The Polynomial Approximation in the Finite Element Method *

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Abstract

In the analysis of a Finite Element Method (FEM) we can describe the shape of a given element by a set of elementary functions known as *shape functions*. The approaches describing these functions are quite different ones. In the plane (x_1, x_2), these functions are a product of lagrangian polynomials when the coordinate system can be chosen with the axes parallel to the sides of the element, otherwise a system of *barycentric coordinates* (sometimes called *area coordinates*) could be introduced.
The aim of this paper is the description and the representation of shape functions when the *element has triangular shape* (the simplest). The representation has been done by using two algorithmic schemes: Neville-Aitken and De Casteljau. For these schemes we have deduced very important properties.

1 Introduction

We shall use the standard multiindex notation: $\mathbb{N}^k$ stands for the set of multiindeces, a real algebraic polynomial $q(x)$ in k variables of degree $\leq p$ will be denoted as a finite sum

$$q(x) = \sum_{|\mathbf{r}| \leq p} \lambda_{\mathbf{r}} x^{\mathbf{r}}$$

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where as usual, $\mathbf{r} \in I\!N^k$, $|\mathbf{r}| = \sum_{i=1}^{k} r_i$ and $x^{\mathbf{r}} = \prod_{i=1}^{k} x_i^{r_i}$. We shall use Π_p as the space of all these polynomials. The dimension of Π_p is $\begin{pmatrix} k+p \\ p \end{pmatrix}$.

Following [7], a *Finite Element* in $I\!R^n$ is a triplet (T, P, Σ) where

(i) T is a closed polyhedron in $I\!R^n$;

(ii) $P \subset \mathcal{C}^s(T) \quad s \in I\!N$ is a finite space of real value functions defined over the set T (we let $N = dim\ P$);

(iii) Σ is a set of linear forms $\phi_i,\ 1 \leq i \leq N$, linearly independent, defined over the space P and by definition it is P-unisolvent. In particular there exist N functions $p_i \in P,\ 1 \leq i \leq N$ that satisfy

$$\phi_j(p_i) = \delta_{ij},\ 1 \leq j \leq N. \tag{1.0.1}$$

In particular the following identity holds

$$p = \sum_{i=1}^{N} \phi_i(p) p_i \quad \forall\, p \,\in\, P.$$

Functions ϕ_i are known as *degrees of freedom of the finite element* T and functions p_i are known as *basis functions of the finite element* T. In engineering literature basis functions are called *shape functions.*
The main characteristic of the spaces P is that they all contain a "full" polynomial space $\Pi_k(P)$ for some $k \geq 1$. This paper is concerned with the case in which $P = \Pi_k(P)$ allowing to consider shape functions of polynomial type.

Given a triangle (2-simplex finite element) $\mathbf{T}$ with vertices $\{T_1, T_2, T_3\}$, that we shall call *original triangle*, there are several techniques that allow us to build a subdivision of it into smaller triangles, that we shall call *derived triangles* (see [1, 8]).
The *order* of the triangle $\mathbf{T}$ is an integer $p > 0$. Given p, we can identify on $\mathbf{T}$ a set of *node points* whose number is $m = \frac{(p+1)(p+2)}{2}$. Using these nodes, we can describe the interpolating polynomial over $\mathbf{T}$ by a set of *shape functions.* In the plane (x, y) these generally are product of lagrangian polynomials $p(x)$ and $p(y)$ (see [12]). When the shape is triangular a better choice is the use of *area coordinates*, $\mathbf{a}_P = (\alpha_1, \alpha_2, \alpha_3)$ for the generic point P $\in \mathbf{T}$, where $\alpha_i = \alpha_i(x, y)$ and $\sum_{i=1}^{3} \alpha_i = 1$.
With the use of barycentric coordinates the i-th polynomial shape function $N_i^{(k)}$ (where k stands

for the degree of the polynomial) can be written as

$$N_i^{(k)}(\mathbf{a}) = \sum_{|\mathbf{r}|\leq k} \lambda_{\mathbf{r}} \mathbf{a}^{\mathbf{r}} \tag{1.0.2}$$

where $\mathbf{r} \in I\!N^3$ and $i = 1, \ldots, m$.

We shall give an example how to build shape functions in section 2.

After a brief survey of barycentric coordinates and shape functions, we shall introduce Neville-Aitken and De Casteljau algorithms for the computation of shape functions. Extensions to some formulae for determinants will be given, as well. Appendix A is a collection of MATLAB routines and functions used to test the algorithms presented.

2 Generality: Area Coordinates and Shape Functions

2.1 Area Coordinates

Let us consider the triangle in Fig.1. While cartesian coordinates are suitable for a rectangle, because they can be taken parallel to the sides, they are not convenient for a triangle. A better choice is represented by *area coordinates* (sometimes called *barycentric coordinates*) that represent a local coordinate system (see [12]).

Let $\mathbf{T}$ be a triangle of vertices $\{T_1, T_2, T_3\}$ and let $(x_i, y_i), i = 1, 2, 3$ be the coordinates of T_i in $(x, y) - plane$.

The following linear system represents the relation between area and cartesian coordinates

$$\alpha_i = \frac{a_i + b_i x + c_i y}{2\Delta} \quad i = 1, 2, 3 \tag{2.1.1}$$

or in matrix form

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{2.1.2}$$

where $\Delta = area\ \mathbf{T}$ and a_i, b_i and c_i are given by the following expressions:

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

where the triple $(i, j, k) \in S_3^{(e)}$ ($S_3^{(e)}$ is the 3-symmetric group of even permutations).

