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Parametric Method for Global Optimization[1,2]

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Abstract. This paper considers constrained and unconstrained parametric global optimization problems in a real Hilbert space. We assume that the gradient of the cost functional is Lipschitz continuous but not smooth. A suitable choice of parameters implies the linear or superlinear (supergeometric) convergence of the iterative method. From the numerical experiments, we conclude that our algorithm is faster than other existing algorithms for continuous but nonsmooth problems, when applied to unconstrained global optimization problems. However, because we solve $2n + 1$ subproblems for a large number n of independent variables, our algorithm is somewhat slower than other algorithms, when applied to constrained global optimization.

Key Words. Global optimization, Lyapunov functions, Lyapunov function methods, Hilbert spaces, differential inclusions, monotone operators, subdifferentials.

1. Introduction

In this paper, we consider the problem of minimizing the functional $V_P(x)$, where x belongs to the whole Hilbert space H or to some bounded closed subset $H_1 \subset H$ and p is an abstract parameter,

$$\inf_{x\in H} V_p(x) = V_p^* \quad \text{or} \quad \inf_{x\in H_1} V_p(x) = V_p^*.$$

We suppose that there exists a Lipschitz continuous gradient $\nabla V_p(x)$ and at least one solution point $x^* \in H$ (respectively $x^* \in H_1$), for which

$$\inf_{x\in H} V_p(x) = V_p(x^*) = V_p^*, \tag{1}$$

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$$\inf_{x \in H_1} V_p(x) = V_p(x^*) = V_p^*. \tag{2}$$

The present paper continues some investigations of Refs. 1–5. We suppose also that all local minima are isolated. We consider first the global unconstrained optimization algorithm and then we continue with the constrained case.

In Ref. 2, we presented a parametric Newton method for optimization problems in Hilbert spaces. This is a modified Newton method for solving the minimization problems, avoiding the calculation of the inverse operators in the infinite-dimensional spaces. This method solves parametric optimization problems (see Refs. 6–15) in a real Hilbert space, supposing that the gradient of the cost functional is a Lipschitz continuous nonsmooth function. This problem could be solved with other methods (see Refs. 16–21).

Under the Lipschitz continuity of $\nabla V_p(x)$, we can construct the Clark subdifferential $\partial^2 V_p(x)$ (see Refs. 2, 22–27) for which $(\partial^2 V_p(x))^* \nabla V_p(x)$ is a multivalued map of H into bounded, closed, and convex subsets of H. Here, $(\partial^2 V_p(x))^*$ is the adjoint map; i.e.,

$$\left\langle \partial^2 V_p(x)x, y \right\rangle \equiv \left\langle x, (\partial^2 V_p(x))^* y \right\rangle, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in H. Using the properties of $\partial^2 V_p(x)$ and the Lipschitz continuity of $\nabla V_p(x)$, the map $(\partial^2 V_p(x))^* \nabla V_p(x)$ is usc (Refs. 2, 22, 23, 25).

We use the following notations:

$|x| = \sqrt{\langle x, x \rangle}$ is the norm of H;

$X = \{x \in H | V_p(x) = V_p^*\}$ is the set of solutions of (1) which does not depend on the disturbances of the parameter p;

$Pr_A(x) = \{y \in H \| x - y| = \min_{z \in A} |x - z|\}$ is the metric projection of x into A $\subset$ H;

$m(V_p(x))$ is the metric projection of the origin into the values of the usc multifunction $(\partial^2 V_p(x))^* \nabla V_p(x)$;

$X_\delta = \{x \in H | \min_{y \in H} |x - y| < \delta\}$ is a δ-neighborhood of X.

We proved in Ref. 2 the following theorem.

Theorem 1.1. Consider the differential inclusion

$$\dot{x} \in -(\partial^2 V_p(x))^* \nabla V_p(x), \quad x(0) = x_k, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where the right-hand side is a usc multifunction with bounded, closed, and convex values, $x_k \notin X_\delta$, and $\delta > 0$ is arbitrarily chosen. Let $|m(V_p(x))| \geq \eta > 0$, $x \notin X_\delta$, and let

$$X = \{x \in H | V_p(x) = V_p^*\} \equiv \{y \in H | \nabla V_p(y) = 0\}.$$

If the differential inclusion (4) admits a solution and the set of the solutions is bounded for $t \geq 0$, then every solution of

$$\dot{x} \in -(\partial^2 V_p(x))^* \nabla V_p(x)/|m(V_p(x))|^2, \quad x(0) = x_k, \quad k = 0, 1, 2, \ldots, \tag{5}$$

can be extended up to the set X in a finite time $T_k \leq (1/2)|\nabla V_p(x_k)|^2$.

We present two iterative procedures. We consider the following procedure for the undisturbed case:

$$x_{k+1} = x_k - |\nabla V_p(x_k)|^2 m(V_p(x_k))/|m(V_p(x_k))|^2, \quad k = 0, 1, 2, \ldots, \tag{7}$$

where x_0 is known. This is a modified Newton method where the calculation of the inverse operator is replaced by the calculation of the metric projection $m(V_p(x))$ of the origin into the values of the usc multifunction $(\partial^2 V_p(x))^* \nabla V_p(x)$.

We consider also the following iterative procedure for the disturbed case:

$$x_{k+1} = x_k - \left|\nabla V_{p_k}^{\varepsilon_k}(x_k)\right|^2 m\left(V_{p_k}^{\varepsilon_k}(x_k)\right)/\left|m\left(V_{p_k}^{\varepsilon_k}(x_k)\right)\right|^2, \quad k = 0, 1, 2, \ldots, \tag{8}$$

where x_0 is known and $\nabla V_{p_k}^{\varepsilon_k}(x)$ and $m(V_{p_k}^{\varepsilon_k}(x))$ denote the approximating values of $\nabla V_{p_k}(x)$ and $m(V_{p_k}(x))$ respectively.

Let us consider now the following problem. We solve a nonlinear equation, with the left-hand side being is a scalar function in the Hilbert space H.

A parametric Lyapunov function method for solving nonlinear systems in Hilbert spaces was presented in Ref. 1. This method considers a nonlinear system of finitely many equations in Hilbert spaces. Using the left-hand sides of the equations, a family of scalar and convex functions is constructed. Under the hypothesis of convexity, these functions are the Lyapunov functions for the differential inclusion for which the right-hand side is the negative value of a maximal monotone operator. Every solution of the differential inclusion solves the nonlinear system of equations in a finite moment of time.

