Translation of the paper by B. Germansky

"On the systems of Fekete-points of an arc of circumference"

done by *L. Brutman and S. De Marchi*

Abstract

This paper is the complete translation in English from Hebrew of the paper *On the systems of Fekete-points of an arc of circumference* by Baruch Germansky, whose small summary in English can be found in the journal *Riveon Lematematika* **3** *(1949), 56-57*, the same issue where appeared the original paper. The translation was required by Stefano De Marchi to Lev Brutman during his stay at the University of Udine in 1997. After his death, we present this translation as a tribute and a "posteriori work" to a nice man and good mathematician as was Prof. Lev Brutman.

1 Part A

Let E be the infinite set of points bounded and closed in the z-plane. Let us take from E, n points, $n \geq 2, z_1, \ldots, z_n$ and let us form the geometric mean of the distances of the points, i.e.let $k = \frac{n(n-1)}{2}$

$$\sqrt[k]{\prod_{1 \leq \mu < \nu \leq n} |z_\mu - z_\nu|}.$$

The maximum of this geometric mean, when $z_1, \ldots, z_n$ vary arbitrarily on E is called the *diameter of the order* n of the set E. This diameter is expected at least for one system of n points from E, and the system is called *system of Fekete-nodes* (or Fsystem of order n of E).

F-systems play a fundamental role in the theory of trasfnite diameter as well as approximation theory and interpolation. F-systems were known until my previous paper only for intervals of R, for circle as well as for any set E whose boundary is the circle. In this work we shall deal with F-systems for the arc of the circle and we shall give some theorems concerning the distribution of the location of the points on the arc, as well as the number of such systems of a given order. As we shall see later, our problem, for a given arc and n sufficiently large, is part of the following general problem:

Problem. 1: `Find F-systems of (n-2)-points on every circle, such that together with two fixed points, in such a way they give relative maxima for the geometric mean on the circle.`

2 Part B

Let us first prove the following theorem.

Theorem 1 *Let A be the arc of the unit circle C in the z-plane with opening $\alpha, 0 < \alpha < 2\pi$. We shall distinguish two cases:*

$$(1) \qquad 0 < \alpha \leq \pi;$$

$$(2) \qquad \pi < \alpha < 2\pi.$$

In the case (1), every F-system of Aincludes both the end points of A. The same will be true in the case (2) for those F-systems of A for which the order n satisfies the following condition $n \geq \frac{2\pi}{2\pi-\alpha}$.
On the other hand, in the case (2) there are no F-systems of A that will include simultaneously both end points if the order satisfies

$$2 \leq n < \frac{2\pi}{2\pi - \alpha}.$$

Since in the case (1), the condition $2 \leq n < \frac{2\pi}{2pi-\alpha}$, is also satisfied and since $n \geq 2$, we get that

$$n \geq \frac{2\pi}{2\pi - \alpha}\ .$$

Therefore we may reformulate the Theorem 1 in a shorter way.

Theorem 2 *Let A be the arc of the unit circle C, with opening* $\alpha, 0 < \alpha < 2\pi$. *In the case (1)* $n \geq \frac{2\pi}{2\pi-\alpha}$ *every F-system of of order n includes both the end points of A. Otherwise, in the case* $2 \leq n < \frac{2\pi}{2\pi-\alpha}$, *no F-systems of order n can include simultaneously both the end points of A.*

Note. We shall see later, that for the case $n \geq 2$, $n < \frac{2\pi}{2\pi-\alpha}$ there are infinite numbers of F-systems of order n while in the case $\frac{2\pi}{2\pi-\alpha} \leq n$ there is only one F-system of order n.

3 Part C

Theorem 2 follows from the following theorem.

