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UNIVARIATE RADIAL BASIS FUNCTIONS WITH COMPACT SUPPORT CARDINAL FUNCTIONS

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We discuss the class of univariate Radial Basis Functions for which the ith cardinal function u_i for interpolation at $x_1 < x_2 < \cdots < x_n$ has support $[x_{i-1}, x_{i+1}]$. We also give an explicit example where it can be proven that the points in an interval $[a, b]$ for which the associated Lebesgue constant is minimal, are *equally spaced*.

1. Introduction

Radial Basis Function interpolation (RBF) is an important method of (multivariate) interpolation of typically scattered data, which has been much used in applications. The *basic* form of RBF is quite simple. Given a function $g : \mathbb{R}^+ \to \mathbb{R}$, the associated RBF interpolant of a data set $\{(x_j, y_j)\} \subset \mathbb{R}^{d+1}$ with n "sites" $x_j \in \mathbb{R}^d$ and function values $y_j \in \mathbb{R}$, is the function of the form

$$s(x) = \sum_{j=1}^{n} a_i g(|x - x_j|) \text{ such that } s(x_i) = y_i, \quad 1 \le i \le n$$

(if it exists).

The theory of RBF has been by now rather well developed (cf. the monographs [6],[3]) but there remain some interesting, and important, questions. One of these is the question of the upper bound for the associated Lebesgue constant of the interpolation process. To describe what this is, it can be shown that the RBF interpolant can be written in Lagrange form

$$s(x) = \sum_{j=1}^{n} y_j u_j(x)$$

where the $u_j(x)$ are the so-called *cardinal* functions, defined by the property that $u_j(x_i) = \delta_{ij}$, the Kronecker delta. If the sites x_i are restricted to lie in a compact set $K \subset \mathbb{R}^d$, then the Lebesgue constant is defined to be

$$\Lambda_n := \max_{x\in K} \sum_{j=1}^{n} |u_j(x)|.$$

The value of Λ_n gives important information on the stability of the interpolation process (see [4] for a discussion of this property for kernels g with limited smoothness). A particularly interesting question is to determine an optimal bound for Λ_n when the sites are "equally spaced" in K, as it is understood that such sites are near optimal (cf. [5]) for RBF interpolation. This remains a largely open and likely difficult problem.

Given this typical difficulty of analyzing multivariate interpolation schemes and procedures, it is often useful to look more carefully at the univariate case for some suggestion as to how the general case might behave. This is the goal of this paper.

2. The case of $g(x) = x$

For the rest of the paper we will be considering only the univariate case, i.e., with sites $x_1 < x_2 < \cdots < x_n$ belonging to some interval $[a, b]$. It is instructive to first consider the case $g(x) = x$, for which the interpolant is of the form

$$s(x) = \sum_{j=1}^{n} a_j|x - x_j|, \quad a_j \in \mathbb{R}. \tag{1}$$

It is easy to see that this interpolation problem is *correct*, i.e., for every set of function values y_i, $1 \le i \le n$, there is a unique s of the form (1) such that

$$s(x_i) = y_i, \quad 1 \le i \le n.$$

This can be done in one of two ways. The first, perhaps the most direct, follows from the fact that the determinant of the associated linear system, also known as the associated Vandermonde determinant, may be computed (cf. [1]) to be

$$\det\left([|x_i - x_j|]_{1\le i,j\le n}\right) = (-1)^{n-1}2^{n-2}\left(\prod_{j=1}^{n-1} h_j\right)\left(\sum_{j=1}^{n-1} h_j\right), \tag{2}$$

where $h_j := x_{j+1} - x_j$. Clearly this is non-zero and hence the interpolation problem is indeed correct.

