ON THE LIMIT UNDER SCALING OF POLYNOMIAL LAGRANGE INTERPOLATION ON ANALYTIC MANIFOLDS

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Abstract

We consider interpolation at points on an analytic manifold $\mathcal{M} \subset I\!R^d$ and describe the limiting behaviour of the associated fundamental Lagrange polynomials as the points coalesce.

1 Polynomial interpolation on Spheres

Univariate polynomial interpolation is a classical and much studied subject. In recent years there has been a growing interest in the corresponding multivariate problem (cf. [3, 4, 5, 6, 7] and references therein) including the important special case of interpolation at points restricted to lie on special surfaces such as a sphere (cf. [2, 8, 9]).

In our recent paper [2] we described the interesting behaviour of polynomial interpolation on spheres as the points coalesce under an angular scaling, a study

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

motivated in part by an application to radial basis functions (cf. [8]). Subsequently we have discovered that this is a special instance of a much more general phenomenon for polynomial interpolation on analytic manifolds, and it is the generalization that is the subject of this current work.

To make our notions more precise, we begin with a description of multivariate polynomial interpolation.

Suppose that $\mathcal{M} \subset I\!R^d$ is compact (we shall later assume that $\mathcal{M}$ is in fact (a piece of) an analytic manifold). We shall denote by $\mathcal{P}_n(\mathcal{M})$ the space of the polynomials of degree at most n in d real variables, restricted to $\mathcal{M}$. As a vector space, $\mathcal{P}_n(\mathcal{M})$ has a dimension d_n and a basis $\{p_1, ..., p_{d_n}\}$.

Now, given a set $X = \{x^1, ..., x^{d_n}\} \subset \mathcal{M}$, of d_n distinct points and a function $f : \mathcal{M} \to I\!R$, the polynomial interpolation problem, is to find a polynomial $p \in \mathcal{P}_n(\mathcal{M})$ such that

$$p(x) = f(x) \ , \qquad x \in X \ .$$

If $p(x)$ is written in the form $p = \sum_{i=1}^{d_n} a_i p_i$, then this amounts to solving the system

$$\sum_{i=1}^{n} a_i p_i(x^j) = f(x^j) \ , \qquad 1 \le j \le d_n \ ,$$

for some coefficients a_i, which is uniquely possible provided the associated Vandermonde determinant

$$VDM(x^1, ..., x^{d_n}) = det \left[p_i(x^j)\right]_{1 \le i,j \le d_n} \neq 0 \ .$$

We say then that the interpolation problem is *unisolvent*. If it is indeed the case, then we may form the *fundamental Lagrange polynomials* defined by

$$l_i(x; X) = \frac{VDM(x^1, ..., x^{i-1}, x, x^i, ..., x^{d_n})}{VDM(x^1, ..., x^{d_n})} \ , \qquad 1 \le i \le d_n \ . \tag{1}$$

They have the property that $l_i(x^j; X) = \delta_{i,j}, \quad 1 \le i,j \le d_n$, and hence the interpolating polynomial may then be succinctly written as

$$p(x) = \sum_{i=1}^{d_n} f(x^i) l_i(x; X) \ .$$

Now to describe the result of [2], suppose that $\mathcal{M} = S^{d-1} = \{x \in I\!R^d \ : \ |x|^2 = 1\}$ is the unit sphere, and write the points of X as

$$X = \{(x^k, y^k) \in S^{d-1}, \ 1 \le k \le d_n\} \ ,$$

where $x^k \in I\!R^{d-1}$ and $y^k \in I\!R$. We may express each point (x^k, y^k) in spherical coordinates by

$$\begin{aligned}
x_2^k &= \cos(\theta_1^k)\sin(\theta_2^k)\\
x_3^k &= \cos(\theta_1^k)\cos(\theta_2^k)\sin(\theta_3^k)\\
\vdots &= \vdots\\
x_1^k &= \cos(\theta_1^k)\cos(\theta_2^k)\cdots\cos(\theta_{d-2}^k)\cos(\theta_{d-1}^k)\\
y^k &= \sin(\theta_1^k)
\end{aligned}$$

We refer to this choice of coordinates by $(x^k, y^k) = (x^k(\theta^k), y(\theta^k)),\ \ \theta^k \in I\!R^{d-1}$.