An algorithm for the computation of the area coordinates corresponding to a triangle of given

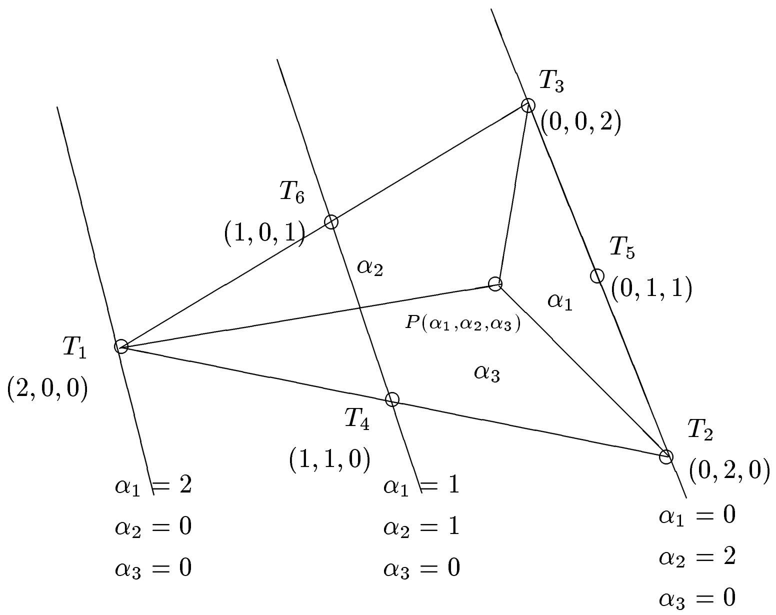

Figure 1: *Area coordinates in a triangle of order 2*

vertices, is the MATLAB procedure **carea1** presented in Appendix A. Relations (2.1.1) and (2.1.2) imply that contours of, let say α_1, are equally placed straight lines parallel to the side joining vertices T_2 and T_3, where $\alpha_1 = 0$ (see Fig. 1). An alternative definition of the barycentric coordinates of a point $P \in \mathbf{T}$ deducible from (2.1.2), is

$$\alpha_P^{(i)} = \frac{area\ PT_jT_k}{\Delta} \tag{2.1.3}$$

where $(i,j,k) \in S_3^{(e)}$. Hence the name of area coordinates.
Proving the following propositions comes easy:

Proposition 2.1 *Area coordinates are positive inside the domain triangle.*

Proposition 2.2 *Area Coordinates are symmetric: each side of the triangle is treated the same way as other ones.*

The use of area coordinates is thus easier and more elegant than that of cartesian coordinates which need more distinctions for each side of the original triangle.

2.2 Shape Functions

As will be stated in the next section, the i-th shape function $N_i^{(k)}$ can be written as product of lagrangian polynomials [12]. In area coordinates we have:

$$N_i^{(k)}(\mathbf{a}) = l_I^I(\alpha_1) l_J^J(\alpha_2) l_K^K(\alpha_3) \tag{2.2.1}$$

where each $l_q^q(\alpha_i)$ with $q \in \{I, J, K\}$, $i = 1, 2, 3$ is given by the usual lagrangian ratio

$$l_q^q(\alpha_i) = \prod_{j=0, j\neq q}^{n(j)} \frac{\alpha_i(x,y) - \alpha_i(x_j, y_j)}{\alpha_i(x_q, y_q) - \alpha_i(x_j, y_j)} \tag{2.2.2}$$

The triplet (I, J, K) identifies the node on $\mathbf{T}$. For example from Fig.1 the node T_1 has $(I, J, K) = (2, 0, 0)$, the node T_4 has $(1, 1, 0)$ and so on. Therfore we consider a triangle $\mathbf{T}$ of order M, only on one of its sides, we shall have the sequence $(M, 0, 0), (M-1, 1, 0), \ldots, (1, M-1, 0), (0, M, 0)$.
By definition, shape functions have *local support.* Moreover, from (1.0.1), shape functions must verify the *interpolating condition*

$$N_i^{(k)}(\alpha_1, \alpha_2, \alpha_3) = 1 \iff \alpha_j = \alpha_j(x_i, y_i) \quad j = 1, 2, 3 \tag{2.2.3}$$

The maximum term involved expanding (2.2.1) is: $\alpha_1^I \alpha_2^J \alpha_3^K$ when $I + J + K = M$.

2.3 The Neville-Aitken Algorithm

From (2.2.1) a shape function in barycentric coordinates can be written

$$N_i^{(M)}(\mathbf{a}) = \sum_{|\mathbf{r}| \leq M} \lambda_{\mathbf{r}} \sigma_{\mathbf{r}}(\mathbf{a}) \tag{2.3.1}$$

where $\sigma_{\mathbf{r}}(\mathbf{a}) = \alpha_1^{r_1} \alpha_2^{r_2} \alpha_3^{r_3}$.