An iterative method for finding at least one of the solutions of the system is presented. This method is based on the Lyapunov function and depends on a vector parameter. A suitable choice of the parameter implies the linear or superlinear convergence.

Let $f(x)$ be a scalar function (it can be nonsmooth) which is defined on the Hilbert space H by the following equation:

$$f(x) = V_{pi}(x) - V_{pi}(x_{i+1}) = 0, \quad i = 0, 1, 2, \ldots, \tag{9}$$

where x_{i+1} is the $(i+1)$th local minimum point of the problem (1). We denote a family of functions depending on p' by

$$W_{p'}(x) = \alpha \left|V_{p_i}(x) - V_{p_i}^*(x_{i+1})\right|^{1+p'} \geq 0, \quad i = 0, 1, 2, \ldots, \tag{10}$$

and also for the case with disturbances a family of functions depending on $p'(\varepsilon')$,

$$W_{p'(\varepsilon')}(x) = \alpha \left| V_{p_i}(x) - V^*_{p_i}(x_{i+1}) \right|^{1+p'(\varepsilon')} \geq 0, \quad i = 0, 1, 2, \ldots, \tag{10'}$$

where $\alpha > 0$ and the parameter $p'(\varepsilon') \in (p' - \varepsilon',\ p' + \varepsilon')$ and $p' - \varepsilon' > -1,\ \varepsilon' > 0$. According to Ref. 22, the generalized directional derivative is

$$f^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} [f(y + \lambda v) - f(y)]/\lambda$$

and the generalized gradient is

$$\partial f(x) = \{\zeta \in H | f^0(x; v) \geq \langle \zeta, v \rangle, \forall v \in H\}.$$

Definition 1.1. See Ref. 22. f is said to be regular at x provided:

(i) For all v, the usual one-side directional derivative $f'(x; v)$ exists.
(ii) For all v, $f'(x; v) = f^0(x; v)$.

We suppose that the function $f(x)$ is regular. We suppose also that functions $W_{p'}(x)$ and $W_{p'(\varepsilon')}(x)$ are locally convex at every local minimum of the cost functional $V_p(x)$. Let us denote the metric projection of the point x on the set X' by $Pr'_X x$; i.e.,

$$Pr'_X x = \left\{ y \in X' |\ |x - y| = \min_{z \in X'} ||x - z|| \right\} \tag{11}$$

We denote by $m(\partial W_{p'(\varepsilon')}(x))$ the metric projection of the origin into the set $\partial W_{p'(\varepsilon')}(x)$.

Let $0 \notin \partial W_{p'(\varepsilon')}(x_0)$. Consider the following differential inclusion (see Ref. 28):

$$\dot{x} \in -\partial W_{p'(\varepsilon')}(x)/|m(\partial W_{p'(\varepsilon')}(x))|, \quad x(0) = x_0, \tag{12}$$

As long as

$$f(x) = V_{p_i}(x) - V^*_{p_i}(x_{i+1})$$

is a regular function, according to Ref. 22 we have

$$\begin{aligned}\partial W_{p'(\varepsilon')}(x) = \alpha \left| V_{p_i}(x) - V^*_{p_i}(x_{i+1}) \right|^{p'(\varepsilon')} \operatorname{sign}\big(V_{p_i}(x) - V^*_{p_i}(x_{i+1})\big) \partial \big(V_{p_i}(x) \\ - V^*_{p_i}(x_{i+1})\big)(1 + p'(\varepsilon')).\end{aligned} \tag{13}$$

When $W_{p'(\varepsilon')}(x)$ is a convex function, the subdifferential $\partial W_{p'(\varepsilon')}(x)$ is a maximal monotone operator and the differential inclusion

$$\dot{x} \in -\partial W_{p'(\varepsilon')}(x), \quad x(0) = x_0, \tag{14}$$

has a unique solution; see Ref 29. We denote by

$$X' = \left\{x \in H \,|\, f(x) = V_{p_i}(x) - V^*_{p_i}(x_{i+1}) = 0\right\}$$

the solution set of equation (9) and let

$$X'_1 = \{x \in H \,|\, 0 \in \partial W_{p'(\varepsilon')}(x)\}$$

be the set of the stationary points of the differential inclusion (12). We use the next theorem (see Ref. 1).

Theorem 1.2. Consider equation (9). Let the function (10) be convex for the parameters $\alpha > 0,\ p' > -1 + \varepsilon',\ \varepsilon' > 0$. Let the solution set X' be not empty. Then:

(i) The solution set X' of (1) coincides with the set for which the origin belongs to the subdifferential $\partial W_{p'(\varepsilon')}(x)$, i.e. $X' = X'_1$.

(ii) For every initial position $x_0 \in H$ and every solutions x(t) of the differential inclusion (12), there exists a finite moment $T \leq W_{p'(\varepsilon')}(x_0)$ (T possibly depends on the solution) for which x (T) solves equation (9).

Under the conditions of the Theorem 1.2, every approximative method which solves the differential inclusion (12) generates an approximate solution of equation (9). In Ref. 29, this approximate solution is obtained using the Iosida approximation, which has a good relation with the Euler implicit scheme for the systems of ODEs.

We consider now two iterative procedures based on (12). For the undisturbed case,

$$x_{k+1} = x_k - W_{p'^k}(x_k) m(\partial W_{p'^k}(x_k))/|m(\partial W_{p'^k}(x_k))|^2, \quad k = 0, 1, \ldots, \tag{15}$$

and for the disturbed case,

$$x_{k+1} = x_k - W_{p'^k(\varepsilon'^k)}(x_k) m(\partial W_{p'^k(\varepsilon'^k)}(x_k))/|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))|^2, \quad k = 0, 1, \ldots. \tag{16}$$

We divide the solution of the unconstrained global minimization problem into two steps:

Step 1. The first step solves equation (1) with the iterative procedure (7) or (8) for the disturbed case of the parametric Newton method for optimization problems in Hilbert space (see Ref. 2). We start at a given initial point x_0^0 and find the local minimum point x_1 using the correspondent iterative procedure (7) or (8). On every next ith step, the iterative procedure starts the iterations at the initial point x_i^0 and finds the local minimum x_{i+1}. On the ith iteration, we look for a solution x_{i+1} different from x_i. The selection of x_0^i insures that. But it is possible also that $V_p(x_i) = V_p(x_{i+1})$. This could happen when the cost functional has the same value at these local minimum points. Then again, we construct the same plane (hyperplane)

$$V = V_p(x_{i+1}) = V_p(x_i)$$

and look for its intersection with the surface (hypersurface) $V_p(x)$; i.e. we solve with the procedure (15) or (16) the equation

$$V_p(x) - V_p(x_{i+1}) = 0$$

at a point x_0^{i+1} different from x_0^{i-1} and x_0^i. We solve now the optimization problem with initial point x_0^{i+1}. If the solution is one of the previous local minimum points, we return to solving the equation $V_p(x) - V_p(x_{i+1}) = 0$ at a point x_0^{i+2} different from x_0^{i-1}, x_0^i, x_0^{i+1}. Again, we solve the optimization problem with the initial point x_0^{i+2} and so on.