Supposing that the associated Vandermonde determinant is non-zero, to these points we associate a *second interpolation problem* for polynomials of degree n on the *paraboloid*

$$V^{d-1} = \{\theta_d = \theta_1^2 + \ldots + \theta_{d-1}^2\} \subset I\!R^d ,$$

with interpolation point set

$$\Theta = \{(\theta^k, |\theta^k|^2)\ :\ 1 \le k \le d_n\} .$$

(We emphasize that these are the angles of our original spherical point set!)

If this second problem is again unisolvent, we denote the corresponding Lagrange polynomials by $L_i(\theta),\ 1 \le i \le d_n$.

Now, consider the points, $X \subset S^{d-1}$ scaled by an angular factor t:

$$\begin{aligned}
x_2^k &= \cos(t\theta_1^k)\sin(t\theta_2^k)\\
x_3^k &= \cos(t\theta_1^k)\cos(t\theta_2^k)\sin(t\theta_3^k)\\
\vdots &= \vdots\\
x_1^k &= \cos(t\theta_1^k)\cos(t\theta_2^k)\cdots\cos(t\theta_{d-2}^k)\cos(t\theta_{d-1}^k)\\
y^k &= \sin(t\theta_1^k)\ ;
\end{aligned}$$

which we refer to as X_t. Then the main result of [2] is

Theorem 1

$$\lim_{t\to 0} l_i(x(t\theta); X_t) = L_i(\theta),\ \ 1 \le i \le d_n .$$

Of course, we might naturally ask what is the effect of the specific type of scaling on the limiting interpolation. As a discussion of other scalings also leads us naturally to the case of surfaces more general than just spheres, we first consider this problem.

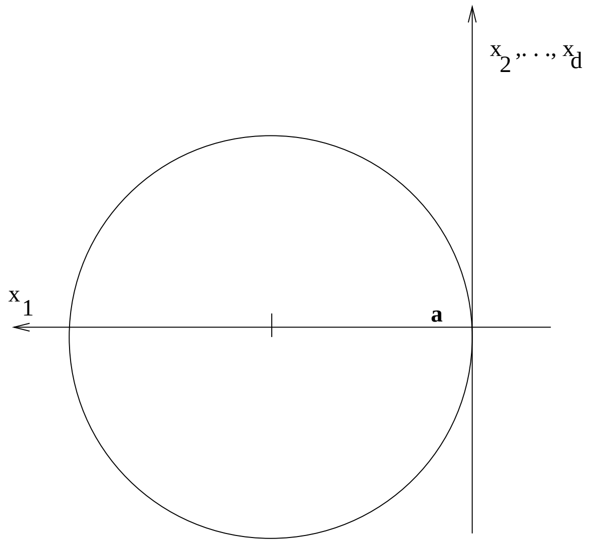

Figure 1: The tangent plane at the point a

Hence, let $a \in S^{d-1}$ be a point to which we wish to have our interpolation points coalesce. Without loss of generality, we may assume that $a = (1, 0, ..., 0)$. We parametrize the hemi-sphere with a as a pole, by considering it as a function above the tangent plane at a. If $x_2, ..., x_d$ are coordinates on this tangent plane with a as the origin, then the equation of the hemi-sphere is

$$x_1 = 1 - \sqrt{1 - (x_2^2 + \cdots + x_d^2)}\,. \tag{2}$$

Now, suppose that our points x^k, $1 \le k \le d_n$ are in the hemi-sphere. If we scale $x_2, ..., x_d$ by the factor t so that

$$x_1(t) = 1 - \sqrt{1 - t^2(x_2^2 + \cdots + x_d^2)} = \frac{t^2}{2}(x_2^2 + \cdots + x_d^2) + \frac{t^4}{8}(x_2^2 + \cdots + x_d^2) + \cdots$$

then, as $t \to 0$, these points again coalesce at a, however at different rates than under angular scaling.

