Lissajous sampling and spectral filtering in MPI applications: the reconstruction algorithm for reducing the Gibbs phenomenon

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I. Introduction

The Magnetic Particle Imaging (MPI) is an emerging medical imaging technology which attracted the interest of different research groups in the last years [14]. The technique of the MPI is based on the detection of a tracer which consists of superparamagnetic iron oxide nanoparticles through the superimposition of different magnetic fields.

When the particles are excited by oscillating magnetic fields, an electromagnetic induction phenomenon is induced and measured. The acquisition of the signal which comes from the particles is performed moving a field-free point along suitable sampling trajectories, using appropriate magnetic gradient fields.

A possible choice is to move along Lissajous curves [13], but the problem of selecting the set of sampling points to take along the curve is not trivial. The first time in which the Lissajous curves have been considered in polynomial approximation was in the debut of the *Padua points* (see [4], [5], [7]). Due to their excellent properties, efforts have been made in order to understand more about the sets of points which can be generated from Lissajous curves.

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In this view, Lissajous node points, which are provided with optimal properties similarly to the Padua points, were introduced and studied [1], [10], [11]. In [2], [3] three-dimensional Lissajous curves have been studied. Using conjectured optimal parameters for these curves, a hyperinterpolation polynomial approximation has been performed, which consists in considering a discretized expansion of a function in series of chosen orthogonal polynomial up to a fixed total-degree.

The polynomial interpolant of the two-dimensional case and the hyperinterpolant of the three-dimensional one are expressed as a Chebyshev series, which is a particular case of Fourier series.

In applications, we typically deal with objects represented by underlying discontinuous functions. It is well-known that the presence of leap points of discontinuity in the function carries the arise of the Gibbs phenomenon.

The Gibbs phenomenon causes distortions in the image reconstruction, providing oscillations near the leap points of the function which affect the whole image as well. This phenomenon has already been treated in literature, for example in [6], [17].

We faced the problem in [9] first using Fourier spectral filters [12] and then introducing the adaptive spectral filtering process, showing its efficiency in some MPI applications.

Here we discuss the reconstruction algorithm presented in [9], describing a possible implementation in Matlab using a modified version of the *Chebfun* 5.3.0 package [21], as we will describe.
Given a discontinuous and piecewise regular function, the reconstruction algorithm consists of the followings steps.

- Obtain a first reconstruction by interpolating the function on the Lissajous nodes.
- Apply the first spectral filtering process.
- Use an edge-detector on the filtered reconstruction in order to find the edges and the distances required for the adaptivity.
- Apply the final adaptive filtering procedure on the first reconstruction.

In the next sections we present the procedure in the two-dimensional case, showing the three-dimensional in the last section.

II. Lissajous approximation

Let $Q_2 = [-1,1]^2$, $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ a vector of relatively prime integers and $\epsilon \in \{1,2\}$, we consider the two-dimensional Lissajous curve $\gamma_\epsilon^{\boldsymbol{n}} : [0, 2\pi] \to Q_2$ defined as

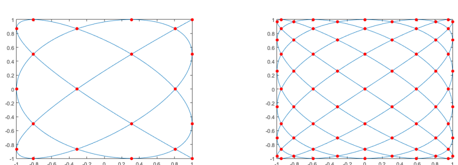
$$\gamma_\epsilon^{\boldsymbol{n}}(t) := (\cos(n_2 t), \cos(n_1 t - (\epsilon - 1)\pi/(2n_2))) \,.$$

The curve $\gamma_\epsilon^{\boldsymbol{n}}$ is called *degenerate* if $\epsilon = 1$, and *non-degenerate* if $\epsilon = 2$.
The set of Lissajous node points associated to the curve $\gamma_\epsilon^{\boldsymbol{n}}$ is given by

$$\mathrm{LS}_\epsilon^{\boldsymbol{n}} := \left\{ \gamma_\epsilon^{\boldsymbol{n}}\left(\tfrac{\pi k}{\epsilon n_1 n_2}\right) : \qquad k = 0, ..., 2\epsilon n_1 n_2 - 1 \right\}.$$