Theorem 3 *When n points* $z_1, \ldots, z_n, n \geq 2$ *vary on C in arbitrary way then the following function*

$$(3) \qquad \Delta(z_1, \ldots, z_n) = \prod_{1 \leq \mu < \nu \leq n} |z_\mu - z_\nu|,$$

assume relative maxima only for those values of the variables $z_1, \ldots, z_n$ *which correspond to the vertices of the regular polygon of order n inscribed in C. Since all these relative maxima are the same, therefore the vertices of this regular polygon are the F-system of order n of the circle C.*

Proof. The short proof of this theorem is due to M. Fekete. For $n \geq 2$, and $z_1, \ldots, z_n$ arbitrary points on C. Denoting in the usual way their conjugates as $\bar{z}_1, \ldots, \bar{z}_n$, we can write

$$(4) \qquad \Delta^2(z_1, \ldots, z_n) = \prod_{1 \leq \mu < \nu \leq n} (z_\mu - z_\nu)(\bar{z}_\mu - \bar{z}_\nu)$$

$$= \prod_{1\le\mu<\nu\le n} (z_\mu - z_\nu)(\frac{1}{z_\mu} - \frac{1}{z_\nu}) =$$

$$= \frac{(-1)^{\frac{n}{2}}}{(z_1 z_2 \dots z_n)^{n-1}} \prod_{1\le\mu<\nu\le n} (z_\mu - z_\nu)^2.$$

Let us denote the **last** rational depending on $z_1, \dots, z_n$ by $F(z_1, \dots, z_n)$, i.e.

$$F(z_1, \dots, z_n) = \frac{(-1)^{\frac{n}{2}}}{(z_1 z_2 \dots z_n)^{n-1}} \prod_{1\le\mu<\nu\le n} (z_\mu - z_\nu)^2. \tag{5}$$

This function is denoted in all the z-plane of $z_1, \dots, z_n$ excluding the points zero ∞, and it coincides according to the above calculations, for

$$z_1 = e^{i\phi_1}, \dots, z_n = e^{i\phi_n}, 0 \le \phi_\nu \le 2\pi, i = 1, 2, \dots, n, \tag{6}$$

with $\Delta^2(z_1, \dots, z_n)$, since for those values of $z_1, \dots, z_n$ the functions $\Delta(z_1, \dots, z_n)$, $\Delta^2(z_1, \dots, z_n)$ and $F(z_1, \dots, z_n)$ assume simultaneously their maximum. Therefore, the vanishing of

$$\frac{\partial F(e^{i\phi_1}, \dots, e^{i\phi_n})}{\partial\phi_1}, \dots, \frac{\partial F(e^{i\phi_1}, \dots, e^{i\phi_n})}{\partial\phi_n}$$

are the necessary conditions for finding the relative maximum of $\Delta(z_1, \dots, z_n)$ on C.

However, for all $1 \le \nu \le n$, we have

$$\frac{\partial F(e^{i\phi_1}, \dots, e^{i\phi_n})}{\partial\phi_\nu} = \frac{\partial F(e^{i\phi_1}, \dots, e^{i\phi_n})}{\partial z\nu}\frac{\partial z_\nu}{\partial\phi_\nu} = \frac{\partial F(z_1, \dots, z_n)}{\partial z\nu} i z_\nu.$$

Therefore, since for $1 \le \nu \le n$, $z_\nu \neq 0$, the conditions for relative maxima are

$$\frac{\partial F(z_1, \dots, z_n)}{\partial z\nu} = 0, \nu = 1, \dots, n. \tag{7}$$

Let us denote the values $z_1, \dots, z_n$, which give the relative maximum to Δ by $\varsigma_1, \dots, \varsigma_n$. Then, a simple computation shows that conditions (7) are equivalent, if we take in account (5), to the following ones:

$$\sum_{\mu=1, mu\neq\nu} \frac{-2}{\varsigma_\mu - \varsigma_\nu} = \frac{n-1}{\varsigma_\nu}, \nu = 1, \dots, n. \tag{8}$$

Let us write

$$p(z) = (z - \varsigma_1)\dots(z - \varsigma_n) = z^n + c_1 z^{n-1} + \dots + c_n, \tag{9}$$

then, the equation (8) can be represented as follows

$$\frac{p''(\varsigma_\nu)}{p'(\varsigma_\nu)} = \frac{n-1}{\varsigma_\nu}, \nu = 1, \dots, n. \tag{10}$$

From this, it follows that the polynomial

$$(n-1)p'(z) - zp''(z),$$

which is in order $(n-1)$ at most, has n distinct zeros, and its zeros $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ are so that the function $\Delta(z_1, \ldots, z_n)$ attains its relative maximum value on C.