Step 2. The second step solves equation (9) with the iterative procedure (15) or (16) of the parametric Lyapunov function method for solving nonlinear systems in Hilbert spaces (see Ref. 1). We find first the initial point for the next iteration process which solves the nonlinear equation (9). For that purpose, we construct a plane for the three-dimensional case or a hyperplane for the multidimensional cases through the point $x_1[x_i$ on every next ith step] $V = V_p(x_1)$ $[V = V_p(x_i)]$ on the ith step. For the three-dimensional case, we construct the plane as we turn the vector with the initial point x_1 and the terminal point $-kx_0^0$ (the initial point is x_i and the terminal point is $-kx_0^{i-1}$ for the ith iteration) around the point x_1 (x_i) in the plane $V = V_p(x_1)$ $[V = V_p(x_i)]$, where k is a positive integer. We take the terminal point of this vector as the initial point for the procedure (15) or (16). If this plane has the only common point x_1 $[x_i]$ with the surface [i.e., no other common point with the surface (hypersurface) $V_p(x)$], then x_1 $[x_i]$ is the global minimum of the problem. If we find a solution different from the $x_1[x_i]$ solution with the procedure (15) or (16), we denote it with x_0^1 $[x_0^i]$ and use it as the initial point for the procedures (7) or (8) of Step 1. It is possible also that $V_p(x_i) = V_p(x_{i+1})$. This could happen when the cost functional has the same value at these local minimum points. Then again, we construct the same plane [hyperplane] $V = V_p(x_{i+1}) = V_p(x_i)$, go to the Step 2, and look for its intersection with the surface [hypersurface] $V_p(x)$; i.e., we solve with the procedure (15) or (16) the equation

$$V_p(x) - V_p(x_{i+1}) = 0 \tag{17}$$

at a point x_0^{i+1} different than x_0^{i-1} and x_0^i. We solve now the optimization problem (Step 1) with initial point x_0^{i+1}. If the solution is one of the previous local minimum points, we return to Step 2 to solve equation (8) with another initial point. The solution point x_0^{i+2} should be different from x_0^{i+1}, x_0^i, x_0^{i+1}. Again, we solve the optimization problem (Step 1) with the initial point x_0^{i+2} and so on.

The global constrained optimization case (2) includes some additional problems:

(P1) We consider one subproblem for the inner points of the bounded closed set H_1 and the subproblems on the boundaries. For example, if every independent variable is located in a closed interval, then we solve $2n + 1$ subproblems if n is the number of independent variables, because there is one subproblem for the inner points and $2n$ subproblems on the boundary planes [hyperplanes].

(P2) The algorithm takes more computer time for every subproblem, because it has to be checked on every step if the new point is still in the bounded set. The solution for a given subproblem could happen to be a boundary point which is not a local minimum.

(P3) The global solution of the problem is the solution point of the above subproblems that gives the minimum value to the cost functional $V_p(x)$.

2. Main Results

Theorem 2.1.

(i) Let us consider the minimizing problem (1) and look for the local minimum nearest to the initial point x_0^i with the iterative procedure (7) where $x \in H$, $\nabla V_p(x)$ is Lipschitz continuous, $m(V_p(x))$ is the metric projection of the origin into the values of the usc multifunction $(\partial^2 V_p(x))^* \nabla V_p(x)$. Let the nearest local minimum (solution set) in the cone of decreasing direction for the given initial point x_0^i be isolated and such that

$$X = \{x \in H | V_p(x) = V_p^*\} \equiv \{y \in H | \nabla V_p(y) = 0\}$$

is invariant with respect to p [if x_0^i coincides with a local minimum, then the first iterative procedure is skipped and we move to (ii)], $x_k \notin X,\ k = 0, 1, 2, \ldots,$ and $|m(V_p(x)| > 0,\ x \notin X$. The $p_k, k = 0, 1, 2, \ldots,$ are chosen via the following equalities:

$$\left|\nabla V_{p_k}(x_k)\right|^2 = \left\langle m\left(V_{p_k}(x_k)\right),\ x_k - Pr_X(x_k)\right\rangle; \tag{18}$$

then,

$$|x_{k+1} - Pr_X(x_{k+1})| \leq |x_k - Pr_X(x_k)|, \quad k = 0, 1, 2, \ldots.$$

(ii) We consider also the nonlinear equation (9) and the iterative procedure (15), where $f(x)$ is a regular function defined in the Hilbert space H. Let there exist a δ'-neighborhood $X'_{\delta'}$ of the set X' for which

$$\langle \xi,\ x - y\rangle \neq 0, \quad \xi \in \partial W_{p'}(x),\ y \in Pr_{X'x},\ x \in X'_{\partial'} \backslash X',$$

where $W_{p'}(x)$ is defined by (10). Let p'^k be any solution of the following equality:

$$W_{p'^k}(x_k) + \langle m(\partial W_{p'^k}(x_k)),\ y_k - x_k\rangle = 0, \tag{19}$$

where $y_k \in Pr'_X x_k$ are arbitrarily chosen. If the initial point $x_0 \in X'_{\delta'}$ and if

$$\varepsilon'^k \leq (q'/2)W_{p'^k(\varepsilon'^k)}(x_k),\ 0 < q < 1,$$

then the following inequality:

$$|x_{k+1} - y_{k+1}| \leq |x_k - y_k| \tag{20}$$

holds for every $y_k \in Pr'_X x_k, k = 0, 1, 2, \ldots.$

(i) and (ii) ensure that the algorithm for the global optimization problem is monotonous.

Proof.