Problem 1 *Does it exist a recurrent form for* (2.3.1) *?*

A first answer is given by the *Neville-Aitken Algorithm.*
In the unidimesional case this algorithm can be expressed (see [11]):

Given a set of points $\mathcal{S} = \{(x_i, y_i) : i = 0, 1, \ldots, n\}$ *let construct the interpolating polynomial* $P \in \Pi_n$ *by means of the following recurrent formulae:*

$$\begin{cases} P_i^0 = y_i \\ \\ P_i^k = \lambda_i^k(x) P_i^{k-1}(x) + \mu_i^k(x) P_{i+1}^{k-1}(x) \quad 1 \leq k \leq i \quad i = 0, 1, 2, ..., n-k \end{cases} \tag{2.3.2}$$

where

$$\lambda_i^k(x) = \frac{x_{i+k} - x}{x_{i+k} - x_i} \qquad \mu_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i}. \tag{2.3.3}$$

Problem 2 *In which way can we apply this algorithm to the representation of lagrangian polynomials for shape functions?*

First let we give an example.

Let suppose to take the values assumed by the first area coordinate α_1: $(2,1,0)$. Our *conjecture is to consider these values as three different points.* From property (2.2.3) of shape functions the values assumed by the polynomial on these points are $(1,0,0)$. Our construction consists in taking the set $\mathcal{S} = \{(2,1),(1,0),(0,0)\}$ and apply (2.3.2) with $k = 0,1,2$. For the node T_4 where α_1 and α_2 are different from zero, we have to consider the following two sets $\mathcal{S}_1 = \{(1,1),(0,0)\}$ and $\mathcal{S}_2 = \{(1,1),(0,0)\}$ corresponding to the area coordinates α_1 and α_2 respectively. It is worthwhile to note that *the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are not ordered sets.* This means that we can take the sets $\mathcal{S}_1' = \{(0,0),(1,1)\}$ and $\mathcal{S}_2' = \{(0,0),(1,1)\}$ which produce the same polynomials.
We can now summarize these steps in an informal algorithm:

Algorithm 1

- Suppose (I,J,K) is the point in which to evaluate the shape function. The values of I, J and K indicate the degree of the i-th lagrangian component of the shape function. (**Example:** $(2,0,0) \Longrightarrow l_2^2(\alpha_1), l_0^0(\alpha_2), l_0^0(\alpha_3)$. This implies the shape function is a polynomial of degree two in α_1 and a constant one equal to 1 in α_2 and α_3).
- By using the Neville-Aitken Algorithm build the shape functions.

  (a) From a given set $\mathcal{S}_i$ corresponding to the i-th node
  compute the lagrangian components of the i-th shape function;
  (b) Repeat (a) for all the node points of the triangle.

At the end we shall have all the shape functions.

Let us go into relations (2.3.2). We can rewrite $P_i^k(x)$ as:

$$P_i^k(x) = \sum_{j=i}^{i+k} \eta_{ji}^k(x) y_j \tag{2.3.4}$$

where $\eta_{ji}^k(x)$ are related to $\lambda_i^k(x)$ and $\mu_i^k(x)$ by the following recurrence

PROPOSITION 2.3

$$\begin{cases} \eta_{ii}^k = \lambda_i^k \eta_{ii}^{k-1} \\ \\ \eta_{ji}^k = \lambda_i^k \eta_{ji}^{k-1} + \mu_i^k \eta_{j\ i+1}^{k-1} \quad j = i+1, \ldots, i+k-1 \\ \\ \eta_{i+k\ i}^k = \mu_i^k \eta_{i+k\ i+1}^{k-1} \end{cases} \tag{2.3.5}$$

with boundary conditions $\eta_{ii}^0 = 1$.

Proof See [6] ■

Let observe that the coefficients η_{ji}^k *are the elementary Lagrange polynomials.*

PROPOSITION 2.4 *Set* $\eta_k = \sum_{j=i}^{i+k} \eta_{ji}^k$. *Then*

$$\eta_k = 1 \ \ \forall\, k \geq 0 \tag{2.3.6}$$

Proof. Well known result on Lagrange polynomials. ■

2.4 Generalized Neville-Aitken Scheme

In the following part we shall use some notations as in [9].
Let $n \in I\!N$ and $I\!K$ be a commutative field of zero characteristic. Let G be a general set of cardinality n at least. We shall denote with $|G|$ its cardinality.

DEFINITION 2.1 *A set of continuous functions* $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ *satisfies the* Haar Condition *on* G *if any determinant*

$$H_{\mathcal{F}}(x_1, \ldots, x_n) = \begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \ldots & \ldots & \ldots \\ f_1(x_n) & \ldots & f_n(x_n) \end{vmatrix} \tag{2.4.1}$$

defined on the set of **distinct** *points* $X = \{x_1, \ldots, x_n\} \subset G$ *is not vanishing.*

DEFINITION 2.2 *The set of functions* $\mathcal{F} = (f_1, f_2, \ldots, f_n)$, *where* $f_i : G \to I\!K$, *is called a* $I\!K$ Chebyshev System on G *(in the following* $I\!K - CS$*)* **iff** *it verifies the Haar condition* (2.4.1).

DEFINITION 2.3 *Given a* $I\!K - CS$. *It is said* **complete** *if and only if for every* $k = n, n-1, \dots, 1$ *the subset* $(f_1, ..., f_k)$ *of* $\mathcal{F}$ *is a* $I\!K - CS$.

Examples of complete $I\!K - CS$ are given in [9].