From this, it follows that

$$(n-1)p'(z) - zp''(z) = 0, \tag{11}$$

is equivalent to

$$\frac{d}{dz}(\frac{p'(z)}{z^{n-1}}) = 0. \tag{12}$$

Therefore

$$p'(z) = const\dot{z}^{n-1}. \tag{13}$$

On the other hand, according to (9), since $p'(z) = nz^{n-1} + (n-1)c_1 z^{n-2} + \ldots + c_{n-1}$ it follows

$$c_1 = c_2 = \ldots = c_{n-1} = 0, \tag{14}$$

that is

$$p(z) = z^n + c_n. \tag{15}$$

However, in accordance to (9),

$$c_n = (-1)^n \varsigma_1 \varsigma_2 \ldots \varsigma_n,$$

therefore, taking into account that $|\varsigma_\nu| = 1, \nu = 1, \ldots, n$ we get

$$|c_n| = 1. \tag{16}$$

From this fact, Theorem 3 follows. □

4 Part D

We shall prove now, with the help of Theorem 3, Theorem 1.

Proof. of Theorem 1.

• Let us start with case 1. When $0 < \alpha \leq \pi$, some Fsystems of order $n, \varsigma_1, \ldots, \varsigma_n$ of A does not include both end points simultaneously. Then the points $\varsigma`_1, \ldots, \varsigma'_n, \varsigma'_\nu = \varsigma_\nu e^{i\epsilon}, \nu = 1, \ldots, n$ with ϵ real and small in absolute value (it simply rotates the points), then these points will be a new Fsystem which is inside A. Hence, these points give a relative maxima to $\Delta(z_1, \ldots, z_n)$ on the circle C. According to Theorem 3, the points $\varsigma`_1, \ldots, \varsigma'_n$ should be the vertices of the regular polygon with n sides, inscribed in C. But, from our hypothesis on $\alpha, 0 < \alpha \leq \pi, n \geq 2$, it follows that inside A there is no room for all vertices of the regular polygon, since the regular polygon which has the smallest number of sides, namely 2, requires that all the openings $\alpha, \alpha > \pi$, in order that all the points will be inside A.

Therefore, our assumption in not true, and the Fsystem which we are dealing with, includes in the case 1 both end points simultaneously.

• Let us turn to the case 2, $\pi < \alpha < 2|pi$.

When $n \geq \frac{2\pi}{2\pi-\alpha}$, the proof that any Fsystem of order n, of A, includes both end points of A simultaneously, is analogous to the proof of the case 1. While in the case $n < \frac{2\pi}{2\pi-\alpha}$ i.e.

$\frac{2\pi}{n} > 2\pi - \alpha$, there is room inside A for the regular polygon having $n$sides and since these vertices of the regular polygon give the maximum of $\Delta(z_1, \ldots, z_n)$ for all circles C, then they give *a fortiori* a maximum of $\Delta(z_1, \ldots, z_n)$ for the arc A. Thus, any system of n vertices of such regular polygon which is located inside A, is the Fsystem of order n of A.

Now, let us point out the fact that any system of n points of A, which is different from the system of the vertices of regular polygon, correspond to the values of $\Delta(z_1, \ldots, z_n)$ which is smaller than the maximum (which was mentioned above) as well as to the fact that this regular polygon can include at most one endpoint of A. Thus, we conclude that no such Fsystem of A which in our case includes both endpoints simultaneously.

This concludes the proof of Theorem 1. □

We also obtained the result that, in the case we have dealt with, there is an innite number of Fsystem of order n of A.

5 Part E

In order to study more carefully the location of the points $\varsigma_1, \ldots, \varsigma_n$ of the Fsystem of order $n, n \geq 3$, of A in the case when 2 points of the system coincide with the endpoints u and v of A (Theorem 1, gives the necessary and sufficient conditions for this). Up to determining these points as a function of n, u, v, we shall consider the following general maximum problem, for which the Fsystem, in the case we mentioned before, is one of its solutions.

Problem. **2** Find n points $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ on the unit circle $C, C = A + \bar{A}$ such that two of them, say ς_1, ς_n coincide correspondingly with u and v, and the remaining $(n-2)$ points $\varsigma_2, \ldots, \varsigma_{n-1}$ give to $\Delta(u, z_2, \ldots, z_{n-1}, v)$ relatively maximum when $z_2, \ldots, z_{n-1}$ vary arbitrary on C. Part of them on A and rest on $\bar{A}$.