We further introduce the following index set associated to the Lissajous nodes

$$\Gamma^{\epsilon \boldsymbol{n}} := \left\{ (i,j) \in \mathbb{N}_0^2 \; : \; \frac{i}{\epsilon n_1} + \frac{j}{\epsilon n_2} < 1 \right\} \cup \{(0, \epsilon n_2)\}.$$



(a) $\gamma_1^{(5,6)}$ (b) $\gamma_2^{(5,6)}$

As described in [11], defining the polynomial space $\Pi^{\epsilon \boldsymbol{n}} := \mathrm{span}\{T_i(x)T_j(y) \; : \; (i,j) \in \Gamma^{\epsilon \boldsymbol{n}}\}$, where T_i is the *i-th Chebyshev polynomial of the first kind*, we express the unique polynomial interpolant of a given function f on $\mathrm{LS}_\epsilon^{\boldsymbol{n}}$ in $\Pi^{\epsilon \boldsymbol{n}}$ as

$$\mathcal{L}^{\boldsymbol{n}} f(x,y) = \sum_{(i,j) \in \Gamma^{\epsilon \boldsymbol{n}}} c_{ij} T_i(x) T_j(y) \,. \qquad (1)$$

This formula is equivalent to a truncated Fourier-Chebyshev series of the function f. In order to perform the spectral filtering process to the interpolant we need to extract the coefficients c_{ij}.
In Matlab, we can efficiently perform this process by using the functions `chebfun2.m` and `chebcoeffs2.m` of the *Chebfun* 5.3.0 package [21]. Actually, we slightly modified the function `chebfun2.m` since it already works also for the *Padua points*, that are Lissajous nodes of the degenerate curve with $\boldsymbol{n} = (n, n+1)$.
More precisely, given the parameters $\boldsymbol{n}, \epsilon$ using the function `lissapts.m` we get the coordinates of the Lissajous nodes $\mathrm{LS}_\epsilon^{\boldsymbol{n}}$ and the related cubature weights necessary for computing the coefficients c_{ij} in the formula (1) (for more details see [4], [11]). The function `chebfun2.m` gives the interpolant $\mathcal{L}^{\boldsymbol{n}} f$ of the function f and through the function `chebcoeffs2.m` we extract the matrix $C = (c_{ij})$ of the coefficients of the interpolant $\mathcal{L}^{\boldsymbol{n}} f$.
In Matlab this is equivalent to run the following lines:

```
x=lissapts(n,e);
fx=f(x(:,1),x(:,2));
ff=chebfun2(fx,Q_2,'lissa',pars);
C=chebcoeffs2(ff);
```

where $Q\_2 = [-1,1]^2$, f is the function to be reconstructed, $\boldsymbol{n} = [n1, n2]$, e is the ϵ parameter of the curve and $pars = [\boldsymbol{n}, e]$.

III. Fourier spectral filters

As observed above, the interpolant (1) can be seen as a Fourier series in the Chebyshev basis. Moreover, since f is a discontinuous function the reconstruction is affected by the Gibbs phenomenon. In the previous section we showed that the coefficients of the series can be easily obtained numerically by Chebfun, hence we can work on them applying spectral filters aimed to reduce the

oscillations due to the Gibbs phenomenon.
A real and even function σ is called a *spectral filter of order* p if:

1) $\sigma(0) = 1$, $\sigma^{(l)}(0) = 0$ for $1 \le l \le p-1$.
2) $\sigma(\eta) = 0$ for $|\eta| \ge 1$.
3) $\sigma(\eta) \in C^{p-1}$, $\eta \in (-\infty, \infty)$.

First consider the one-dimensional case. Given the Fourier series

$$\mathcal{S}_N f(x) = \sum_{k=-N}^{k=N} c_k(f) e_k(x) \ ,$$

letting

$$\sigma_k := \sigma\left(\frac{k}{N}\right) \ , \ -N \le k \le N \ ,$$

the filtered Fourier series is then

$$\mathcal{S}_N^\sigma f(x) = \sum_{k=-\infty}^{\infty} \sigma_k c_k(f) e_k(x) \ .$$

We can extend this procedure to the ν-dimensional case through a tensor product extension, obtaining

$$\mathcal{S}_N^\sigma f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^\nu} \boldsymbol{\sigma_k} c_{\boldsymbol{k}}(f) e_{\boldsymbol{k}}(\boldsymbol{x}) \ , \qquad (2)$$

where

$$\boldsymbol{\sigma_k} := \sigma\left(\frac{k_1}{N}\right) \cdot ... \cdot \sigma\left(\frac{k_\nu}{N}\right) \ .$$