(i) Let us consider

$$0 < A_k = |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2,\ \text{where } y_k = Pr_X(x_k),$$
$$k = 0, 1, 2, \ldots.$$

Using (7) and (18), we can write

$$\begin{aligned}A_k &= |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2\\ &\leq |x_{k+1} - y_k|^2/|x_k - y_k|^2\\ &= 1 + \left(\left|\nabla V_{p_k}(x_k)\right|^2/\left[\left|m(V_{p_k}(x_k))\right|^2|x_k - y_k|^2\right]\right)(|\nabla V_{p_k}(x_k)|^2\\ &\quad - 2\left\langle m(V_{p_k}(x_k)), x_k - y_k\right\rangle)\\ &= 1 - \left(\left\langle m\left(V_{p_k}(x_k)\right),\ x_k - y_k\right\rangle\right)^2/\left[|m\left(V_{p_k}(x_k)\right)|^2|x_k - y_k|^2\right] < 1.\end{aligned}$$

(ii) As long as $f(x)$ is regular function, the generalized gradient $\partial W_{p'(\in')}(x)$ is a weakly compact and convex set; see Ref. 22. Thus, $m(\partial W_{p'(\varepsilon')}(x))$ is well defined. Under (11) and (20), for any $y_k \in Pr'_X x_k,\ k = 0, 1, 2, \ldots,$ we obtain

$$\begin{aligned}0 \leq A'_k &= |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2 \leq |x_{k+1} - y_k|^2/|x_k - y_k|^2\\ &= (1/|x_k - y_k|^2)|y_k - x_k + W_{p'^k}(x_k)m(\partial W_{p^k}(x_k))/|m(\partial W_{p'^k}(x_k))|^2\\ &= (W_{p'^k}(x_k))^2/[|m(\partial W_{p'^k}(x_k))|^2|x_k - y_k|^2] + 1\\ &\quad + 2W_{p'^k}(x_k)(m(\partial W_{p'^k}(x_k)), y_k - x_k)/\\ &\quad [|m(\partial W_{p'^k}(x_k))|^2|x_k - yx_k|^2].\end{aligned}$$

Under the equality (19), we complete the proof:

$$\begin{aligned}0 \leq A'_k &\leq (W_{p'^k}(x_k))^2/[|m(\partial W_{p'^k}(x_k))|^2|x_k - y_k|^2] + 1\\ &\quad + 2(-W_{p'^k}(x_k))\, W_{p'^k}(x_k)/[|m(\partial W_{p'^k}(x_k))|^2|x_k - y_k|^2]\\ &= 1 - \frac{(W_{p'^k}(x_k))^2}{[|m(\partial W_{p'^k}(x_k))|^2|x_k - y_k|^2]}\\ &= 1 - (B'_k)^2 < 1,\end{aligned}$$

where

$$B_k' = (W_{p'^k}(x_k))/[|m(\partial\, W_{p'k}(x_k))||x_k - y_k|].$$

Thus, for every $x_k \in X_{\delta'}'\backslash X',\ k = 1,\ 2, \ldots,$ we obtain

$$0 \le A_k' < 1 - (B_k)^2 < 1. \tag{21}$$

Then, the procedure (15) is monotonous. From (i) and (ii), we conclude that the global minimization algorithm is monotonous. □

The next theorem refers to the disturbed case when we use the procedure (8). We denote by $\nabla\, V_{p_k}^{\varepsilon_k}(x)$ and $m(V_{p_k}^{\varepsilon_k}(x))$ the approximating values of $\nabla V_{p_k}(x)$ and $m(V_{p_k}(x))$, respectively.

Theorem 2.2.
Let the conditions of the Theorem 2.1 be fulfilled except (18) and (19).

(i) Consider the iterative procedure (8). If the p_k are chosen by the following inequalities:

$$\left|\nabla\, V_{p_k}^{\varepsilon_k}(x_k)\right|^2 \le (1+q)\left\langle m\left(V_{p_k}^{\varepsilon_k}(x_k)\right),\ x_k - x^*\right\rangle, \quad 0 < q < 1, \tag{22}$$

then the iterative procedure (8) is monotonous.

(ii) We consider also the nonlinear equation (9) and the iterative procedure (16) where $f(x)$ is a regular function defined in the Hilbert space H. Let there exist a δ'-neighborhood $X_{\delta'}'$ of the set X' for which $\langle \xi,\ x - y\rangle \ne 0,\quad \xi \in \partial W_{p'(\varepsilon')}(x),\quad y \in Pr_{X'}x,\quad x \in X_{\delta'}'\backslash X'$, where $W_{p'(\varepsilon')}(x)$ is defined by (10′). Let $p'^k(\varepsilon'^k)$ be any solution of the following inequality:

$$W_{p'^{k(\varepsilon'^k)}}(x_k) + \langle m(\partial W_{p'^k(\varepsilon'^k)}(x_k)),\, y_k - x_k\rangle < \varepsilon'^k, \tag{23}$$

where $y_k \in Pr_X' x_k$ are arbitrary chosen. If the initial point $x_0 \in X_{\delta'}'$ and if

$$\varepsilon'^k \le (q'/2) W_{p'k(\varepsilon'^k)}(x_k),\ 0 < q < 1,$$

then the following inequality:

$$|x_{k+1} - y_{k+1}| \le |x_k - y_k| \tag{24}$$

holds for every $y_k \in Pr_X' x_k, k = 0, 1, 2, \ldots$. If $W_{p'(\varepsilon')}(x)$ is a convex function, then the inequality (23) is sufficiently fulfilled with any $\varepsilon' > 0$. In this case, (23) is superfluous.

(i) and (ii) ensure that the algorithm for the global optimization problem is monotonous.

Proof.
(i) As was done in the proof of the Theorem 2.1, by (8) and (22) we have

$$\begin{aligned} A_k &= |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2 \\ &\le |x_{k+1} - y_k|^2/|x_k - y_k|^2 \\ &= 1 + \left(\left|\nabla V_{p_k}^{\varepsilon_k}(x_k)\right|^2 / \left[\left|m\left(V_{p_k}^{\varepsilon_k}(x_k)\right)\right|^2 |x_k - y_k|^2\right]\right)\left(\left|\nabla V_{p_k}^{\varepsilon_k}(x_k)\right|^2 \right. \\ &\quad \left. - 2\left\langle m\left(V_{p_k}^{\varepsilon_k}(x_k)\right),\ x_k - y_k\right\rangle\right) \\ &\le 1 + \left(\left|\nabla V_{p_k}^{\varepsilon_k}(x_k)\right|^2 / \left[\left|m(V_{p_k}^{\varepsilon_k}(x_k))\right|^2 |x_k - y_k|^2\right]\right)(1 + q - 2) \\ &\quad \left\langle m(V_{pk}^{\varepsilon_k}(x_k)),\ x_k - y_k\right\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} A_k &\le 1 - (1-q)\left(\left|\nabla V_{p_k}^{\varepsilon_k}(x_k)\right|^2 / \right. \\ &\quad \left. \left[\left|m\left(V_{p_k}^{\varepsilon_k}(x_k)\right)\right|^2 |x_k - y_k|^2\right]\right)\left\langle m\left(V_{p_k}^{\varepsilon_k}(x_k)\right), x_k - y_k\right\rangle < 1. \end{aligned} \tag{25}$$