We shall refer to the present scaling as *tangential scaling.*

In order to understand the behaviour of the interpolation problem as $t \to 0$, we work with a particular basis of polynomials. Any polynomial $p \in \mathcal{P}_n(I\!R^d)$ of degree n in d real variables, may be written in the form

$$p(x_1, ..., x_d) = \sum_{k=0}^{n} a_k(x_1, ..., x_{d-1}) x_d^k\,,$$

where $a_k \in \mathcal{P}_{n-k}(I\!R^{d-1})$.

On the sphere, S^{d-1}, $x_d^2 = 1 - x_1^2 - \cdots - x_{d-1}^2$ and thus on replacing *even* powers of x_d by the previous expression we have that any $p \in \mathcal{P}_n(S^{d-1})$ may be written in the form

$$p(x_1, ..., x_d) = p_1(x_1, ..., x_{d-1}) + x_d p_2(x_1, ..., x_{d-1}) \ ,$$

for some $p_1 \in \mathcal{P}_n(I\!R^{d-1})$ and $p_2 \in \mathcal{P}_{n-1}(I\!R^{d-1})$.

It follows that a basis for $\mathcal{P}_n(S^{d-1})$ is

$$\{(x_1, ..., x_{d-1})^\alpha \mid |\alpha| \le n\} \cup \{x_d(x_1, ..., x_{d-1})^\beta \mid |\beta| \le n-1\} \ .$$

To calculate the $\lim_{t\to 0} l_i(x(t); X_t)$ we use formula (1) and expand $x_1^k(t)$ in a Taylor series

$$x_1(t) = \frac{t^2}{2}(x_2^2 + \cdots + x_d^2) + \frac{t^4}{8}(x_2^2 + \cdots + x_d^2) + \cdots$$

Each monomial $(x_1, ..., x_{d-1})^\alpha$ becomes

$$x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} = \frac{t^{2\alpha_1}}{2^{\alpha_1}} \left\{ (x_2^2 + \cdots + x_d^2)^{\alpha_1} x_2^{\alpha_2} \cdots x_{d-1}^{\alpha_{d-1}} + \mathcal{O}(t^2) \right\} \ ; \tag{3}$$

while each $(x_1, ..., x_{d-1})^\beta x_d$ becomes

$$(x_1^{\beta_1} \cdots x_{d-1}^{\beta_{d-1}}) x_d = \frac{t^{2\beta_1}}{2^{\beta_1}} \left\{ (x_2^2 + \cdots + x_d^2)^{\beta_1} x_2^{\beta_2} \cdots x_{d-1}^{\beta_{d-1}} x_d + \mathcal{O}(t^2) \right\} \ . \tag{4}$$

Now the ith row of the determinant $VDM(x^1, \cdots, x^{d_n}) = \det[p_i(x^j)]$ consists of the ith basis function (in this case ith monomial) evaluated at all the points. Hence, the row corresponding to $(x_1, \cdots, x_{d-1})^\alpha$ has a common factor of $t^{2\alpha_1}/2^{\alpha_1}$ and that corresponding to $x_d(x_1, \cdots, x_{d-1})^\beta$ has a common factor of $t^{2\beta_1}/2^{\beta_1}$. Since each such factor appears in both the numerator and denominator of (1), we may cancel off the powers of t, $\frac{t^{2\alpha_1}}{2^{\alpha_1}}$, $\frac{t^{2\beta_1}}{2^{\beta_1}}$ in (1). Then, letting $t \to 0$ we have, from (3) and (4), that $\lim_{t\to 0} l_i(x(t); X_t)$ is a ratio of Vandermonde determinants (1), for the basis

$$\{(x_2^2 + \cdots + x_d^2)^{\alpha_1} x_2^{\alpha_2} \cdots x_{d-1}^{\alpha_{d-1}} \mid |\alpha| \le n\} \cup \{(x_2^2 + \cdots + x_d^2)^{\beta_1} x_2^{\beta_2} \cdots x_{d-1}^{\beta_{d-1}} x_d \mid |\beta| \le n-1\} \ ;$$

and the points

$$(x^k)' := (x_2^k, ..., x_d^k), \ \ 1 \le k \le d_n \ .$$

In [2] it is shown that this is precisely a basis for $\mathcal{P}(I\!R^d)$ restricted to the paraboloid $x_1 = x_2^2 + \cdots + x_d^2$, i.e. essentially to V^{d-1} mentioned above.