For a complete Chebyshev system the following characterization is valid [9]:

PROPOSITION 2.5 *Let* $(f_1, f_2, \dots, f_n)$ *be a n-set of functions* $f_j : G \to I\!K$. *The following statements are equivalent:*

(i) $(f_1, f_2, \dots, f_n)$ *is* $I\!K - CS$;

(ii) for every function $f : G \to I\!K$ *and every set* $G_n = \{x_1, x_2, \dots, x_n\} \subset G$ *with* $|G_n| = n$, *it exists a unique linear combination*

$$p_n f := pf \left[\begin{array}{c} f_1, f_2, \dots, f_{n-1}, f_n \\ x_1, x_2, \dots, x_{n-1}, x_n \end{array} \right]$$

such that

$$p_n f(x_i) = f(x_i) \quad i = 1, 2, \dots, n$$

(iii) Every non trivial combination $P_n(x) = \sum_{i=1}^{n} c_i f_i(x)$, *has up in* G *to n-1 zeros. Such a polynomial will be called a* **C***-polynomial.*

Proof. The proof is based on Cramer's rule. ∎

The greatest monomial coefficient of the interpolant $p_n f$, in analogy with Newton's interpolating formula, is obtained generalizing the divided differences to a $I\!K - CS$.

DEFINITION 2.4 The Generalized Divided Difference *of a function* f *with* simple knots $\{x_1, x_2, ..., x_n\}$ *based on the* $I\!K - CS$ $(f_1, f_2, \dots, f_n)$ *can be expressed by*

$$\left[\begin{array}{c} f_1, f_2, \dots, f_n \\ x_1, x_2, \dots, x_n \end{array} \middle| f \right] := \frac{\left| \begin{array}{c} f_1, f_2, \dots, f_{n-1}, f \\ x_1, x_2, \dots, x_{n-1}, x_n \end{array} \right|}{\left| \begin{array}{c} f_1, f_2, \dots, f_{n-1}, f_n \\ x_1, x_2, \dots, x_{n-1}, x_n \end{array} \right|} \qquad (2.4.2)$$

Set the *interpolation error*

$$r_n f := rf\left[\begin{array}{c} f_1, f_2, \ldots, f_{n-1}, f_n \\ x_1, x_2, \ldots, x_{n-1}, x_n \end{array}\right] := f - p_n f.$$

By using (2.4.2), it can be rewritten as

$$r_n f(x) = \frac{\left|\begin{array}{c} f_1, f_2, \ldots, f_n, f \\ x_1, x_2, \ldots, x_n, x \end{array}\right|}{\left|\begin{array}{c} f_1, f_2, \ldots, f_n \\ x_1, x_2, \ldots, x_n \end{array}\right|} \quad x \in G \tag{2.4.3}$$

The following theorem is the Generalized Neville-Aitken Scheme for a set of functions that is a Chebyshev system on a field $I\!K$ (cf. [9]).

THEOREM 2.1 *Let m and n be two positive integers.*
Let $\mathcal{F} = (f_1, \ldots, f_n)$ and $\mathcal{F}' = (f_1, \ldots, f_n, \ldots, f_{n+m})$ be two $I\!K - CS$ on a set G. Then for any $f : G \to I\!K$, any $G_{n+m} = \{x_1, x_2, \ldots, x_{n+m}\} \subset G$ such that $|G_{n+m}| = n + m$ and any $x \in G$, there holds the following recurrence

$$pf\left[\begin{array}{c} f_1, f_2, \ldots, f_n, \ldots, f_{n+m} \\ x_1, x_2, \ldots, x_n, \ldots, x_{n+m} \end{array}\right](x) = \sum_{j=0}^{m} \lambda_j(x) pf\left[\begin{array}{c} f_1, f_2, \ldots, f_n \\ x_{j+1}, \ldots, x_{j+n} \end{array}\right](x) \tag{2.4.4}$$

where the coefficients $(\lambda_j)_{j=0,\ldots,m}$ are independent functions of f.
Moreover, $\sum_{j=0}^{m} \lambda_j(x) = 1 \quad \forall x \in G.$
Set

$$\gamma_{0,j} = 1 \quad j = 0, ..., m$$

$$\gamma_{k,j} = rf_{n+k}\left[\begin{array}{c} f_1, f_2, \ldots, f_n \\ x_{j+1}, \ldots, x_{j+n} \end{array}\right] \begin{array}{l} {\scriptstyle j=0,\ldots,m} \\ {\scriptstyle k=1,\ldots,m} \end{array} \tag{2.4.5}$$

then $\forall x \in G \backslash \{x_2, ..., x_{n+m-1}\}$

$$N(x) = det(\gamma_{k,j}(x)) \neq 0 \quad j = 0, ..., m \quad k = 0, ..., m \tag{2.4.6}$$

and uniquely $\forall x \in G \backslash \{x_2, ..., x_{n+m-1}\}$ *and* $j = 0, ..., m$

$$\lambda_j(x) = \frac{(-1)^j}{N(x)} det(\gamma_{k,l}(x)) \begin{array}{l} {\scriptstyle k=1,\ldots,m} \\ {\scriptstyle l=0,\ldots,j-1,j+1,\ldots,m} \end{array} \tag{2.4.7}$$

where for $x \in \{x_2, ..., x_{n+m-1}\}$, the 'coefficients' $\lambda_j(x) \in I\!K$ can be chosen such that

$$\sum_{j=0}^{m} \lambda_j(x) = 1 \tag{2.4.8}$$

if $m \geq 2$ *and* $p = 2, ..., m$

$$\begin{aligned} &\lambda_j(x_{n+p-1}) = 0 \quad {\scriptstyle j=0,\ldots,p-2} \\ &\lambda_j(x_p) = 0 \quad {\scriptstyle j=p,\ldots,m} \end{aligned} \tag{2.4.9}$$