A second, more instructive way, is to give formulas for the cardinal functions u_j. In fact, since $|x-x_i|$ is piecewise linear, our interpolant is just the ordinary "connect the dots" piecewise linear interpolant of the data. The cardinal functions must then be

$$u_j(x) = \begin{cases} 0 & \text{if } x \le x_{j-1} \\ \frac{x-x_{j-1}}{x_j-x_{j-1}} & \text{if } x_{j-1} < x \le x_j \\ \frac{x_{j+1}-x}{x_{j+1}-x_j} & \text{if } x_j < x \le x_{j+1} \\ 0 & \text{if } x > x_{j+1} \end{cases} \tag{3}$$

for $2 \le j \le n-1$, i.e., x_j an *interior* point. For the two boundary points x_1 and x_n the formulas are slightly different:

$$u_1(x) = \begin{cases} \frac{x_2-x}{x_2-x_1} & \text{if } x_1 \le x \le x_2 \\ 0 & \text{if } x > x_2, \end{cases} \tag{4}$$

$$u_n(x) = \begin{cases} 0 & \text{if } x \le x_{n-1} \\ \frac{x-x_{n-1}}{x_n-x_{n-1}} & \text{if } x_{n-1} < x \le x_n. \end{cases} \tag{5}$$

These are none other than the classical piecewise linear "hat" functions. What remains to show is that these "hat" functions can be written in the form (1). But this is easy. Indeed, one may check that

$$\begin{aligned} u_j(x) = {} & \frac{1}{2(x_j-x_{j-1})}|x-x_{j-1}| - \frac{x_{j+1}-x_{j-1}}{2(x_{j+1}-x_j)(x_j-x_{j-1})}|x-x_j| \\ & + \frac{1}{2(x_{j+1}-x_j)}|x-x_{j+1}| \end{aligned} \tag{6}$$

for $2 \le j \le n-1$, i.e., x_j an interior point. Note that the formula (6) is defined for all $x \in \mathbb{R}$, but is *identically* zero outside the interval $[x_{j-1}, x_{j+1}]$.

The boundary points x_1 and x_n are again slightly different. For u_1, if we were to add a site $x_0 < x_1$, then the same formula as (6) with $j = 1$ would hold for u_1. However, this would involve a multiple of $|x - x_0|$ which is *not* in the basis of allowed translates given by (1). But, restricted to the interval $[x_1, x_n]$, $|x - x_0| = (x - x_0)$, which way be expressed as

$$\begin{aligned} |x-x_0| = x - x_0 &= \frac{x_n-x_0}{x_n-x_1}(x-x_1) + \frac{x_1-x_0}{x_n-x_1}(x_n-x) \\ &= \frac{x_n-x_0}{x_n-x_1}|x-x_1| + \frac{x_1-x_0}{x_n-x_1}|x-x_n|, \end{aligned}$$

again restricted to $[x_1, x_n]$.

It follows that

$$
\begin{aligned}
(7)\qquad u_1(x) &= \frac{1}{2(x_1-x_0)}|x-x_0| \\
&\quad - \frac{x_2-x_0}{2(x_2-x_1)(x_1-x_0)}|x-x_1| + \frac{1}{2(x_2-x_1)}|x-x_2| \\
&= \frac{1}{2(x_1-x_0)}\left\{\frac{x_n-x_0}{x_n-x_1}|x-x_1| + \frac{x_1-x_0}{x_n-x_1}|x-x_n|\right\} \\
&\quad - \frac{x_2-x_0}{2(x_2-x_1)(x_1-x_0)}|x-x_1| + \frac{1}{2(x_2-x_1)}|x-x_2| \\
&= -\frac{x_n-x_2}{2(x_2-x_1)(x_n-x_1)}|x-x_1| + \frac{1}{2(x_2-x_1)}|x-x_2| \\
&\quad + \frac{1}{2(x_n-x_1)}|x-x_n|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(8)\qquad u_n(x) &= \frac{1}{2(x_n-x_1)}|x-x_1| + \frac{1}{2(x_n-x_{n-1})}|x-x_{n-1}| \\
&\quad - \frac{x_{n-1}-x_1}{2(x_n-x_1)(x_n-x_{n-1})}|x-x_n|.
\end{aligned}
$$