We shall solve the Problem 2 by giving a common characterization of maximal systems $\varsigma_1, \ldots, \varsigma_n$, which give the solution of the maximum problem above, by finding the differential equation for this polynomial

$$p(z) = \prod_{\nu=1}^{n} (z - \varsigma_\nu) = z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n, n \geq 3 \tag{17}$$

and $\varsigma_1 = u, \varsigma_n = v$.

Since $\varsigma_2, \ldots, \varsigma_{n-1}$ give to $\Delta(u, z_2, \ldots, z_{n-1}, v)$ a relative maximum when $z_2, \ldots, z_{n-1}$ vary on C, as before, by means of the following identity

$$\Delta^2(u, z_2, \ldots, z_{n-1}, v) = F(u, z_2, \ldots, z_{n-1}, v)$$

when the absolute value of $u, z_2, \ldots, z_{n-1}, v$ are equal to 1 (compare with Part C), it follows necessarily

$$\frac{\partial F(u, z_2, \ldots, z_{n-1}, v)}{\partial z_\nu} = 0, \nu = 0, \ldots, n-1$$

when $z_\nu = \varsigma_\nu, \nu = 2, \ldots, n-1$.

Therefore, compare with Part C, for $p(z)$ the following equation is true:

$$(n-1)p'(\varsigma_\nu) - \varsigma_\nu p''(\varsigma_\nu) = 0 \nu = 2, \ldots, n-1.$$

In other words, the points $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ are the zero points of the polynomial

$$(n-1)p'(z) - \varsigma_\nu p''(z) = \sum_{\mu=1}^{n-1} \mu(n-\mu)c_\mu z^{n-1-\mu}, \tag{18}$$

of degree $n-2$ in z. On the other hand, they are the zero points of the polynomial

$$\frac{p(z)}{(z-u)(z-v)} = z^{n-2} + \dots.$$

Therefore, $p(z)$ satisfies the differential equation

$$(n-1)p'(z) - \varsigma_\nu p''(z) = \lambda \frac{p(z)}{(z-u)(z-v)}, \tag{19}$$

where λ is constant corresponding to the maximal system from which we started. In addition to this $p(z)$ satisfies the boundary conditions

$$p(u) = 0, p(v) = 0. \tag{20}$$

6 Part F

Now, we shall prove that conversely the value λ determines the polynomial $p(z)$ or alternately the maximum system $\varsigma_1, \dots, \varsigma_n$ when (17), (19) and (20) are satisfied. Indeed, in (19) we compared the coefficients by the powers $z^{-1}, z^0, \dots, z^{n-2}$ of both sides of this equation. Taking into account (18) and the following expansion of the right hand side by decreasing powers of z

$$\lambda \frac{p(z)}{(z-u)(z-v)} = \lambda p(z) \sum_{\rho=0}^{\infty} \frac{\omega_\rho}{z^{\rho-2}} = \tag{21}$$

$$= \lambda z^{n-2} + \lambda(c_1 + \omega_1) z^{n-3} + \lambda(c_2 + \omega_2) z^{n-4} + \dots$$

$$\dots + \lambda(c_{\sigma-1} + \omega_1 c\sigma - 2 + \dots + \omega_{\sigma-2} c_1 + \omega_{\sigma-1}) z^{n-\sigma-1} + \dots$$

$$\dots + \lambda(c_{n-2} + \omega_1 cn - 3 + \dots + \omega_{n-3} c_1 + \omega_{n-2}) +$$

$$+ \lambda(c_{n-1} + \omega_1 cn - 2 + \dots + \omega_{n-2} c_1 + \omega_{n-1}) + \dots$$

where

$$\omega_0 = 1, \omega_1 = u + v, \omega_2 = u^2 + uv + v^2, \dots \tag{22}$$

$$\omega_{\sigma-1} = u^{\sigma-1} + u^{\sigma-2} v + \dots + uv^{\sigma-2} + v^{\sigma-1}, \dots$$

$$\omega_{n-1} = u^{n-1} + u^{n-2} v + \dots + uv^{n-2} + v^{n-1}.$$

Then, we obtain the following representation

$$1(n-1)c_1 = \lambda \tag{23}$$

$$2(n-2)c_2 = \lambda(c_1 + \omega_1)$$
$$3(n-3)c_3 = \lambda(c_2 + \omega_1 c_1 + \omega_2)$$
$$\sigma(n-\sigma)c_\sigma = \lambda(c_{\sigma-1} + \omega_1 c_{\sigma-2} + \ldots + \omega_{\sigma-2}c_1 + \omega_{\sigma-1})$$
$$(n-1)c_{n-1} = \lambda(c_{n-2} + \omega_1 c_{n-3} + \ldots + \omega_{n-3}c_1 + \omega_{n-2})$$

$$0 = \lambda(c_{n-1} + \omega_1 c_{n-2} + \ldots + \omega n - 2c_1 + \omega_{n-1}). \tag{24}$$