Proof See [9]. An alternative proof based on Sylvester's identity for determinants is given in [3]. ■

Remark: substituting $f = f_{n+k}$ in (2.4.4) for $k = 1, .., m$ we can see that necessarily

$$\sum_{j=0}^{m} 1 \, \lambda_j(x) = 1 \tag{2.4.10}$$

$$\sum_{j=0}^{m} \gamma_{k,j}(x)\lambda_j(x) = 0 \quad {\scriptstyle k=1,\ldots,m} \tag{2.4.11}$$

Definition 2.5 *Given the interpolating polynomial* $P_i^k(x)$ *obtained with the Neville-Aitken Scheme* (2.3.2) *or* (2.4.4)*, a set* $\mathcal{W} = \{w_0, w_1, ...\} \subset I\!R$ *and a set* $\Gamma = \{\gamma | \ \gamma : \mathcal{W} \to I\!R\}$*, we define the functional*

$$\mathcal{P}_i^k(\gamma) = \sum_{j=i}^{i+k} \eta_{ji}^k \gamma(w_j) \tag{2.4.12}$$

As in [6], we call it **Reference functional associated to** $P_i^k(x)$*.*

Obviously, $\mathcal{P}_i^k(\gamma) = P_i^k$ if we take $\gamma(w_j) = y_j$. It means that $\gamma(w)$ is the interpolating polynomial at the set $\mathcal{A} = \{(w_j, y_j) : \ j = 0, 1, \ldots\}$.
We are interested in the following problem:

Problem 3 *Does it exist a subspace* $\Gamma^k = \{\gamma_0, \ldots, \gamma_k\}$ *of* Γ *such that*

$$\mathcal{H}_{\Gamma^k}(w_i, ..., w_{i+k}) = \begin{vmatrix} \gamma_0(w_i) & \gamma_0(w_{i+1}) & \ldots & \gamma_0(w_{i+k}) \\ \gamma_1(w_i) & \gamma_1(w_{i+1}) & \ldots & \gamma_1(w_{i+k}) \\ \ldots & \ldots & & \ldots \\ \gamma_k(w_i) & \gamma_k(w_{i+1}) & \ldots & \gamma_k(w_{i+k}) \end{vmatrix} \neq 0 \tag{2.4.13}$$

and

$$\mathcal{P}_i^k(\gamma_j) = \begin{cases} 1 & j = 0 \\ & \\ 0 & j = 1, 2, 3, ..., k \end{cases} \quad ? \tag{2.4.14}$$

We shall refere to the subspace Γ^k as **Characteristic Space** of $\mathcal{P}_i^k$.

THEOREM 2.2 *Given the interpolating polynomial of Lagrange $P_i^k(x)$ and its associated reference functional $\mathcal{P}_i^k$, the characteristic space Γ^k exists and we have*

$$\Gamma^k = span < \gamma_i > \quad i = 0, ..., k$$

where $\gamma_0(w) = 1$ *and* $\gamma_i(w) = x^i - w^i \quad i = 1, ..., k$

Proof. From theorem 2.1 we know that the functions $\gamma_{k,j}(x), \quad k, j = 0, ..., m$ given in (2.4.5) satisfy relations (2.4.10) and (2.4.11) that are equivalent to (2.4.14).
In the case of lagrangian polynomials these functions have the expression $\gamma_{k,j}(x) = x^k - x_j^k \;\; for \; k > 0$. If we fix x, we may consider $\gamma_{k,j}$ only as functions of x_j, i.e. $\gamma_{k,j}(x) = \gamma_k(x_j)$. Moreover, the $k+1$ functions $\gamma_0(w) = 1$ and $\gamma_i(w) = x^i - w^i \quad i = 1, ..., k$ satisfy the Haar condition and are linearly independent. They span a subspace of dimension $k+1$ of all the functions from $W = \{w_i, ..., w_{i+k}\}$ to $I\!R$. ■

2.5 The De Casteljau Algorithm

So far, we have seen that the Neville-Aitken Scheme can only be applied to the lagrangian component of (2.3.1). Our next step consists in looking for a global scheme which allows to compute shape functions in their most general form. Since we are considering barycentric coordinates, there is a general approach that comes from the theory of *"Bézier patches"* and its computational algorithm: the *De Casteljau Algorithm*.
Some notation. We denote the generic point of the original triangle $\mathbf{T}$, b_{ijk} by $b_{\mathbf{l}}$. This means that $|\mathbf{l}| = i + j + k$. We always assume $i, j, k \geq 0$ and $i, j, k \in I\!N$.
A Bézier patch of degree n on a triangle, in the $(u, v)-$plane, can be represented as (see [2])

$$b(u, v) = \sum_{|\mathbf{l}|=n} B_{\mathbf{l}}^n(\mathbf{a}) b_{\mathbf{l}} \tag{2.5.1}$$

with obvious meaning. The $B_{\mathbf{l}}^n(\mathbf{a})$ is the l-th Bernstein polynomial of degree n

$$B_{\mathbf{l}}^n(\mathbf{a}) = \begin{pmatrix} n \\ \mathbf{l} \end{pmatrix} \alpha_1^i \alpha_2^j \alpha_3^k = \frac{n!}{i!j!k!} \alpha_1^i \alpha_2^j \alpha_3^k \tag{2.5.2}$$

where $|\mathbf{l}| = n = i + j + k$.
We shall assume that $B_{\mathbf{l}}^n(\mathbf{a}) = 0$ if some of the components of the triplet (i, j, k) is negative. The points $b_{\mathbf{l}}$ are known as *Bézier points.*
Bernstein polynomials satisfy the following recurrence:

PROPOSITION 2.6

$$B_{\mathbf{l}}^r(\mathbf{a}) = \alpha_1 B_{\mathbf{l}-\mathbf{e_1}}^{r-1} + \alpha_2 B_{\mathbf{l}-\mathbf{e_2}}^{r-1} + \alpha_3 B_{\mathbf{l}-\mathbf{e_3}}^{r-1} \quad |\mathbf{l}| = r \tag{2.5.3}$$

Proof. It follows from (2.5.2) by induction on r. ■

The *De Casteljau Algorithm*, well known for the computation of Bézier curves, can be generalized to Bézier patches as follows:

Algorithm 2

- **Given:** a triangle of order n with nodes $b_{\mathbf{l}} \in I\!R^3$ ($|\mathbf{l}| = n$) and a point $P \in (u, v) - plane$ with barycentric coordinates $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)$.
- **Set:**

$$\begin{cases} b^0_{\mathbf{l}}(\mathbf{a}) = b_{\mathbf{l}} \quad |\mathbf{l}| = n \\ b^r_{\mathbf{l}}(\mathbf{a}) = \alpha_1 b^{r-1}_{\mathbf{l}+\mathbf{e_1}}(\mathbf{a}) + \alpha_2 b^{r-1}_{\mathbf{l}+\mathbf{e_2}}(\mathbf{a}) + \alpha_3 b^{r-1}_{\mathbf{l}+\mathbf{e_3}}(\mathbf{a}) \\ \qquad\qquad\qquad\qquad r = 1, ..., n \quad |\mathbf{l}| = n - r \end{cases} \tag{2.5.4}$$

where $\mathbf{e_1} = (1, 0, 0), \mathbf{e_2} = (0, 1, 0)$ and $\mathbf{e_3} = (0, 0, 1)$.

The equivalence between the De Casteljau Algorithm (2.5.4) and the Bézier polynomial (2.5.1) is stated as follows.

PROPOSITION 2.7 *The generic point computed by the de Casteljau Algorithm* (2.5.4)*, can be expressed by Bernstein polynomials. It results*

$$b^r_{\mathbf{i}}(\mathbf{a}) = \sum_{|\mathbf{j}|=r} B^r_{\mathbf{j}}(\mathbf{a}) b_{\mathbf{i}+\mathbf{j}} \quad |\mathbf{i}| = n - r \tag{2.5.5}$$

We want to look for determinant formulae in this scheme as we did for the Lagrange polynomials. First of all, we define

$$\mathcal{B}^r_{\mathbf{i}}(\gamma) = \sum_{|\mathbf{j}|=r} B^r_{\mathbf{j}}(\mathbf{a}) \gamma(\mathbf{s}) \quad |\mathbf{i}| = n - r \tag{2.5.6}$$

the *reference functional* associated to $b^r_{\mathbf{i}}$.

THEOREM 2.3 *There exists* a "weak" characteristic space *associated to the functional* $\mathcal{B}^r_{\mathbf{i}}$*.*

Proof. The proof follows the idea developed in the previous section about the Neville-Aitken Scheme for Lagrange polynomials.

Let us note that 3-variate Bernstein polynomials of given degree k are a basis for the space of 3-variate polynomials of degree k. They are linearly independent and on a set of distinct points they satisfy the Haar condition. Moreover they are a Chebyshev system on $I\!R$.

It follows, we are in the hypotheses of theorem 2.1, therefore we can determine functions $\gamma_{k,j}$ satisfying (2.4.10) and (2.4.11) equivalent to (2.4.14). ■

Comments

We have found a solution by using an interpolating procedure starting from an approximating scheme. This is the reason why we have called the associated characteristic space a "weak" characteristic space. This idea can be generalized saying that if we are considering an approximating scheme we may work on an associated interpolating space for finding the characteristic space.

Example

We apply theorem 2.1 with $n = 1$ and $m = 2$. In this case the dimension d of the space of 3-variate polynomials of degree one is 3. The set $\mathcal{F}$ is formed of 3 base functions that are the Bernstein polynomials of degree 1 in 3 variables, i.e. $\mathcal{F} = (B^1_{100}, B^1_{010}, B^1_{001}) = (\alpha_1, \alpha_2, \alpha_3)$.

To apply (2.4.10) and (2.4.11) we have to compute the functions $\lambda_l(\alpha_1, \alpha_2, \alpha_3)$ $l = 0, 1, 2$ and the 6 functions $\gamma_{k,j}(\alpha_1, \alpha_2, \alpha_3)$ $k = 1, 2$ $j = 0, 1, 2$.

With $|G| = n + m = 3$, we take as G the set formed of three distinct points $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ with $\mathbf{a_1} = (a_{11}, a_{12}, a_{13})$, $\mathbf{a_2} = (a_{21}, a_{22}, a_{23})$ and $\mathbf{a_3} = (a_{31}, a_{32}, a_{33})$ [1].