The interpolation points of the limiting interpolation problem are

$$(y^k, x_2^k, ..., x_d^k), \quad 1 \le k \le d_n,$$

where $y^k := (x_2^k)^2 + \cdots + (x_d^k)^2$.

Clearly, we may think of these as the projections of the original points onto the tangent plane at a, and then lifted to the paraboloid.

In summary, we have that the interpolation problem on the sphere S^{d-1} becomes, *in the limit as the points coalesce*, an interpolation problem on the paraboloid V^{d-1} with interpolation points the original spherical angles, in the case of angular scaling and with interpolation points the tangent plane coordinates of the original points in the case of tangential scaling.

The reader who finds the presentation somewhat sketchy may refer to [2] for supporting details.

2 The general case

At this point we are ready to discuss a more general limiting process. Suppose that $\pi : S^{d-1} \to I\!R^{d-1}$ is an analytic coordinate system on S^{d-1}, valid on a neighbourhood, $\mathcal{U}$, of our point a, so that a is the origin: i.e. $\pi(a) = 0$. We of course assume that all of our points $x^k \in \mathcal{U}$.

Let $y^k := \pi(x^k)$ be the coordinates of x^k. Then the points

$$x^k(t) := \pi^{-1}(t\, y^k)\ , \tag{5}$$

have the property that $x^k(1) = x^k$ and $\lim_{t\to 0} x^k(t) = a$.

Now, we ask for the limit of the interpolation problem as $t \to 0$ under this general scaling. For the answer we must understand what is really happening when we expand the monomials in t as in (2). To recapitulate, since the leading factor of $t^{2\alpha_1}$ multiplies the entire row in the Vandermonde determinant corresponding to that basic element, and since this is the case in both the numerator and denominator of the expression for the Lagrange polynomials (1), we may *cancel* this factor. This leaves, as $t \to 0$, in that case

$$(x_2^2 + \cdots + x_d^2)^{\alpha_1} x_2^{\alpha_2} \cdots x_{d-1}^{\alpha_{d-1}} ,$$

which is precisely the *leading homogeneous term* in the Taylor expansion at the origin of the function

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{d-1}^{\alpha_{d-1}} ,$$

where $x_1 = x_1(x_2, ..., x_d)$ is given by (2).

In [4], the authors refer to the leading homeogenous term of a function analytic in a neighbourhood of the origin, as the *least* of f and write $f_\downarrow$ for this term. More generally if W is a vector space of such functions, then

$$W_\downarrow := \mathrm{span}\{f_\downarrow \ : \ f \in W\} .$$

They proved the following theorem:

Theorem 2 ([4]) *If W is finite dimensional, then $W_\downarrow$ is a homogeneous polynomial space of the same dimension as W.*

Using this result, we can state the following result.

Theorem 3 *Suppose that $X = \{x^k \ : \ 1 \le k \le d_n\} \subset S^{d-1}$, is a set of distinct points which is made to coalesce at the point $a \in S^{d-1}$ under the scaling given by (5). Denote the scaled points by X_t.*

We may consider two interpolation problems.

1. *Interpolation at the x^k by polynomials from $\mathcal{P}_n(S^{d-1})$. If this problem is unisolvent, we denote the corresponding Lagrange polynomials by $l_i(\cdot; X)$.*
2. *Interpolation at the points $Y = \{y^k \in I\!R^{d-1}\}$ by polynomials from $(\mathcal{P}_n(S^{d-1}) \circ \pi^{-1})_\downarrow$. If the problem is unisolvent, we denote the corresponding Lagrange polynomials by $L_i(\cdot; Y)$.*

Then, in the case that the second problem is unisolvent

$$\lim_{t \to 0} l_i(x(t); X_t) = L_i(y; Y) ,$$

where $y := \pi(x(1))$.

It is instructive to note that in general the resulting basis $(\mathcal{P}(S^{d-1}) \circ \pi^{-1})_{\downarrow}$ is not necessarily $\mathcal{P}(V^{d-1})$ as in the case of angular and tangential scaling.