(ii) As long as $f(x)$ is a regular function, the generalized gradient $\partial W_{p'(\varepsilon')}(x)$ is a weakly compact and convex set (see Ref. 22). Thus, $m(\partial W_{p'(\varepsilon')}(x))$ is well defined. Under (11) and (24), for any $y_k \in pr'_X x_k$, $k = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned} 0 \le A'_k &= |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2 \\ &\le |x_{k+1} - y_k|^2/|x_k - y_k|^2 = (1/|x_k - y_k|^2)|y_k - x_k \\ &\quad + W_{p'^k(\varepsilon'^k)}(x_k)\, m(\partial W_{p^{k(\varepsilon'^k)}}(x_k))/|m(\partial V_{p'^{k(\varepsilon'^k)}}(x_k))|^2 \\ &= (W_{p'^k(\varepsilon'^k)}(x_k))^2/[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2|x_k - y_k|^2] + 1 + 2W_{p'^k(\varepsilon'^k)} \\ &\quad \times (x_k)(m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k)), y_k - x_k)/[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2|x_k - yx_k|^2]. \end{aligned}$$

Under the inequality

$$\varepsilon'^k < (q'/2) W_{p'^{k(\varepsilon'^k)}}(x_k)$$

and (23), we complete the proof:

$$\begin{aligned} A'_k &\le \left(W_{p'^k(\varepsilon'^k)}(x_k)\right)^2 / \left[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2 |x_k - y_k|^2\right] + 1 \\ &\quad + 2\,(\varepsilon'^k - W_{p'^{k(\varepsilon'^k)}}(x_k))\, W_{p'^{k(\varepsilon'^k)}}(x_k) / \left[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2 |x_k - y_k|^2\right] \\ &= 1 - \left(W_{p'^k(\varepsilon'^k)}(x_k)\right)^2 / \left[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2 |x_k - y_k|^2\right] \\ &\quad + 2\,\varepsilon'^k \left(W_{p'^k(\varepsilon'^k)}(x_k)\right) / \left[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2 |x_k - y_k|^2\right] \\ &\le 1 - (1 - q')\left(W_{p'^k(\varepsilon'^k)}(x_k)\right)^2 / \left[|m(\partial W_{p'^{k(\varepsilon'^k)}}(x_k))|^2 |x_k - y_k|^2\right] \\ &= 1 - (1 - q')(B'_k)^2 < 1. \end{aligned}$$

Thus, for every $x_k \in X'_{\delta'} \backslash X'$, $k = 1, 2, \ldots,$ we obtain

$$0 \le A'_k < 1 - (1 - q')(B'_k)^2 < 1. \tag{26}$$

Then, the procedure (16) is monotonous and the global algorithm is monotonous. □

We use the following technical lemma (see Ref. 2).

Lemma 2.1. Let the conditions of the Theorem 1.1 be fulfilled and let $x(\mathrm{t})$ be a solution of (5) with initial condition $x(0) = x_k$ for which $x(T_k) = x^* \equiv X \equiv \{y \in H | \nabla V_p(y) = 0\}$ is the unique minimum point of the problem (1). Suppose that P is a compact set, $(\partial^2 V_p(x))^* \nabla V_p(x)/|\nabla V_p(x)|$ is uniformly continuous at every point (p, x^*), $p \in P$, and the following limit is single-valued:

$$\lim_{p \to p^*, x \to x^*} (\partial^2 V_p(x))^* \nabla V_p(x)/|\nabla V_p(x)| = u(p^*) \neq 0. \tag{27}$$

Then, for every $\varepsilon > 0$ and all sufficiently large k, the following inequalities are valid:

$$\begin{aligned}
&\left|\left\langle m(V_{p_k}(x_k))/|\nabla V_{p_k}(x_k)|, x_k - x^*\right\rangle\right| \\
&\quad \ge \int_0^{T_k} ds/|\nabla V_{p_k}(x(s))| \left(1 - \varepsilon/|u(p^*)| - \varepsilon^2 - \varepsilon|u(p^*)|\right), \\
&\left[|m(V_{p_k}(x_k))|/|\nabla V_{p_k}(x_k)|\right] |x_k - x^*| \\
&\quad \le \int_0^{T_k} [1/|\nabla V_{p_k}(x(s))|]\, ds \left(1 + \varepsilon/|u(p^*)| + \varepsilon^2 + \varepsilon|u(p^*)|\right).
\end{aligned}$$

We present next two convergence theorems.

Theorem 2.3.

(i) Consider the minimizing problem (1) and the iterative procedure (7), where $x \in H$, $\nabla V_p(x)$ is Lipschitz continuous, $m(V_p(x))$ is the metric projection of the origin into the values of the u.s.c. multifunction $(\partial^2 V_p(x))^* \nabla V_p(x)$, $x^* \equiv X \equiv \{y \in H | \nabla V_p(y) = 0\}$ is the unique isolated local minimum point in the cone of decreasing direction for the given initial point x_0^1, the p_k belong to a compact set P and are chosen by

$$|\nabla V_{p_k}(x_k)|^2 = \left\langle m(V_{p_k}(x_k)), x_k - x^*\right\rangle, \tag{28}$$

$k = 0, 1, 2, \ldots$ Let for every $\delta > 0$, there exists $\eta(\delta) > 0$ for which $|m(V_{p_k}(x))| \ge \eta(\delta)$, $|x - x^*| \ge \delta$, $k = 0, 1, 2, \ldots.$ Let $(\delta^2 V_p(x))^* \nabla V_p(x)/|\nabla V_p(x)|$ be uniformly continuous at every point (p, x^*), $p \in P$ and the limit (27) is single-valued.

If there exists a solution of (5) for every p_k, then the procedure (7) has a superlinear rate of convergence.