Remark. Note that each u_j is a combination of just three translates, $|x-x_{j-1}|$, $|x-x_j|$ and $|x-x_{j+1}|$. This also holds for u_1 and u_n if we make the cyclic identification $x_0 = x_n$ and $x_{n+1} = x_1$. This is reflected in the fact that the inverse of the Vandermonde matrix $[|x_i - x_j|]$ is cyclically tridiagonal, i.e., is tridiagonal, except that the $(1, n)$ and $(n, 1)$ entries are also non-zero. □

3. The case of $g''(x) = \lambda^2 g(x)$

We will show that, remarkably, for $g(x)$ a solution of the differential equation

$$g''(x) = \lambda^2 g(x), \quad \lambda \in \mathbb{C}$$

the cardinal functions have the same structure (6), as for the simple linear function $g(x) = x$. Specifically, we show that the cardinal functions for interpolants of the form

$$(9)\qquad s(x) = \sum_{j=1}^{n} a_j g(|x-x_j|)$$

are a linear combination of three consecutive translates of $g(|x|)$ and are supported in $[x_{j-1}, x_{j+1}]$. Moreover, we will then show that this is the *unique* class of such functions.

To begin, note that the $\lambda = 0$ case corresponds (essentially) to $g(x) = x$ and hence we assume that $\lambda \neq 0$. We may then write

$$g(x) = ae^{\lambda x} + be^{-\lambda x} \tag{10}$$

for some $a, b \in \mathbb{C}$. We first observe that the interpolation problem for functions of the form (9) is correct provided $b \neq a$ and $ae^{\lambda x_n} \neq \pm be^{\lambda x_1}$. This follows easily from the following formula for the associated Vandermonde determinant.

Theorem 1. *For $g(x)$ of the form (10) we have*

$$det([g(|x_i - x_j|)]_{1\le i,j\le n}) =$$
$$(b-a)^{n-2} e^{-2\lambda \sum_{j=1}^n x_j} \left(\prod_{j=1}^{n-1} (e^{2\lambda x_{j+1}} - e^{2\lambda x_j}) \right) \left(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_n} \right).$$

Proof. This follows from the same types of calculations as in Proposition 2.1 of [1] where the formula for the $b = -a$ case is given. □

Proposition 3.1. *For functions $g(x)$ of the form (10) with a, b so that the interpolation problem is correct, we have for $2 \le j \le n-1$,*

$$u_j(x) = A_1 g(|x - x_{j-1}|) + A_2 g(|x - x_j|) + A_3 g(|x - x_{j+1}|)$$

where

$$A_1 = -\frac{e^{\lambda x_{j-1}} e^{\lambda x_j}}{(e^{2\lambda x_j} - e^{2\lambda x_{j-1}})(b-a)},$$
$$A_2 = \frac{(e^{2\lambda x_{j+1}} - e^{2\lambda x_{j-1}}) e^{2\lambda x_j}}{(e^{2\lambda x_{j+1}} - e^{2\lambda x_j})(e^{2\lambda x_j} - e^{2\lambda x_{j-1}})(b-a)},$$
$$A_3 = -\frac{e^{\lambda x_j} e^{\lambda x_{j+1}}}{(e^{2\lambda x_{j+1}} - e^{2\lambda x_j})(b-a)}.$$

Proof. It is easily verified that the stated formula for u_j has all the properties of a cardinal function. Note that, again, u_j is *identically* zero outside the interval $[x_{j-1}, x_j]$. □

The cardinal functions for the boundary points x_1 and x_n have exactly the same structure as those for $g(x) = x$, i.e, (7) and (8).

Proposition 3.2. *For functions $g(x)$ of the form (10) with a, b so that the interpolation problem is correct, we have*

$$u_1(x) = B_1 g(|x - x_1|) + B_2 g(|x - x_2|) + B_3 g(|x - x_n|)$$
$$u_n(x) = C_1 g(|x - x_1|) + C_2 g(|x - x_{n-1}|) + C_3 g(|x - x_n|)$$

where

$$B_1 = \frac{e^{2\lambda x_1}(b^2 e^{2\lambda x_2} - a^2 e^{2\lambda x_n})}{(b-a)(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_n})(e^{2\lambda x_2} - e^{2\lambda x_1})},$$
$$B_2 = -\frac{e^{\lambda x_2} e^{\lambda x_1}}{(b-a)(e^{2\lambda x_2} - e^{2\lambda x_1})},$$
$$B_3 = -\frac{e^{\lambda x_1} e^{\lambda x_n} ab}{(b-a)(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_n})}$$