By using (2.4.5) we get

$$\gamma_{1,j}(\mathbf{a}) = \frac{a_{j+1,1}}{a_{1,1}}\alpha_2 - \frac{a_{j+1,2}}{a_{1,1}}\alpha_1 \quad {\scriptstyle j=0,1,2} \tag{2.5.7}$$

$$\gamma_{2,j}(\mathbf{a}) = \frac{a_{j+1,1}}{a_{1,1}}\alpha_3 - \frac{a_{j+1,3}}{a_{1,1}}\alpha_1 \quad {\scriptstyle j=0,1,2} \tag{2.5.8}$$

From these formulae, the point $\mathbf{a_1}$ may be chosen in such a way that $a_{1,1} \neq 0$. Then

$$N(\mathbf{a}) = \frac{1}{a_{11}^2}\left(\alpha_1^2 D_1 + \alpha_1\alpha_2 D_2 + \alpha_1\alpha_3 D_3\right) \tag{2.5.9}$$

where

$$\begin{cases} D_1 = a_{12}(a_{23} - a_{33}) + a_{22}(a_{33} - a_{13}) + a_{32}(a_{13} - a_{23}) \\ D_2 = a_{11}(a_{33} - a_{23}) + a_{21}(a_{13} - a_{33}) + a_{31}(a_{23} - a_{13}) \\ D_3 = a_{11}(a_{22} - a_{32}) + a_{21}(a_{32} - a_{12}) + a_{31}(a_{12} - a_{22}) \end{cases} \tag{2.5.10}$$

Functions $\lambda_j(\alpha_1, \alpha_2, \alpha_3)$ $j = 0, 1, 2$ are then given

$$\begin{aligned} \lambda_0(\mathbf{a}) &= \frac{\alpha_1^2(a_{33}a_{22} - a_{23}a_{32}) + \alpha_1\alpha_2(a_{31}a_{23} - a_{21}a_{33}) + \alpha_1\alpha_3(a_{21}a_{32} - a_{22}a_{31})}{N'(\mathbf{a})} \\ \lambda_1(\mathbf{a}) &= -\frac{\alpha_1^2(a_{12}a_{33} - a_{13}a_{32}) + \alpha_1\alpha_2(a_{31}a_{13} - a_{11}a_{33}) + \alpha_1\alpha_3(a_{11}a_{32} - a_{12}a_{31})}{N'(\mathbf{a})} \\ \lambda_2(\mathbf{a}) &= \frac{\alpha_1^2(a_{12}a_{23} - a_{22}a_{13}) + \alpha_1\alpha_2(a_{21}a_{13} - a_{11}a_{23}) + \alpha_1\alpha_3(a_{11}a_{22} - a_{12}a_{21})}{N'(\mathbf{a})} \end{aligned} \tag{2.5.11}$$

[1] Although *area coordinates* are not independent coordinates, we prefere to consider 3-variate polynomials

where $N'(\mathbf{a}) = a_{11}^2 N(\mathbf{a})$.

By definition it is easy to verify $\sum_{j=0}^{2} \lambda_j(\mathbf{a}) = 1$ and $\sum_{j=0}^{2} \gamma_{k,j}(\mathbf{a})\lambda_j(\mathbf{a}) = 0 \quad k = 1,2$.

It is worthwhile to note that the functions $\lambda_j(\mathbf{a})$ have the property $\lambda_{j+1}(\mathbf{a_i}) = 1 \;\; j = 0,1,2 \; i = 1,2,3$ if $i = j+1$ otherwise they are vanishing. This property suggests that these functions can be seen as *shape functions* and their expressions (2.5.11) give a way to compute them.

Moreover the functions $\gamma(\mathbf{s})$ required in theorem 2.3 have the following expressions $\gamma_0(\mathbf{s}) = 1$, $\gamma_k(s_{j,1}, s_{j,2}, s_{j,3}) = s_{j,1}\alpha_{k+1} - s_{j,2}\alpha_1 \;\; j = 0,1,2, \; k = 1,2$.

To conclude this section we would like to identify some points that justify the preference to use this algorithm instead of the Neville-Aitken's.

The De Casteljau recursion formula together with Theorem 2.7, give a new recursion scheme to describe shape functions.

The differences, in comparison with the Neville-Aitken Algorithm, can be summarized as follows

(i) *The globality of this expression.* It means that we can describe the interpolating polynomial only considering the barycentric coordinates of the interpolating points.

(ii) *The recursion scheme 2.5.4 is not triangular.* With reference to [6], a recursion scheme is called *triangular* if there is an initialization step followed by a two-terms recursion part (for example the Neville-Aitken Scheme). The recursion scheme (2.5.4) is called a *three-terms recursion scheme* since it involves 1 term in the inizialization step and 3 terms in the recursion part.

(iii) *Expression* (2.5.5) *can be seen as a system of equations.*

Given a triplet $\mathbf{l} \in I\!N^3$, $\mathbf{l}$=(i,j,k), let $|\mathbf{l}| = i+j+k$ be its length. We use the notation $m(\mathbf{l})$ for the total number of triplets with length $|\mathbf{l}|$.

Rewriting expression (2.5.5) in matrix notation as $Ac = b$, we see that A is a $m(\mathbf{i})$ x $m(\mathbf{j})$ *matrix* whose elements are $b_{\mathbf{i+j}}$. This matrix is symmetric if $m(\mathbf{i}) = m(\mathbf{j})$. c is the vector formed by the base Bernstein polynomials valued in $\mathbf{a}$ and b is the vector of the resulting Bézier polynomials.