From (23) we can calculate the values $c_1, c_2, \ldots, c_{n-1}$ as polynomials $q_1(\lambda), q_2(\lambda), \ldots, q_{n-1}(\lambda)$ of exact degree $1, 2, \ldots, n-1$ with the coefficients that are the polynomials in $\omega_1, \omega_2, \ldots, \omega_{n-2}$ whose coefficients are again rational positive numbers depending on n. These polynomials, $q_\sigma(\lambda), \sigma = 1, \ldots, n-1$, have a factor λ at least. That is,

$$c_\sigma = q_\sigma(\lambda) = \frac{\lambda^\sigma}{\sigma!(n-\sigma)(n-\sigma+1)\ldots(n-1)} + \ldots = \lambda r_\sigma(\lambda), 1 \leq \sigma \leq n-1. \tag{25}$$

If we substitute instead of $c_\sigma, \sigma = 1, \ldots, n-1$, the polynomials $q_\sigma(\lambda)$ in equation (24), we get that λ satisfies necessarily the following algebraic equation:

$$\lambda(q_{n-1}(\lambda) + \omega_1 q_{n-2}(\lambda) + \ldots + \omega_{n-2}q_1(\lambda) + \omega_{n-1}) = 0, \tag{26}$$

that is of exact degree n.

We claim that λ satisfies also the following equation:

$$q_{n-1}(\lambda) + \omega_1 q_{n-2}(\lambda) + \ldots + \omega_{n-2}q_1(\lambda) + \omega_{n-1} = 0. \tag{27}$$

This fact is trivial if $\lambda \neq 0$. Otherwise, if $\lambda = 0$ we see from (18) and (19) that necessarily

$$c_1 = c_2 = \ldots = c_{n-1} = 0. \tag{28}$$

Therefore,

$$p(z) = z^n + c_n. \tag{29}$$

On the other hand, from (29), together with (20), we get

$$u^n - v^n = p(u) - p(v) = 0.$$

Thus, in accordance with (22), we have

$$\omega_{n-1} = u^{n-1}v + \ldots + uv^{n-1} = \frac{u^n - v^n}{u - v} = 0,$$

and from this, taking into account (25) we found that $\lambda = 0$ is the root of (27). Equation (25) shows that $c_1, \ldots, c_{n-1}$ are determined uniquely by λ.

In order to prove our claim in a simple way, we have also to calculate c_n as a function of λ. To this end, we put 0 in both sides of (19) instead of z. We shall obtain, by using (18), the equation

$$(n-1)c_{n-1} = \lambda\frac{c_n}{uv}$$

In accordance with (25) it gives

$$\lambda c_n = \lambda uv(n-1)r_{n-1}(\lambda). \tag{30}$$

Then, the required expression for c_n follows:

$$c_n = (n-1)uvr_{n-1}(\lambda), \tag{31}$$

under the condition $\lambda \neq 0$. But this is also correct when $\lambda = 0$. In fact, as we have seen before, for $\lambda = 0$ by using (29) and (20) we get

$$c_n = -u^n = -v^n. \tag{32}$$

On the other hand, the polynomials $q_\sigma(\lambda), \sigma = 1, \ldots, n-1$ are related by the following identity:

$$(n-1)q_{n-1}(\lambda) = \lambda(q_{n-2}(\lambda) + \omega_1 q_{n-3}(\lambda) + \ldots + \omega_{n-3}q_1(\lambda) + \omega_{n-2}).$$

From this, in view of (25) the following identity follows:

$$(n-1)r_{n-1}(\lambda) = q_{n-2}(\lambda) + \omega_1 q_{n-3}(\lambda) + \ldots + \omega_{n-3}q_1(\lambda) + \omega_{n-2}. \tag{33}$$