and

$$C_1 = -\frac{e^{\lambda x_1} e^{\lambda x_n} ab}{(b-a)(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_n})},$$
$$C_2 = -\frac{e^{\lambda x_{n-1}} e^{\lambda x_n}}{(b-a)(e^{2\lambda x_n} - e^{2\lambda x_{n-1}})},$$
$$C_3 = \frac{e^{2\lambda x_n}(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_{n-1}})}{(b-a)(b^2 e^{2\lambda x_1} - a^2 e^{2\lambda x_n})(e^{2\lambda x_n} - e^{2\lambda x_{n-1}})}.$$

Proof. Again, once given these formulas are easy to verify. Notice that if either a or b are zero, then $C_1 = B_3 = 0$ and u_1 and u_2 are linear combinations of just *two* translates. In this case the inverse of the Vandermonde matrix is exactly tridiagonal. □

In Figure 1 we show an example with the plot of the cardinals made by using the previous formulas.

4. The uniqueness of this class of functions

Theorem 2. *Suppose that $g : \mathbb{R}^+ \to \mathbb{R}$ is analytic. Suppose further that for any $x_1 < x_2 < \cdots < x_n$, the cardinal functions for interpolation of the form (9) can be given as a linear combination of three consecutive translates, i.e., there exist constants α_j, β_j and γ_j such that*

$$u_j(x) = \alpha_j g(|x - x_{j-1}|) + \beta_j g(|x - x_j|) + \gamma_j g(|x - x_{j+1}|),$$

$2 \le j \le n-1$. *Suppose further that u_j has support in the interval $[x_{j-1}, x_{j+1}]$. Then there exists a $\lambda \in \mathbb{C}$ such that*

$$g''(x) = \lambda^2 g(x).$$

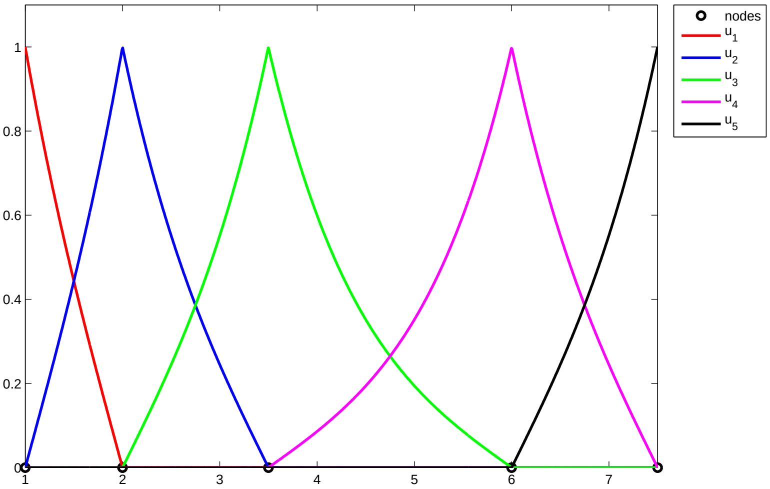

Figure 1: Cardinal functions for the nodes [1, 2, 3.5, 6, 7.5]

Proof. Since the x_i are arbitrary, for every three points $t_1 < t_2 < t_3$ we must be able to find constants α, β, γ, not all zero, such that

$$s(x) := \alpha g(|x - t_1|) + \beta g(|x - t_2|) + \gamma g(|x - t_3|)$$

has support inside $[t_1, t_3]$. In other words,

$$\alpha g(x - t_1) + \beta g(x - t_2) + \gamma g(x - t_3) \equiv 0, \quad x \geq t_3 \tag{11}$$

$$\alpha g(t_1 - x) + \beta g(t_2 - x) + \gamma g(t_3 - x) \equiv 0, \quad x \leq t_1. \tag{12}$$