Example 1 Consider the sphere $S^2 \subset I\!R^3$ and the point $a = (1,0,0)$. Let $\pi(x,y,z) = (y+z, y+2z)$ so that $\pi^{-1}(y,z) = (1-\sqrt{1-(2y-z)^2-(z-y)^2}, 2y-z, z-y)$. Then as

$$1-\sqrt{1-(2y-z)^2-(z-y)^2} = \frac{1}{2}(5y^2-6yz+2z^2)+\cdots$$

it follows that, e.g. $(\mathcal{P}_1(S^2) \circ \pi^{-1})_{\downarrow}$ is $\mathrm{span}\{1, y, z, 5y^2-6yz+2z^2\}$ which differs from $\mathcal{P}_1(V^2) = \mathrm{span}\{1, y, z, y^2+z^2\}$.

$\diamond\diamond$

Now, the above theorem makes no use of the special properties of a sphere and hence the same result holds with the sphere replaced by a *general analytic manifold*. To be precise we may state the following theorem which is the natural extension of the previous one.

Theorem 4 *Suppose that $\mathcal{M} \subset I\!R^d$, is an analytic manifold of dimension m, with local analytic coordinates $\pi : \mathcal{M} \to I\!R^m$, valid in a neighbourhood, $\mathcal{U}$, of $a \in \mathcal{M}$. Suppose further that $X = \{x^k \ : \ 1 \le k \le d_n := dim(\mathcal{P}_n(\mathcal{M}))\} \subset \mathcal{U}$ is a set of distinct points which is made to coalesce at the point a under the scaling given by*

$$X_t = \{x^k(t) := \pi^{-1}(t\,y^k) \ : \ 1 \le k \le d_n\}\,.$$

Here the y^k are the coordinates of x^k, i.e. $y^k := \pi(x^k)$.

Then we may consider two interpolation problems:

1. *Interpolation at the x^k by polynomials from $\mathcal{P}_n(\mathcal{M})$. If this problem is unisolvent, we denote the corresponding Lagrange polynomials by $l_i(\cdot; X)$.*

2. *Interpolation at the points $Y = \{y^k \in I\!R^{m-1}\}$ by polynomials from $(\mathcal{P}_n(\mathcal{M}) \circ \pi^{-1})_{\downarrow}$. If the problem is unisolvent, we denote the corresponding Lagrange polynomials by $L_i(\cdot; Y)$.*

Then, in the case that the second problem is unisolvent,

$$\lim_{t\to 0} l_i(x(t); X_t) = L_i(y; Y) \ ,$$

where $y := \pi(x(1))$.

It should be noted that the limiting basis of $(\mathcal{P}_n(\mathcal{M}) \circ \pi^{-1})_\downarrow$ may not always be expressible as the restriction of polynomials to an associated surface, i.e. there need not always be an analogue of the paraboloid for the case of angular scaling on the sphere.

Example 2 Take $\mathcal{M}$ to be the curve $y(x) = x^3 + x^2$. It is readily seen that

$$\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, 3x^8 + x^9\}$$

is a basis for $\mathcal{P}_3(\mathcal{M})$.

Hence consider 9 distinct points $(x_k, x_k^2 + x_k^3) \in \mathcal{M}, \ \ 1 \le k \le 9$ and scale them into the origin tangentially, i.e. by

$$X_t = \{(tx_k, t^2(x_k^2 + tx_k^3) \ : \ 1 \le k \le 9\} \ .$$

The limit basis is thus

$$\mathrm{span}\{1, x, x^2, \ldots, x^7, 3x^8 + x^9\}_\downarrow = \mathrm{span}\{1, x, x^2, \ldots, x^7, x^8\} \ .$$

Now, we *claim* that this latter is not the restriction of $\mathcal{P}_3(I\!R^2)$ to **any** curve Γ of degree 3.

Proof. Suppose to the contrary that such a Γ does exist. Then, since the dimension of $\mathrm{span}\{1, x, x^2, \ldots, x^7, x^8\} = 9 < 10 = \dim \mathcal{P}_3(I\!R^2)$, there exists a cubic $p(x, y) \in \mathcal{P}_3(I\!R^2)$ such that $p(x, y) = 0$ on Γ, i.e. Γ is a cubic curve. Now $y \in \mathcal{P}_3(I\!R^2)$ so there exists a polynomial $q(x)$ of degree 8 so that

$$y - q(x) \equiv 0, \ \ \text{on} \ \ \Gamma \ .$$

But since $y - q(x)$ is always irreducible, it must be the case that $\deg(q) \le 3$. In fact, $\deg(q) = 3$ since otherwise $\dim(\mathcal{P}_3(\Gamma)) \le 7$ (since then at least $y - q(x) = 0$, $x(y - q(x))$ and $y(y - q(x))$ are all 0 on Γ).