(ii) Consider also the nonlinear equation (9) where $f(x)$ is a regular function defined on the Hilbert space H and the Lyapunov function $W_{p'}(x)$ is from (10). Let the conditions of Theorem 1.2(ii) be fulfilled. Let $m(\partial W_{p'^k}(x_k))$ be a continuous function at the solutions set X. Let the solution set $X' \neq \emptyset$ of (9) coincide with the set for which origin belongs to $\partial W_{p'}(x)$, i.e. $X' = X'_1$. Let there exist a δ'-neighborhood $X'_{\delta'}$ of the set X' for which $x_0 \in X'_{\delta'} \backslash X'$ and $\langle \xi, x - y \rangle \neq 0, \xi \in \partial W_{p'}(x)$, $y \in Pr'_X x$, $x \in X'_{\delta'} \backslash X'$, and

$$\inf \lim_{k\to\infty} |m(\partial W_{p'^k}(y_k))| \neq 0, \tag{29}$$

where inf is taken at all sequences $y_k \in X'_{\delta'} \backslash X'$, $k = 1, 2, \dots$. Consider the process (15). Let p'^k be chosen under the (19). We recall that $m(\partial W_{p'^k}(x))$ is a continuous function at the solution set X'. Then, the procedure (15) has a supergeometric convergence rate and therefore the global optimization algorithm has a supergeometric rate of convergence.

Proof.

(i) According to Ref. 30, the iterative procedure (7) has a superlinear rate of convergence if

$$\lim_{k\to\infty} |x_{k+1} - x^*|/|x_k - x^*| = 0.$$

We have (see the proof of Theorem 1.1)

$$\begin{aligned} A_k &- |x_{k+1} - x^*|^2/|x_k - x^*|^2 \\ &\leq 1 - \big(\langle m(V_{p_k}(x_k)), x_k - x^*\rangle\big)^2 \big/ \Big[\big|m\big(V_{p_k}(x_k)\big)\big|^2 |x_k - x^*|^2\Big] \\ &= 1 - (B_k)^2. \end{aligned}$$

Thus, $B_k \leq 1$ and, under the conditions of the theorem, the limit of B_k is equal to 1, i.e.,

$$\lim_{k\to\infty} B_k = \lim_{k\to\infty} \big|\langle m(V_{p_k}(x_k)), x_k - x^*\rangle\big| \big/ \big[\big|m(V_{p_k}(x_k))\big| |x_k - x^*|\big] = 1. \tag{30}$$

By the Lemma 2.1, for any $\varepsilon > 0$ and all sufficiently large k, we have

$$B_k \geq \frac{\int_0^{T_k} [1/|\nabla V_{p_k}(x(s))|]\, ds \,\big(1 - \varepsilon/|u(p^*)| - \varepsilon^2 - \varepsilon|u(p^*)|\big)}{\int_0^{T_k} [1/|\nabla V_{p_k}(x(s))|]\, ds \,\big(1 + \varepsilon/|u(p^*)| + \varepsilon^2 + \varepsilon|u(p^*)|\big)}.$$

Thus, we can write

$$\lim_{k\to\infty} B_k \geq \big(1 - \varepsilon/|u(p^*)| - \varepsilon^2 - \varepsilon|u(p^*)|\big) \big/ \big(1 + \varepsilon/|u(p^*)| + \varepsilon^2 + \varepsilon|u(p^*)|\big).$$

As long as $\varepsilon > 0$ is arbitrarily chosen, the proof is complete and (7) has superlinear rate of convergence.

(ii) According to Ref. 30 the convergence rate is supergeometric if, for all $y_k \in Pr'_X x_k$, $k = 0, 1, \ldots,$ it has

$$\lim_{k\to\infty} A'_k = \lim_{k\to\infty} |x_{k+1} - y_{k+1}|^2/|x_k - y_k|^2 = 0.$$

Under the continuity of $m(\partial W_{p'^k}(x))$ and (19) we obtain

$$\lim_{k\to\infty} B'_k \geq \lim_{k\to\infty} |m(\partial W_{p'^k}(x_k(t_k)))|/|m(\partial W_{p'^k}(x_k))| = r = 1.$$

Under (21), we have $B'_k \leq 1$ and by (29) we obtain

$$0 \leq \limsup_{k\to\infty} A'_k \leq 1 - \liminf_{k\to\infty} (B_k)^2 < 1 - r < 1. \tag{31}$$

By (31), we obtain

$$\lim_{k\to\infty} A'_k \leq 1 - \lim_{k\to\infty} (B'_k)^2 = 1 - 1 = 0;$$

consequently, the procedure (15) has a supergeometric rate of convergence. It follows then from (i) and (ii) that the global minimization algorithm will have supergeometric convergence rate.

□

Theorem 2.4.

(i) Let the conditions of Theorem 2.2 (i) and Theorem 2.3 (i) be fulfilled. Suppose that, if the p_k are chosen by the following additional inequalities:

$$|\nabla V^{\varepsilon_k}_{p_k}(x_k)|^2 \geq q \left\langle m(V^{\varepsilon_k}_{p_k}(x_k)),\ x_k - y_k \right\rangle, \quad 0 < q < 1, \tag{32}$$

and ε'^k is chosen under the following inequality:

$$\varepsilon'^k < (q/2)\, W_{p'^k(\varepsilon'^k)}(x_k).$$

Then, the iterative procedure (8) has an asymptotic linear rate of convergence with a constant $\sqrt{1 - q(1 - q)}$.

(ii) Consider also the nonlinear equation (9), where $f(x)$ is a regular defined on Hilbert space H and the Lyapunov function $W_{p'(\varepsilon')}(x)$ is from (10′). Let the conditions of Theorem 1.2 (ii) be fulfilled. Let the solution set $X' \neq \emptyset$ of (9) coincide with the set for which the origin belongs to the subdifferential $\partial W_{p'(\varepsilon')}(x)$, i.e. $X' = X'_1$. Let there exist a δ'-neighborhood $X'_{\delta'}$ of the set X' for which $x_0 \in X'_{\delta'} \backslash X'$ and

$$\langle \xi, x - y \rangle \neq 0, \quad \xi \in \partial W_{p'(\varepsilon')}(x),\ y \in pr'_X x, \quad x \in X'_{\delta'} \backslash X', \text{ and}$$
$$\inf \lim_{k\to\infty} |m(\partial W_{p'^k(\varepsilon'^k)}(yk))| \neq 0, \tag{33}$$

where inf is taken at all sequences $y_k \in X'_{\delta'} \backslash X'$, $k = 1, 2, \ldots$. Consider the process (16). Let ε'^k and $p'^k(\varepsilon'^k)$ be chosen under (23) and the following inequalities:

$$\varepsilon'^k < (q'^k/2) W_{p'^k(\varepsilon'^k)}(x_k).$$

Then, the procedure (16) has a geometric convergence rate and from (i) and (ii) the global minimization algorithm has a geometric convergence rate.