This observation implies that we could seek for a *characteristic space* of $\mathcal{B}_{\mathbf{l}}^r$ by orthogonalization.

3 Conclusions

It is interesting to note that the algorithm for the Generalized Neville-Aitken polynomial as suggested in [9] is an application of the *E-Algorithm*, as verified in [4, 5]; furthermore such algorithm is of

interpolating nature.
Instead the De Casteljau Algorithm appears more flexible to give an approximate solution over a triangle. An improved version with reduced computational complexity of this algorithm, is the *VSC Algorithm* based on a modified formula of Bernstein-Bézier polynomials [10].

3.1 Acknowledgements

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Appendix A

*** Procedure carea1

This procedure determines the system of barycentric coordinates of a generic point P, relatively to the given triangle T of verteces x(i),y(i) (i=1,2,3)

```
clear;
x(1)=input('x coordinate of the first point: ');
y(1)=input('y coordinate of the first point: ');
x(2)=input('x coordinate of the second point: ');
y(2)=input('y coordinate of the second point: ');
x(3)=input('x coordinate of the third point: ');
y(3)=input('y coordinate of the third point: ');
a1(1,:)=[x(1), y(1)];
a1(2,:)=[x(2), y(2)];
a1(3,:)=[x(3), y(3)];
aa=[a1(1,:)',a1(2,:)',a1(3,:)'];
aa1=[aa,
1, 1, 1]
area=1/2*det(aa1); **** triangle area
pe=[2 3,
3 1,
1 2]; **** permutation of remaining indeces
xp=input('x coordinate point P: ');
yp=input('y coordinate point P: ');
for i=1:3
a(i)=a1(pe(i,1),1)*a1(pe(i,2),2)-a1(pe(i,2),1)*a1(pe(i,1),2);
b(i)=a1(pe(i,1),2)-a1(pe(i,2),2);
c(i)=a1(pe(i,2),1)-a1(pe(i,1),1);
mat(i,:)=[a(i), b(i), c(i)];
end;
u=1/(2*area)*(mat)*[1 xp yp]';
disp('Required barycentric coordinates ');
u;
```

The following function determines
all triplets (i,j,k) such that $i + j + k = n$
n=input parameter (is the triple length)

```
function [ tern1, l ] = tern(n) ;
l=1;
for i=n:-1:0
  for j=0:n-i;
    if n ≠ 0
      tern1(l,:)= [i/n, (n-i-j)/n, j/n] ;
    elseif n==0
      tern1(l,:)=[0,0,0];
```

```
    end;
  b(l,:)= [ i,n-i-j,j ];
  l=l+1;
end;
end;
l=l-1;
```

*** Procedure DeCaste

This procedure implements the De Casteljau Algorithm. Given a triangle of order n, it computes the Bernstein-Bézier polynomial in $a = (a_1, a_2, a_3)$.

```
n= input('Triangle Order: ');
[br,m] = tern(n);
for i=1:m
    x(i)=br(i,1);
    y(i)=br(i,2);
end;
v=1;
disp('We are considering normalized barycentric coordinates, i.e. such that a1 + a2 + a3 = 1');
a1 = input('1°area coordinate :');
a2 = input('2°area coordinate :');
a3 = 1 - a1 - a2;
    for j=0:n
            g(j+1)=0.5*(j+1)*(j+2);
    end;
    for r=1:n
            k=1;
            [br1,m] = tern(n - r);
            clear x;
            clear y;
            for l=1:m
                    x(l)=br1(l,1);
                    y(l)=br1(l,2);
            end;
            v=v+1;
            br1=br;
            for l=1:m
            if l==1
            br(l,:)=a1*(br1(l,:))+a2*(br1(l+1,:))+a3*(br1(l+2,:));
            end;
            if (g(k) < & l ≤ g(k+1))
                    br(l,:) = a1 * (br1(l,:)) + a2 * (br1(l + 2 + k - 1,:)) + a3 * (br1(l + 2 + k,:));
            end;
            if rem(l,g(k+1))==0
                    k=k+1;
            end;
```

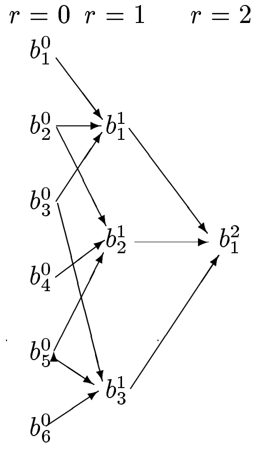

Figure 2: *De Casteljau Table for n=2*

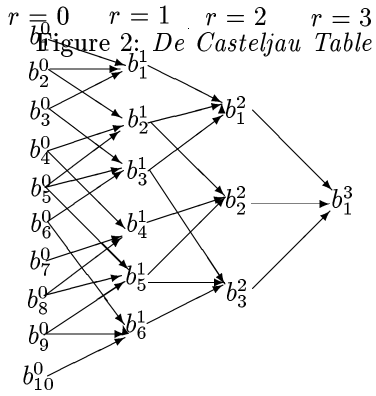

Figure 3: *De Casteljau Table for n=3*

```
                end;
          end;
end;
disp(' ** Results ** ');
disp(br(1,:));
```

The way in which the procedure DeCaste computes the Bernstein-Bézier polynomial can be seen in Fig. 2 (for the case $n = 2$) and Fig. 3 (for the case $n = 3$).