Consider first (11). Since g is assumed to be analytic then this must be an identity for all $x \in \mathbb{R}$. We consider several cases. First, it can certainly not be the case that two of the coefficients α, β, γ are zero, for then g would be identically zero. Secondly, if one of them is zero, for example, say $\gamma = 0$, then

$$\alpha g(x - t_1) + \beta g(x - t_2) \equiv 0.$$

Setting $s := x - t_1$ and $y := t_2 - t_1 > 0$, we would have

$$\alpha g(s) + \beta g(s - y) \equiv 0.$$

In other words g would be such that for all $y > 0$ there existed α and β both non-zero, such that

$$\alpha g(s) + \beta g(s - y) \equiv 0.$$

In fact, since $g \not\equiv 0$ there must exist $s_0 \in \mathbb{R}$ such that $g(s_0) \neq 0$ so that

$$\alpha g(s_0) + \beta g(s_0 - y) = 0$$

and so the vector $\langle \alpha, \beta \rangle = k \langle g(s_0 - y), -g(s_0) \rangle$ for some multiple k. It follows that

$$g(s_0 - y) g(s) - g(s_0) g(s - y) \equiv 0. \tag{13}$$

Hence,

$$g(s) = \frac{g(s_0)}{g(s_0 - y)} g(s - y). \tag{14}$$

Moreover, differentiating (13) with respect to y we obtain

$$-g'(s_0 - y) g(s) - g(s_0) g'(s - y) \equiv 0$$

so that

$$\begin{aligned} g'(s - y) &= -\frac{g'(s_0 - y)}{g(s_0)} g(s) \\ &= -\frac{g'(s_0 - y)}{g(s_0)} \frac{g(s_0)}{g(s_0 - y)} g(s - y) \quad \text{by (14)} \\ &= -\frac{g'(s_0 - y)}{g(s_0 - y)} g(s - y). \end{aligned}$$

This implies that $g'(s - y)/g(s - y)$ is independent of s and hence $g(s) = Ke^{ks}$ for some constants K and k. Such a g is already in our class (with $\lambda = k$ and $b = 0$), but it is easily seen that in this case (with one coefficient zero) the second equation (12) can *not* also be satisfied. In other words, *three* translates are needed to construct the cardinal functions.

Lastly, consider the case when all three of α, β and γ are non-zero. In (11) and (12) we may set $s := x - t_1$, $y_1 := x_2 - x_1 > 0$ and $y_2 := x_3 - x_1 > y_1$ to obtain

$$\begin{aligned} \alpha g(s) + \beta g(s - y_1) + \gamma g(s - y_2) &\equiv 0, \\ \alpha g(-s) + \beta g(-s + y_1) + \gamma g(-s + y_2) &\equiv 0. \end{aligned}$$

If we then solve for $g(s)$ and $g(-s)$ we arrive at the equivalent conditions that for every $y_2 > y_1 > 0$ there must exist constants c_1 and c_2, both non-zero, such that

$$g(s) \equiv c_1 g(s - y_1) + c_2 g(s - y_2), \tag{15}$$

$$g(-s) \equiv c_1 g(-s + y_1) + c_2 g(-s + y_2). \tag{16}$$

The coefficients c_1 and c_2 can again be obtained from (two) certain specific values of s and hence are smooth functions of y_1 and y_2. Differentiating (15) with respect to y_1 and y_2 we obtain

$$\begin{aligned}(17)\qquad 0 &\equiv \frac{\partial c_1}{\partial y_1} g(s-y_1) - c_1 g'(s-y_1) + \frac{\partial c_2}{\partial y_1} g(s-y_2),\\ 0 &\equiv \frac{\partial c_1}{\partial y_2} g(s-y_1) + \frac{\partial c_2}{\partial y_2} g(s-y_2) - c_2 g'(s-y_2).\end{aligned}$$

It follows that $g'(s-y_1)$ and $g'(s-y_2)$ are both in the two-dimensional space, $\mathrm{span}(g(s-y_1), g(s-y_2))$.

