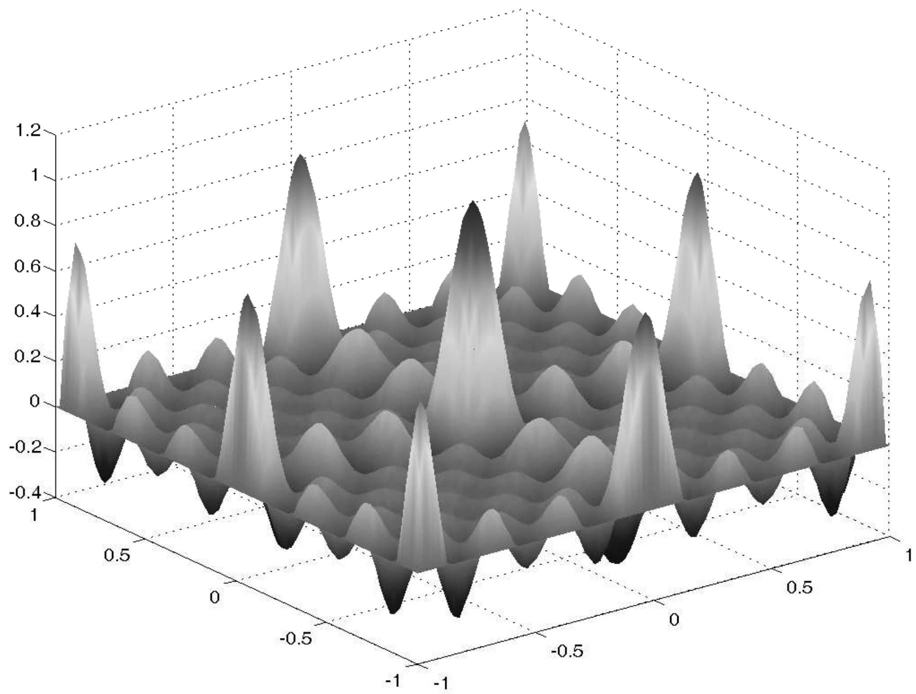

Figure 5: Lagrange basis function on 225 data points, Gaussian kernel with scale 0.4

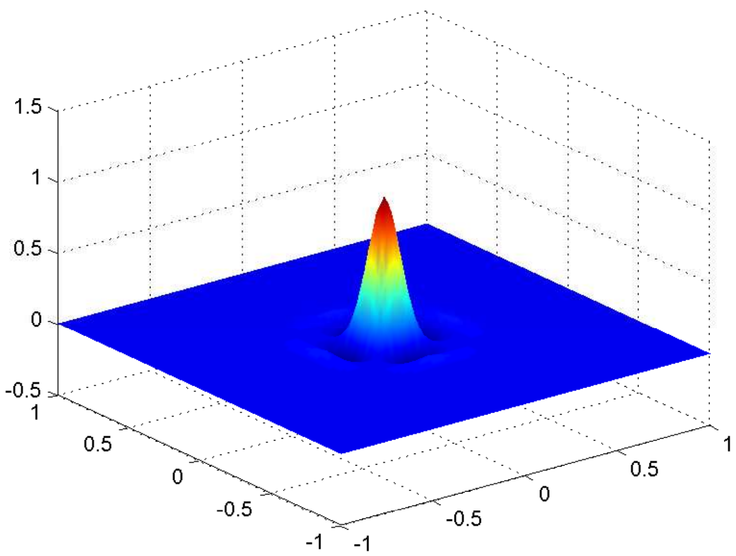

Figure 6: Lagrange basis function on 225 data points, Sobolev/Matern kernel with scale 320

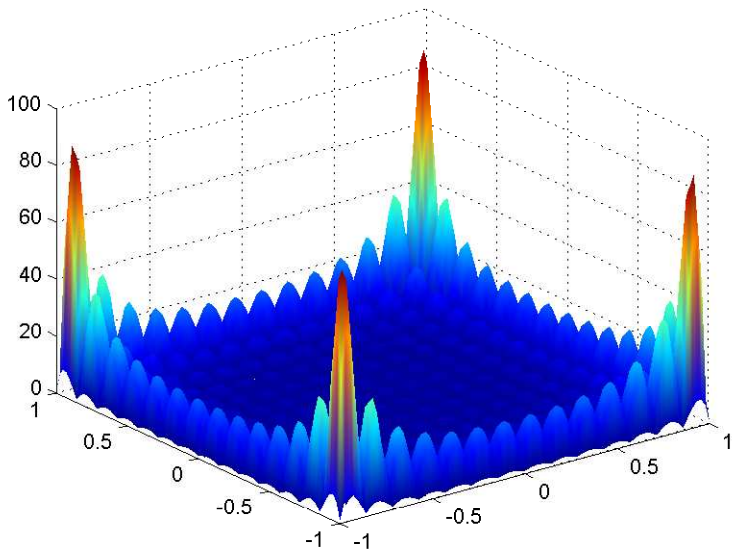

Figure 7: Lebesgue function on 225 regular data points, Gaussian kernel with scale 0.4

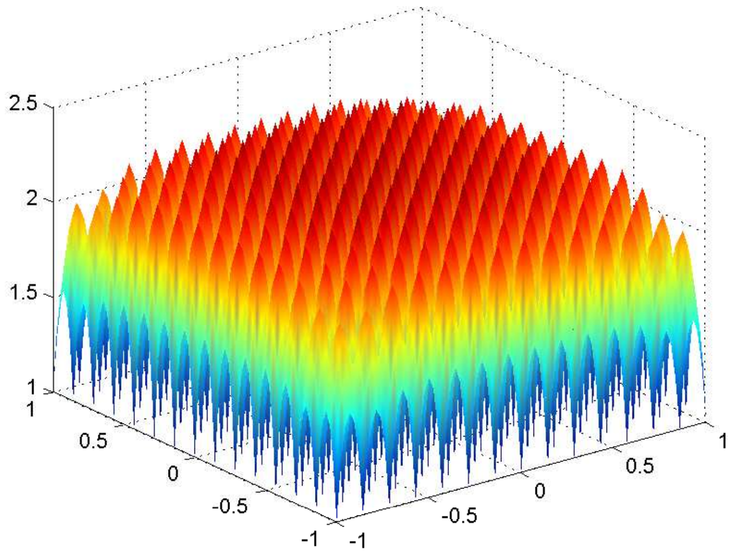

Figure 8: Lebesgue function on 225 regular data points, Sobolev/Matern kernel with scale 320

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