Proof.

(i) Under (25) and (32), we have

$$A_k \leq 1 - (1-q) \left(\left| \nabla V_{p_k}^{\varepsilon_k}(x_k) \right|^2 / [|m(V_{p_k}^{\varepsilon_k}(x_k))|^2 |x_k - y_k|^2] \right) \langle m(V_{p_k}^{\varepsilon_k}(x_k)),\ x_k - y_k \rangle$$
$$\leq 1 - q(1-q) \langle m\left(V_{p_k}^{\varepsilon_k}(x_k)\right),\ x_k - y_k \rangle^2 / \left[|m(V_{p_k}^{\varepsilon_k}(x_k))|^2 |x_k - y_k|^2 \right]$$
$$= 1 - q(1-q) B_k^2.$$

According to Lemma 2.1,

$$\lim_{k\to\infty} B_k \leq \lim_{k\to\infty} \langle m\left(V_{p_k}^{\varepsilon_k}(x_k)\right),\ x_k - y_k \rangle / \left[|m(V_{p_k}^{\varepsilon_k}(x_k))|^2 |x_k - y_k| \right] = 1.$$

Thus,

$$\lim_{k\to\infty} \sqrt{A_k} = \lim_{k\to\infty} |x_{k+1} - y_{k+1}| / |x_k - y_k| \leq \sqrt{1 - q(1-q)}.$$

(ii) According to Ref. 30, the convergence rate is geometric if, for all $y_k \in pr'_X x$, $k = 0, 1, \ldots$, we have

$$\lim_{k\to\infty} A'_k = \lim_{k\to\infty} |x_{k+1} - y_{k+1}|^2 / |x_k - y_k|^2 \leq \text{const} < 1.$$

For any k, if $x_k \in X'$, the statement of the theorem is trivial. Let $x_k \notin X'$, $k = 0, 1, \ldots$. As long as $x \in X'_{\delta'}$, under Theorem 2.2 [see (24)] we obtain that, for all $k = 0, 1, \ldots$, $x_k \in X'_{\delta'} \backslash X'$. By the conditions of the theorem (see Theorem 1.2), for every initial point $x_k(0) = x_k$, there exists a moment $T_k \leq W_{p'^k(\varepsilon'^k)}(x_k) < \infty$ for which the respective solution $x_k(\cdot)$ of (12) satisfies the inclusion $x_k(T_k) \in X'$. Now, for all $k = 0, 1 \ldots$, we can write as in the previous theorem the following inequalities:

$$B'_k = W_{p'^k}(\varepsilon'^k)(x_k) / [|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| |x_k - y_k|]$$
$$\geq - \int_0^{Tk} \langle m(\partial W_{p'^k(\varepsilon'^p)}(x_k(t))), \dot{x}_k(t) \rangle\, dt / [|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| |x_k - x_k(T_k)|]$$
$$= - \int_0^{Tk} \langle m(\partial W_{p'^k(\varepsilon'^k)}(x_k(t))), \dot{x}_k(t) \rangle\, dt / [|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| | \int_0^{Tk} \dot{x}_k(t) dt |]$$

$$\geq T_k/[|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| \int_0^{T_k} (1/|m(\partial W_{p'^k(\varepsilon'^k)}(x_k(t)))|)dt].$$

For every $\xi \in \partial W_{p'^k(\varepsilon'^k)}(y)$, we have

$$|\xi| \geq |m(\partial W_{p'^k(\varepsilon'^k)}(y))|.$$

By the mean theorem, $t_k \in (0, T_k)$, and (33), we obtain

$$\begin{aligned}\liminf_{k\to\infty} B'_k &\geq \liminf_{k\to\infty} T_k/\left[|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| \int_0^{T_k} (1/|m(\partial W_{p'^k(\varepsilon'^k)}(x_k(t)))|)dt\right] \\ &= \liminf_{k\to\infty} T_k/\left[|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| \int_0^{T_k} (1/|\xi(t_k)|)\, dt\right] \\ &\geq \liminf_{k\to\infty} |m(\partial W_{p'^k(\varepsilon'^k)}(x_k(t_k)))|/|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| = \sqrt{r} \neq 0.\end{aligned}$$

Under (26), we have $B'_k \leq 1$ and by (33) we obtain

$$0 \leq \limsup_{k\to\infty} A'_k \leq 1 - (1-q')\liminf_{k\to\infty}(B'_k)^2 < 1 - (1-q')r < 1. \qquad (34)$$

Thus, the iterative procedure (32) has at least a geometric convergence rate with a parameter not greater than $\sqrt{1-r+rq'}$. Hence, it follows from (i) and (ii) that the global minimization algorithm is with a geometric convergence rate. □

Corollary 2.1. Let the conditions of Theorem 2.4 be fulfilled. Let suppose that, in Theorem 2.4 (ii), $m(\partial W_{p'^k(\varepsilon'^k)}(x))$ is a continuous function at the solutions set X' and $\varepsilon'^k < (q'^k/2)\, W_{p'^k(\varepsilon'^k)}(x_k)$, where $0 < q' < 1$. Then, the procedure (16) has a supergeometric convergence rate. Therefore, the global minimization algorithm has a geometric convergence rate for the procedure (8) and a supergeometric rate of convergence for the procedure (16).

Proof. According to Ref. 30, the convergence rate is supergeometric if, for all $y_k \in pr'_X x_k$, $k = 0, 1, \ldots,$

$$\lim_{k\to\infty} A'_k = \lim_{k\to\infty} ||x_{k+1} - y_{k+1}||^2/||x_k - y_k||^2 = 0.$$

Under the additional condition for $m(\partial W_{p'^k(\varepsilon'^k)}(x))$ to be continuous and (23), we obtain

$$\lim_{k\to\infty} B'_k \geq \lim_{k\to\infty} |m(\partial W_{p'^k(\varepsilon'^k)}(x_k(t_k)))|/|m(\partial W_{p'^k(\varepsilon'^k)}(x_k))| = 1.$$

But from (34), then

$$\lim_{k\to\infty} A'_k \leq 1 - \lim_{k\to\infty} (B'_k)^2(1-q'^k) = 1 - r \lim_{k\to\infty}(1-q'^k) = 0.$$

□

3. Numerical Examples

Example 3.1. Unconstrained Global Optimization Ease.

$$\min_{(x,y)} V(x, y) = (4 - 2.1x^2 + (1/3)x^4)x^2 + xy + 4(y^2 - 1)y^2.$$

The solution points are two:

$$(x_1^*, y_1^*) = (-0.09, 0.71), \quad (x_2^*, y_2^*) = (0.09, -0.71),$$

where

$$V(-0.09, 0.71) = V(0.09, -0.71) = -1.03.$$

Different initial points give one of the solution points. The program solving the unconstrained global optimization case will be included in the next package ftp://plato.asu.edu/pub/other software/. There will be also a link to it on http://plato.la.asu.edu/topics/problems/global.html. The problem was solved with exactness 10^{-6} in less then 2 sec.