Now, differentiate (17) with respect to s to obtain

$$0 \equiv \frac{\partial c_1}{\partial y_1} g'(s-y_1) - c_1 g''(s-y_1) + \frac{\partial c_2}{\partial y_1} g'(s-y_2).$$

It follows that the *three* functions $g(s-y_1)$, $g'(s-y_1)$ and $g''(s-y_1)$ are all members of the same *two* dimensional space, $\mathrm{span}(g(s-y_1), g(s-y_2))$, and hence are linearly dependent. Equivalently, $g(s)$ is the solution of some second order, constant coefficient, differential equation. By considering (16) it can be seen (after somewhat tedious calculations) that, in fact, the coefficient of $g'(s)$ must be zero, and indeed $g''(s) = \lambda^2 g(s)$ for some $\lambda \in \mathbb{C}$, as claimed. □

Remark. In the conclusion of Theorem 2 it may be that $\lambda = 0$ or not. If $\lambda = 0$ then $g''(x) = 0$, i.e., $g(x) = ax + b$ for some constants a, b. This is essentially the case considered in §2. If $\lambda \neq 0$ then g has the form $g(x) = ae^{\lambda x} + be^{-\lambda x}$, for some constants a, b.

5. The cardinal functions in piecewise form

There remains another surprise.

Theorem 3. *Suppose that $x_1 < \cdots < x_n$ and that $g(x) = ae^{\lambda x} + be^{-\lambda x}$ is such that the interpolation problem is correct. Then, independently of the values of a and b,*

$$u_j(x) = e^{\lambda(x_j - x)} \begin{cases} \frac{e^{2\lambda x} - e^{2\lambda x_{j-1}}}{e^{2\lambda x_j} - e^{2\lambda x_{j-1}}} & \textit{if } x \in [x_{j-1}, x_j] \\ \frac{e^{2\lambda x} - e^{2\lambda x_{j+1}}}{e^{2\lambda x_j} - e^{2\lambda x_{j+1}}} & \textit{if } x \in [x_j, x_{j+1}] \\ 0 & \textit{otherwise} \end{cases} \quad 2 \le j \le n-1\,,$$

$$u_1(x) = e^{\lambda(x_1 - x)} \begin{cases} \frac{e^{2\lambda x} - e^{2\lambda x_2}}{e^{2\lambda x_1} - e^{2\lambda x_2}} & \textit{if } x \in [x_1, x_2] \\ 0 & \textit{otherwise} \end{cases} \,,$$

$$u_n(x) = e^{\lambda(x_n - x)} \begin{cases} \frac{e^{2\lambda x} - e^{2\lambda x_{n-1}}}{e^{2\lambda x_n} - e^{2\lambda x_{n-1}}} & \text{if } x \in [x_{n-1}, x_n] \\ 0 & \text{otherwise} \end{cases} .$$

Proof. This is just a straightforward calculation. We leave the details to the reader. □

From this it follows easily that for $\lambda \in \mathbb{R}$, the $u_j(x)$ are non-negative on $[x_1, x_n]$ and, in particular that the Lebesgue function

$$\sum_{j=1}^{n} |u_j(x)| = \sum_{j=1}^{n} u_j(x).$$

Using the formulas of Theorem 3 we then easily arrive at

Proposition 5.1. *Suppose that $x_1 < x_2 < \cdots x_n$ and that $g(x) = ae^{\lambda x} + be^{-\lambda x}$ for $\lambda \in \mathbb{R}$, is such that the interpolation problem is correct. Then, independently of the values of a and b,*

$$\sum_{j=1}^{n} |u_j(x)| = \frac{e^{\lambda x} + e^{\lambda(x_j + x_{j+1} - x)}}{e^{\lambda x_j} + e^{\lambda x_{j+1}}}, \quad x \in [x_j, x_{j+1}].$$

In particular,

$$\max_{x_1 \le x \le x_n} \sum_{j=1}^{n} |u_j(x)| = 1.$$

Proof. That the maximum is one follows from the fact that the formula for the Lebesgue function restricted to $[x_j, x_{j+1}]$ equals 1 at x_j and x_{j+1}, and has a strictly positive second derivative. □

6. The case of λ complex

A particularly instructive case is when $\lambda = i$ with $a = -i/2$ and $b = i/2$ so that $g(x) = \sin(x)$. If we make the restriction that $x_n - x_1 < \pi$, then the determinant of Theorem 1 will be non-zero and the interpolation problem is correct.