Then, on the one hand $y^3 \in \mathcal{P}_3(I\!R^2)$, so that $y^3 \equiv p_8(x)$ on Γ, for some polynomial p_8 of degree 8, and on the other hand, for some $c \ne 0$,

$$y^3 = q^3(x) = cx^9 + \cdots \quad \text{on} \ \ \Gamma \ .$$

It follows that $x^9 \in \mathrm{span}\{1, x, ..., x^8\}$ on Γ which is impossible. $\square$

◇◇

Example 3 A second, non-algebraic, example results from taking $\mathcal{M}$ to be the curve $y(x) = e^x$.

In this case, as $\mathcal{M}$ is non-algebraic, $\mathcal{P}_n(\mathcal{M}) = \mathcal{P}_n(I\!R^2)$, hence $d_n = \dim(\mathcal{P}_n(\mathcal{M}) = \frac{(n+2)(n+1)}{2}$. We take d_n points $(x_k, e^{x_k}),\ 1 \le k \le d_n$ and scale them into $a = (0,1)$ via

$$X_t = \{\ (tx_k, e^{tx_k})\ :\ 1 \le k \le d_n\ \}\ .$$

The coordinate of the generic point (x, e^x) is $\pi(x, e^x) := x$ with $\pi^{-1}(x) = (x, e^x)$.

We *claim* that $(\mathcal{P}_n(I\!R^2)(x, e^x))_\downarrow$ is just the span$\{1, x, x^2, ..., x^{d_n-1}\}$.

Proof. To see this, take the monomial basis for $\mathcal{P}_n(I\!R^2)$, $\ m_{ij}(x,y) := x^i y^j$. We denote the resulting composition by $f_{ij}(x)$, i.e.

$$f_{ij}(x) := x^i (e^x)^j = x^i e^{jx}, \quad 0 \le i + j \le n\ .$$

We wish to show that for $x^k,\ 0 \le k \le d_n - 1$ there is a linear combination of the f_{ij}, that is an

$$f_k(x) := \sum_{i+j \le n} a_{ij} f_{ij}(x)\ ,$$

so that $f_k(x) = x^k +$ higher order terms. For then it would follow that $x^k \in (\mathcal{P}_n(I\!R^2)(x, e^x))_\downarrow$ for *each* $k,\ 0 \le k \le d_n - 1$.

But, since $\dim(\mathcal{P}_n(I\!R^2)(x, e^x))_\downarrow = d_n$ by [4, Prop. 1], it then would follow that $(\mathcal{P}_n(I\!R^2)(x, e^x))_\downarrow = \mathrm{span}\{1, x, ..., x^{d_n-1}\}$.

To establish the *existence* of the f_k, we may expand each $f_{ij}(x)$ in a Taylor series of degree $d_n - 1$ about the origin, i.e.

$$f_{ij}(x) = \sum_{k=0}^{d_n-1} \frac{f_{ij}^{(k)}(0)}{k!} x^k + R_{ij}(x)\ ,$$

where $R_{ij}(x) = \alpha_{ij} x^{d_n}$+h.o.t.s.