For more details see [9] and for some examples of filters see [12]. We stress that our applications are for $\nu \le 3$ and the number of coefficients is finite and depends on $\Gamma^{\epsilon \boldsymbol{n}}$.
Going back to Matlab and in the two-dimensional setting, we rearrange the sum in (2) in a different fashion. Precisely, chosen a filter function σ, the filtering process consists of a pointwise vector-matrix multiplication between the filter matrix

$$\Sigma = (s_{ij}) := \sigma\left(\frac{i-1}{N_1}\right) \sigma\left(\frac{j-1}{N_2}\right) \qquad (3)$$

where $1 \le i \le N_1 + 1$, $1 \le j \le N_2 + 1$, and the matrix C of the coefficients c_{ij} of the interpolant (1). The parameters N_1, N_2 are properly chosen so that Σ has the same dimension of C.
After that, the function `chebfun2.m` allows us to recover the filtered approximating polynomial from the new coefficients. Specifically:

```
C=chebcoeffs2(ff);
i=[0,..,N_1];
j=[0,..,N_2];
S=sigma(i/N_1)'*sigma(j/N_2);
C_f=C.*S;
ff_new=chebfun2(C_f,Q_2,'coeffs');
```

IV. Adaptive filtering

As shown in [15], Fourier spectral filters reduce the distortions caused by the Gibbs phenomenon, but they also provide a general smoothing in the image, causing in particular a loss of focalisation near the discontinuities.
A possible solution is to consider a filter function gifted with more adaptivity. Slightly modifying the filter vector in

$$\sigma_k = \sigma\left(\frac{|k|}{N}\right) \quad , \quad -N \le k \le N \ .$$

we consider then the following filter function

$$\sigma^p(x) = \begin{cases} \exp\left(\dfrac{x^p}{x^2-1}\right) & |x| < 1 \ , \\ 0 & |x| \ge 1 \ , \end{cases}$$

observing that we are allowed to let $p \in \mathbb{R}$, $p > 0$. This is a fundamental step for our discussion, since the parameter $p = p(x, N)$ is the key for the adaptivity.
First, we can construct a two-dimensional adaptive filter

$$\boldsymbol{\sigma_k^p} = \sigma_{k_1}^{p_1} \sigma_{k_2}^{p_2} \ .$$

Then, let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be the nearest point of discontinuity with respect to $\boldsymbol{x} = (x_1, x_2)$ in the euclidean norm. For $i = 1, 2$, we call $d_i(x_i) = |x_i - \xi_i|$. We state the following result (whose proof is in [9]).

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a piecewise analytic function. Then, defining*

$$\boldsymbol{p} = (p_1, p_2) = ((N\eta_1^* d_1(x_1))^{1/2}, (N\eta_2^* d_2(x_2))^{1/2}) \ ,$$

where η_1^*, η_2^* are suitable parameters, we get the asymptotic exponential decay of the error $|f - \mathcal{S}_N^{\sigma^p} f|$ away from the points of discontinuity of f, where

$$\mathcal{S}_N^\sigma f = \sum_{\boldsymbol{k} \in \mathbb{Z}^\nu} \boldsymbol{\sigma_k^p} c_{\boldsymbol{k}}(f) e_{\boldsymbol{k}}(\boldsymbol{x})$$

This adaptive filtering process generates some striped distortions in the reconstruction, as observed in [15]. In order to solve this problem, we can consider a different unique adaptive parameter $p = p_1 = p_2$ which depends on the two-dimensional euclidean distance $d(\boldsymbol{x})$ from the nearest discontinuity. We conjecture what follows.

Conjecture 2. *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a function such that:*

- $\Phi(0) = 0$,
- Φ *is a regular and increasing function in* $[0, +\infty)$,
- Φ *has a saturation property, that is there exists* $\epsilon > 0$ *such that*
$$\Phi(x) \geq x$$
for $x \in [0, \epsilon]$.