The formulas for the cardinal functions given in Theorem 3 still hold and indeed simplify to

$$u_j(x) = \begin{cases} \frac{\sin(x - x_{j-1})}{\sin(x_j - x_{j-1})} & \text{if } x \in [x_{j-1}, x_j] \\ \frac{\sin(x_{j+1} - x)}{\sin(x_{j+1} - x_j)} & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases} ; \quad 2 \le j \le n-1 ,$$

$$u_1(x) = \begin{cases} \frac{\sin(x-x_2)}{\sin(x_1-x_2)} & \text{if } x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases},$$

$$u_n(x) = \begin{cases} \frac{\sin(x-x_{n-1})}{\sin(x_n-x_{n-1})} & \text{if } x \in [x_{n-1}, x_n] \\ 0 & \text{otherwise} \end{cases}.$$

It follows easily that $u_j(x) \geq 0$ on $[x_1, x_n]$ under our assumption that $x_n - x_1 < \pi$. Hence, the formula for the Lebesgue function given by Proposition 5.1 is also still valid. Indeed, it may easily be simplified to

$$\sum_{j=1}^{n} |u_j(x)| = \frac{\cos(x - \frac{x_j+x_{j+1}}{2})}{\cos(\frac{x_{j+1}-x_j}{2})}, \quad x \in [x_j, x_{j+1}].$$

The maximum is clearly attained at the midpoint $x = (x_j + x_{j+1})/2$ at which

$$\sum_{j=1}^{n} |u_j(x)| = \frac{1}{\cos(\frac{x_{j+1}-x_j}{2})}.$$

Hence

$$\begin{aligned} \Lambda_n &:= \max_{x_1 \leq x \leq x_n} \sum_{j=1}^{n} |u_j(x)| \\ &= \max_{1 \leq j \leq n-1} \frac{1}{\cos(\frac{x_{j+1}-x_j}{2})} \\ &= \frac{1}{\cos(\max_{1 \leq j \leq n-1} \frac{x_{j+1}-x_j}{2})}. \end{aligned} \tag{18}$$

Consequently we have

Theorem 4. *Suppose that $g(x) = \sin(x)$. Then, among all distributions of points $a = x_1 < x_2 < \cdots < x_n = b$ in the interval $[a, b]$ with $b - a < \pi$, the one for which Λ_n is uniquely minimized is the equally spaced one, i.e, for $x_j = a + (j-1)(b-a)/(n-1)$, $1 \leq j \leq n$.*

Proof. For any other distribution one of the spacings $x_{j+1} - x_j$ must be greater than the average spacing $(b-a)/(n-1)$. Hence, by (**??**),

$$\Lambda_n = \frac{1}{\cos(\max_{1 \leq j \leq n-1} \frac{x_{j+1}-x_j}{2})} > \frac{1}{\cos(\frac{b-a}{2(n-1)})},$$

the value of the Lebesgue constant for the equally spaced distribution. □

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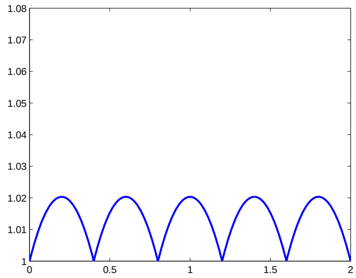

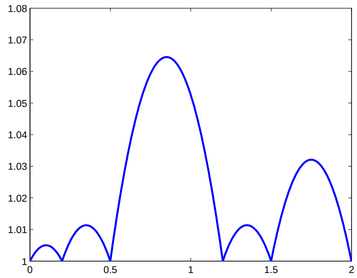

Figure 2: Lebesgue functions for $\lambda = i$ and equally spaced points (left) and non-equally spaced points [0 0.2 0.5 1.2 1.5 2] (right)

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