If we put in (33) the value 0 instead of λ, we shall have, in view of (25)

$$(n-1)r_{n-1}(0) = \omega_{n-2}. \tag{34}$$

However, by (22) and (32)

$$\omega_{n-2} = \frac{u^{n-1} - v^{n-1}}{u-v} = \frac{-v^n v^{-1} - (-u^n)u^{-1}}{u-v} = \frac{c_n v^{-1} - c_n u^{n-1}}{u-v} = \frac{c_n}{uv}.$$

This, together with (34) shows that (31) is satisfied also for $\lambda = 0$. Notice, that since

$$uv = \omega_1^2 - \omega_2,$$

the representation (31) can also be written in the form

$$c_n = (n-1)(\omega_1^2 - \omega_2)r_{n-1}(\lambda).$$

Namely, c_n can be represented as a polynomial in λ with the coefficients which are polynomials in $\omega_1, \ldots, \omega_{n-2}$ with the coefficients that are rational positive numbers depending on n.

7 Part G

Now we shall take into account that the algebraic equation (27) has no more than $n-1$ distinct roots and therefore there are no more than $n-1$ maximal systems $\varsigma_1, \ldots, \varsigma_n$ of the type we are dealing with.

On the other hand, it is obvious that there is one maximal system from the following $n-2$ types.

$M_0 : \varsigma_1^{(0)}, \varsigma_2^{(0)}, \ldots, \varsigma_n^{(0)}$,

with $u = \varsigma_1^{(0)}, v = \varsigma_n^{(0)}$ where no one of the
$n-2$ points $\varsigma_2^{(0)}, \dots, \varsigma_{n-1}^{(0)}$ does not belong
to $\bar{A}$ (that is they belong to A).

$M_1 : \varsigma_1^{(1)}, \varsigma_2^{(1)}, \dots, \varsigma_n^{(1)}$,
with $u = \varsigma_1^{(1)}, v = \varsigma_n^{(1)}$ where one point of the
$n-2$ points $\varsigma_2^{(1)}, \dots, \varsigma_{n-1}^{(1)}$ belongs
to $\bar{A}$ and the rest to A.
and so on. At the step k

$M_k : \varsigma_1^{(k)}, \varsigma_2^{(k)}, \dots, \varsigma_n^{(k)}$,
with $u = \varsigma_1^{(k)}, v = \varsigma_n^{(k)}$ where k points of the
$n-2$ points $\varsigma_2^{(k)}, \dots, \varsigma_{n-1}^{(k)}$ belong
to $\bar{A}$ and the rest $n-k-2$ to A.
At the step $n-2$

$M_{n-2} : \varsigma_1^{(n-2)}, \varsigma_2^{(n-2)}, \dots, \varsigma_n^{(n-2)}$,
with $u = \varsigma_1^{(n-2)}, v = \varsigma_n^{(n-2)}$ where all the
$n-2$ points $\varsigma_2^{(n-2)}, \dots, \varsigma_{n-1}^{(n-2)}$ belong
to $\bar{A}$.

Notice that these maximal systems are determined from the fact that $\Delta(u, z_2, \dots, z_{n-1}, v)$ is a continuous bounded function of the variables $z_2, \dots, z_{n-1}$ when $z_2, \dots, z_{n-1}$ vary on C under the constraint that k of them are located on $\bar{A}$ and the rest $n-k-2$ on A.

From this, it follows that there are uniquely determined maximal systems $M_k, \quad k = 0, 1, \dots, n-2$. Moreover, notice that if the opening α of A, satisfies the condition

$$\frac{2\pi}{2\pi - \alpha} \leq n$$

then, by Theorem 1, every F-system of order n of A is necessarily a maximal system of type M_0 of C.

Then, the following theorem is true.