Hence, it suffices for our purposes to show that there are constants b_{ij} such that for each $k,\ \ 0 \le k \le d_n - 1$

$$x^k = \sum_{i+j \le n} b_{ij} P_{ij}(x)$$

where

$$P_{ij}(x) = \sum_{s=0}^{d_n-1} \frac{f_{ij}^{(s)}(0)}{s!} x^s \ .$$

This is certainly the case if the $d_n \times d_n$ matrix

$$F_n = \left(\frac{f_{ij}^{(s)}(0)}{s!} \right)$$

(with row index the pair ij and column index s) is *non-singular.* But this is true since $\det(F_n) = \prod_{s=0}^{d_n-1} \frac{1}{s!}$ times the *Wronskian* of the d_n functions $\{f_{ij}\}_{i+j\le n}$ evaluated at the origin, which is non-zero since the $f_{ij}(x)$ are the d_n linearly independent solutions of the constant coefficient linear ODE with characteristic polynomial

$$\lambda^{n+1}(\lambda-1)^n(\lambda-2)^{n-1}\cdots(\lambda-n) \ .$$

This proves that $(\mathcal{P}_n(I\!\!R^2)(x,e^x))_\downarrow = \mathrm{span}\{1, x, x^2, ..., x^{d_n-1}\}$. $\square$

◇◇

Although in general it is difficult to explicitly determine least spaces $(\mathcal{P}_n(\mathcal{M})\circ\pi^{-1})_\downarrow$ given $\mathcal{M}$ and π, in certain circumstances, generalizing the case of a sphere, it is possible to do so.

Theorem 5 *Suppose that $\mathcal{M} \subset I\!\!R^d$ is an algebraic hypersurface given by $p(x) = 0$ with p irreducible and that $a \in \mathcal{M}$ is a smooth point. Without loss of generality we may assume that $a = 0$ and that the tangent plane to $\mathcal{M}$ at a is given by $x_d = 0$. Take the coordinate $\pi(x)$ of $x \in \mathcal{M}$ to be $\pi(x) := (x_1, \cdots, x_{d-1})$. Then if, near $a = 0$, $\mathcal{M}$ may be expressed by*

$$x_d = f(x'), \qquad x' := (x_1, \cdots, x_{d-1})$$

where $f(x') = h(x')+$ higher order terms, with $h(x')$ *a homogeneous polynomial of $deg(h) = deg(p)$, then*

$$(\mathcal{P}_n(\mathcal{M}) \circ \pi^{-1})_\downarrow = \mathcal{P}_n(x_d = h(x')).$$

Proof. Since both $x_d - h(x')$ and $p(x)$ are irreducible of the same degree, $\mathcal{P}_n(x_d = h(x'))$ and $\mathcal{P}_n(\mathcal{M})$ have the same dimension, d_n say. Moreover, clearly $\mathcal{P}_n(x_d = h(x'))$ has a basis of "monomials" of the form $(x')^i(h(x'))^j$. But each such $(x')^i(h(x'))^j$ may be realized as $((x')^i(f(x'))^j)_\downarrow$, seeing that $f(x')_\downarrow = h(x')$. Thus

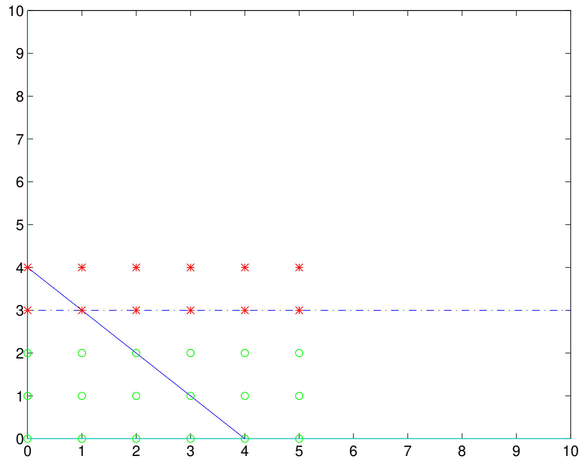

Figure 2: The set H: lattice points above the dashed line

$(\mathcal{P}_n(\mathcal{M})\circ\pi^{-1})_\downarrow \supset \mathcal{P}_n(x_d = h(x'))$, and in fact, they are equal since they both have dimension d_n. □

In general it is still possible to calculate any desired such least space, as we illustrate in the following example.