Hence there exists at least one function with the previous properties, possibly dependent on the setting of the experiments, such that using the adaptive parameter

$$p = \eta N \Phi(d(\boldsymbol{x}))$$

we can improve the final result of the process in terms of resolution of Gibbs phenomenon and image reconstruction.

A possible family of functions which have the described properties and which we consider for our experiments is

$$\Phi_\beta(x) = x^\beta \ ,$$

where $0 < \beta < 1$. Therefore we can define and use a new parameter

$$p_\beta = \eta N (d(\boldsymbol{x}))^\beta \ .$$

The implementation of the adaptive filtering in Matlab is mostly similar to the one presented in the previous section. In addiction, we need first to know the position of the discontinuities in order to obtain the distances required for the adaptivity. Therefore, after a first non-adaptive filtering we use an edge-detector in the filtered image. In our applications we used the Canny edge-detector [8] which is implemented in Matlab as a default function, but one could consider other solutions.

After that, we have to repeat the filtering process described in the previous section for each possible distance, since the key for adaptivity is that the adaptive filter function changes with the point of evaluation. Due to this fact, the adaptive filtering process is computationally havier than the non-adaptive one.

We wrote a code and did some experiments and simulations with different functions, Lissajous curves of increasing grades and with some simulated data from a MPI scanner.

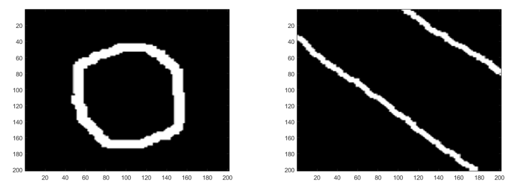


Figure 2: Original MPI phantoms

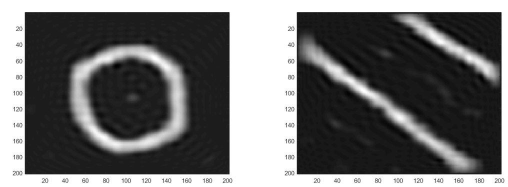


Figure 3: First reconstruction. SSIM= 0.665, SSIM= 0.616.

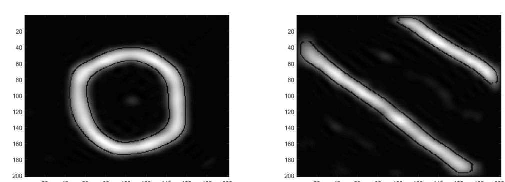


Figure 4: Final reconstruction using the adaptive filtering. SSIM= 0.701, SSIM= 0.649.

V. The three-dimensional case

Moving to the three-dimensional case, let $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{N}^3$, we consider the Lissajous curve in the cube Q_3 defined as

$$\gamma_{\boldsymbol{a}}(t) = (\cos(a_1 t), \cos(a_2 t), \cos(a_3 t)) \ ,$$

where $t \in [0, \pi]$.

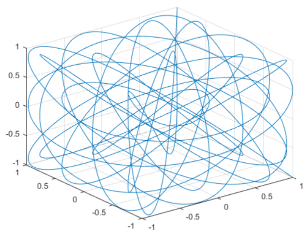


Figure 5: $t \to (\cos(30t), \cos(33t), \cos(37t))$.

As described in [2], [3], restricting to admissible triples it is possible to approximate a function $f : [-1,1]^3 \to \mathbb{R}$ through a series of orthonormal polynomials up to total-degree m,

$$\mathcal{H}_m f(\boldsymbol{x}) = \sum_{0 \le i+j+k \le m} c_{ijk} \hat{\phi}_{ijk}(\boldsymbol{x}) \ .$$

where $\hat{\phi}_{ijk}(\boldsymbol{x}) = \hat{T}_i(x_1)\hat{T}_j(x_2)\hat{T}_k(x_3)$ with $\hat{T}_l$ the normalized Chebyshev polynomial of first kind. $\mathcal{H}_m f$ is known as *hyperinterpolant of* f.

We can express then $\mathcal{H}_m f(\boldsymbol{x})$ as a Fourier series and extend the implementations of the filtering processes of the two-dimensional case to this setting. Due to computational constraints (hardware limitations), we considered a rough discretization in the code and we could not perform detailed experiments in this setting. Nevertheless, we observed that the process is still efficient and its validity is guaranteed by some considerations on the tensor product pattern (cf. [15], [20]).

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