Theorem 4 *Let n be any natural number, $n \geq 2$, and let A be the arc of the unit circle C with opening α which satisfies $\frac{2\pi}{2\pi-\alpha} \leq n (or (\frac{n-1}{n})2\pi \geq \alpha > 0)$ then there is a unique F-system of order n belonging to A.*

8 Part H

From the uniqueness of the maximal systems $M_k, k = 0, 1, \dots, n-2$, it follows the fact that if the arc A is symmetric with respect to the real axis of the $z$plane, namely $v = \bar{u}$, then also the points of the maximal systems are symmetric, too. Therefore also the coefficients $c_\nu^{(k)}$ of the polynomials

$$(35) \qquad p_k(z) = \prod_{\nu=1}^{n} (z - \varsigma_\nu^{(k)}) = z^n + c_1^{(k)} z^{n-1} + \dots + c_{n-1}^{(k)} z + c_n^{(k)},$$

are also real numbers satisfying the conditions

$(36) c_\nu^{(k)} = (-1)^{n-k} c_{n-\nu}^{(k)}, \nu = 1, 2, \dots, n-1. \qquad c_n^{(k)} = (-1)^{n-k}$

And since, in the case $n \geq \frac{2\pi}{2\pi-\alpha}$ the maximal system M_0 is an F-system of order n belonging to A, the condition $v = \bar{u}$ implies that in the case the F-system of order n of A is also symmetric with respect to real axis of the $z$plane, then the coefficients of this polynomial $c_1^{(0)}, \ldots, c_n^{(0)}$ are real numbers satisfying

$$(37) c_\nu^{(0)} = (-1)^n c_{n-\nu}^{(0)}, \nu = 1, 2, \ldots, n-1. \qquad c_n^{(0)} = (-1)^n$$

This condition guarantees obviously that $\omega_1, \ldots, \omega_{n-1}$ are real numbers and therefore by equation (27) the roots of them, $\lambda_0, \ldots, \lambda_{n-2}$ determine the algebraic equation with real coefficients, the roots of which, are also real numbers (distinct!) represented as follows:

$$\lambda_k = (n-1)c_1^{(k)} \tag{38}$$

as seen from the first representation on the system (23).

It is worth to note also that if

$$u = e^{i\phi}, v = e^{-i\phi}, \alpha = 2\phi$$

then the numbers $\omega_1, \ldots, \omega_{n-2}$ are expressed in the following simple form

$$\omega_\nu = \frac{\sin((\nu+1)\phi)}{\sin(\phi)}, \quad \nu = 1, \ldots, n-2. \tag{39}$$

9 Part I

In these two last sections, we shall present some remarks which are essentially due to Fekete.

In the symmetric case, $\bar{u} = v$, we saw that instead of solving (27) we can solve two other algebraic equations of smaller order whose coefficients are polynomials of the variables $\omega_1, \ldots, \omega_{n-2}$. Indeed, from the representation (37) it follows that for even n and odd $k, \nu = \frac{n}{2}$,

$$2c_\nu^{(k)} = 0,$$

which by (25) is equivalent to

$$r_{\frac{n}{2}}(\lambda_k) = 0. \tag{40}$$

While, in the case k even, the same representation (37) gives for $\nu = \frac{n}{2} - 1$

$$c_{\frac{n}{2}-1}^{(k)} = c_{\frac{n}{2}+1}^{(k)},$$

or

$$r_{\frac{n}{2}-1}(\lambda_k) = r_{\frac{n+1}{2}}(\lambda_k). \tag{41}$$

In the same way we obtained, under the condition that n is **odd** and k is **odd** (k is **even**)

$$r_{\frac{n-1}{2}}(\lambda_k) \pm r_{\frac{n}{2}+1}(\lambda_k) = 0. \tag{42}$$

Therefore the calculation of λ_k, which determine the system M_k may be performed by solving the pairs of algebraic equation of degree $[\frac{n}{2}]$ at most in all cases. In other words, the polynomial which is the left hand side of (27) is decomposed in all the cases into 2 polynomial factors.

10 Part L

However the very interesting question is the following one:

Which root among the $n-1$ roots of (27) belongs to the system M_0? Namely, in view of the conditions which we know for F-systems of A, which do belong to M_1? Which to M_2, and so on?

Moreover, with no calculations, just with the help oh these roots at all the $n-1$ polynomials $p(z)$ and finding the zero points?

The answer to this question is the following.

It is enough to order the roots of (27) in the case the arc A is located to the right to the arc $\bar{A}$ in increasing order, then the first one will belong to M_0, the second to M_1, and so on. The largest root,the last in the sequence, will belong to M_{n-2}.

This fact, was discovered by M. Fekete already in 1938 and the proof is based on the continuous change of the systems M_k as we change ϕ (or α) and the uniqueness of these systems for all specific $\phi, 0 < \phi < \pi$. These results will be published very soon.