Example 4 Let us consider the curve displayed in Figure 3, i.e. $y(y-1)^2 - x^2 = 0$ where near $x = 0$ we may write $y = y(x)$. By using standard methods of algebra, cf. [1, Ch.2, §5], it is possibile to find a Gröbner basis for any particular ideal $\mathcal{I}$. In this case, since the ideal generated is principal, it is simply the function $y(y-1)^2 - x^2$ itself. To find a basis for the polynomials restricted to the curve, we again resort to standard arguments (cf. [1, Ch.5, §3]). For any consistent order, the leading term in our polynomial is y^3. From the theory of Gröbner bases, it follows that the leading term for any polynomial $p \in \mathcal{I}$ must be divisible by y^3. Hence, we consider the diagram $H = \{(0,3) + \mathbb{Z}_+^2\}$, of such leading terms (see Figure 2), which consists of all monomials divisible by y^3. By Proposition 1 of [1, Ch.5, §3], the monomials corresponding to the *complement of H* form a basis for the polynomials restricted to our curve. Let us, for example, consider the basis of all bivariate polynomials of degree 4 in x, y restricted to our curve. From Figure 2 this is the set generated by

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, x^4, x^3y, x^2y^2\ \} \ .$$

We wish to compute the least space for this set for $y = y(x)$ determined implicitly by the curve.

This requires the calculation of the derivatives of $y(x)(y(x)-1)^2 - x^2 = 0$ at $x = 0$, by means of implicit differentiation. Any number of coefficients for the function $y(x)$ can be easily generated. Since $y(0) = 0$, we find the following expansion :

$$y(x) = x^2 + 2x^4 + 8x^6 + 34x^8 + \text{higher order terms} .$$

The *de Boor-Ron algorithm* (as described in [5]), can be used to calculate any particular least space. In this case, substituing $y(x)$ with its expansion, we can reproduce all the 12 powers $\{1, x, ..., x^{11}\}$. From the fact that the least space has the same dimension as the original one, it follows that this is the desired space. In fact, we get:

$$\begin{aligned}
(y(x))_\downarrow &= x^2 , \\
(xy(x))_\downarrow &= x^3 , \\
(y^2(x))_\downarrow &= x^4 , \\
(xy^2(x))_\downarrow &= x^5 , \\
(x^2y^2(x))_\downarrow &= x^6 .
\end{aligned}$$

In order to get all other powers, again by the algorithm, we repeat the above process as follows.

$$\begin{aligned}
(xy(x) - x^3)_\downarrow &= 2x^5 , \\
(y^2(x) - x^4)_\downarrow &= 4x^8 , \\
(x^2y(x) - x^4)_\downarrow &= 2x^6 , \\
(x^3y(x) - x^5)_\downarrow &= 2x^7 , \\
(x^2y^2(x) - x^6)_\downarrow &= 4x^{10} , \\
(xy^2(x) - x^5)_\downarrow &= 20x^9 .
\end{aligned}$$

To get the power x^{11} we must apply the algorithm three times to the monomial $x^3y(x)$, that is $(((x^3y(x) - x^5) - 2x^7) - 8x^9)_\downarrow = 34x^{11}$.

$\diamond\diamond$

Remark. The previous example is instructive because it points out two important aspects of the results presented in the paper. For an algebraic surface given either by $p_1(x) = p_2(x) = \cdots = p_{d-m}(x) = 0$, or in parametric form $x_i = x_i(u_1, \cdots, u_m)$, $1 \le i \le d$, and a smooth point a on it, we have seen that

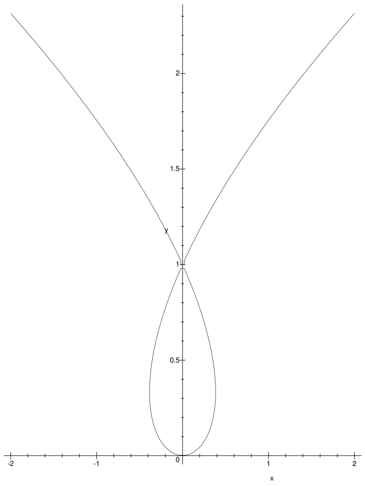

Figure 3: The curve $y(y-1)^2 = x^2$ in $[-2,2] \times [0,3]$

there is a method for finding the appropriate least space using a Gröbner basis for the ideal of the variety so generated and the de Boor-Ron algorithm for finding a least space at the point a. It is worthwhile noting that we must ensure that in order to find an expansion it is necessary that the Jacobian associated to the functions p_i, $1 \leq i \leq d-m$ must be non zero at a.

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