The same problem with the same exactness was solved with CG-plus (conjugate gradient algorithm) in 6 sec; see http://www-neos.mcs.anl.gov/neos/solvers/UCO:CGPLUS/. The same problem with the same exactness was solved with NMTR (trust region version of Newton's method) in 3 sec; see

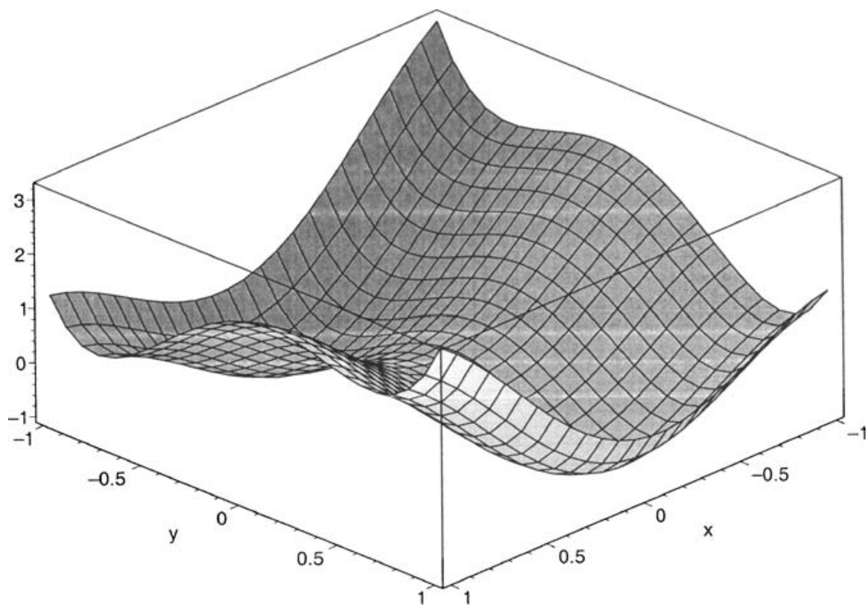

Fig. 1. Uncostrained optimization example.

http://www-neos.mcs.anl.gov/neos/solvers/UCO:NMTR/. We can generalize that the program realization of our algorithm for the unconstrained global optimization case is working better (faster) than the program realizations of the other algorithms for the continuous cases. The smoothness is not necessary.

Example 3.2. Constrained Global Optimization Ease.

$$\min_{-1 \le x, y, z \le 1} -|V(x, y, y)|,$$

where

$$V(x, y, z) = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (z - x)(z - y)(y - x).$$

To solve this problem, we solve seven subproblems. One subproblem refers to the inner points of the cube; six subproblems refer to its six boundary sides. Then, the global solution is the minimum value of the global solutions of the seven subproblems. The solution points are six:

$$(x_1^*, y_1^*, z_1^*) = (-1, 1, 0), \quad (x_2^*, y_2^*, z_2^*) = (-1, 0, 1),$$
$$(x_3^*, y_3^*, z_3^*) = (0, -1, 1), \quad (x_4^*, y_4^*, z_4^*) = (0, 1 - 1),$$

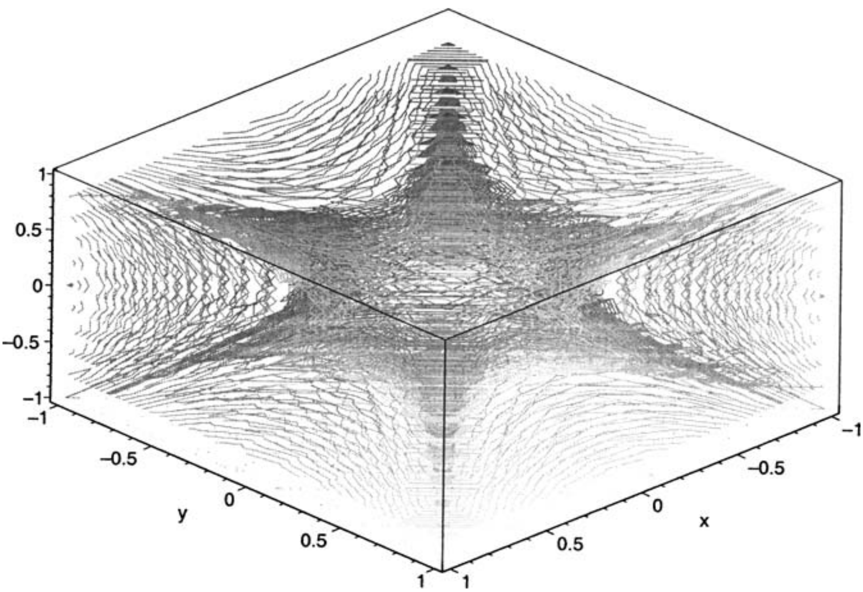

Fig. 2. Costrained optimization example.

$$(x_5^*, y_5^*, z_5^*) = (1, -1, 0), \quad (x_6^*, y_6^*, z_6^*) = (1, 0, -1).$$

The algorithm finds only one of them.

Remark 3.1. This example is important for the solution of a more general problem: $\min_{x \in E} -|V(x)|$, where V is a generalized Vandermonde determinant and E is a compact set. This is a general approach of finding Fekete points.

The problem was solved with exactness 10^{-6} in 18 sec. The same problem was solved with the same exactness with BLMVM (limited memory variable metric method) in 10 sec; see http://www-neos.mcs.anl.gov/neos/solvers/BCO:BLMVM/. The same problem was solved with the same exactness with L-BFGS-B (limited memory BFGS algorithm) in 7 sec; see http://www-neos.mcs.anl.gov/neos/solvers/BCO:L-BFGS-B/ The same problem was solved with the same exactness with TRON (trust region Newton method) in 5 sec; see http://www-neos.mcs.anl.gov/neos/solvers/BCO:TRON/. We generalize that the program realization of our algorithm for the constrained case is slower than the program realizations of the other algorithms for the continuous nonsmooth cases.

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