Polynomials arising in factoring generalized Vandermonde determinants: an algorithm for computing their coefficients

Stefano De Marchi
Dipartimento di Matematica e Informatica,
University of Udine, Via delle Scienze, 206
33100 Udine (Italy), e-mail: demarchi@dimi.uniud.it

Abstract

We consider generalized Vandermonde determinants of the form:

$$V_{s;\mu}(x_1,...,x_s) = |x_i^{\mu_k}|\,, \quad 1 \le i,k \le s\,,$$

where the x_i are distinct points belonging to an interval $[a,b]$ of the real line, the index s stands for the order, the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$. These determinants can be factored as a product of the classical Vandermonde determinant and a homogeneous symmetric function of the points involved, that is a *Schur function*. On the other hand, we show that when $x = x_s$ in the resulting polynomial, depending on the variable x, the Schur function can be factored as a two-factors polynomial: the first is the constant $\prod_{i=1}^{s-1} x_i^{\mu_1}$ times the (monic) polynomial $\prod_{i=1}^{s-1}(x - x_i)$ while the second is a polynomial $P_M(x)$ of degree $M = m_{s-1} - s + 1$.

Our main result is then the computation of the coefficients of the monic polynomial $P_M(x)$. We present an algorithm for the computation of the coefficients of P_M based on the *Jacobi-Trudi identity* for Schur functions.

Keywords: generalized Vandermonde matrices, Schur functions, interpolation, Toeplitz matrices.

AMS subject classification: 15A21, 05E05, 65D05, 47B35

1 Introduction

The study of generalized Vandermonde determinants goes back already at the end of twenties when Heineman proved some interesting formulas for different types of generalized Vandermode determinants (cf. [1]). Later on a few papers were dedicated to the subject but most part of them concentrate on finding explicit formulas for *particular* types of generalized Vandermonde determinants (see e.g. [2, 3, 4] and references there in). In 1975, King [5], provided some useful formulas based on Schur functions, for factoring most general Vandermonde determinants of the same general type we are considering here. There he proved that these Schur functions can be expressed as linear combination with positive coefficients of symmetric monomial functions. Thanks to these results we explored the possibility of explicitly computing the coefficients of these factors and in this paper we present our investigations.

We start from these observations. Let $X_s = \{x_1,...,x_s\}$ be a set of $s \ge 2$ *distinct points* belonging to an interval $[a,b]$ of the real axis. Let us denote by $VDM(x_1,...,x_s)$ the *classical*

Vandermonde determinant of order s, that is:

$$VDM(x_1, ..., x_s) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{s-1} \\ 1 & x_2 & \dots & x_2^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_s & \dots & x_s^{s-1} \end{vmatrix} .$$

It is well-known that (cf. [6, p. 25, (2.1.7)]):

$$VDM(x_1, ..., x_s) = VDM(x_1, ..., x_{s-1}) \left(\prod_{1 \le i \le s-1} (x_s - x_i) \right) . \tag{1}$$

Now, let $V_{s-1}(x) = VDM(x_1, ..., x_{s-1}, x)$ be the polynomial of degree $s-1$ obtained when x is used instead ofx_s. Then, formula (1) says that $V_{s-1}(x)$ is the product of the classical Vandermonde of order $s-1$ times a monic polynomial of degree $s-1$, that is $\prod_{i=1}^{s-1}(x - x_i)$.

This idea can be used to compute *generalized* Vandermonde determinants of the form:

$$V_{s;\mu}(x_1, ..., x_s) = |x_i^{\mu_k}| , \quad 1 \le i, k \le s , \tag{2}$$

where the x_i are s distinct points belonging to an interval $[a, b]$ of the real line; s stands for the order and the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$. Notice that the sequence μ recalls the monomial basis on which (2) is based. In fact, while the classical Vandermonde determinant of order s is based on the monomials $\{1, x, ..., x^s\}$, the generalized ones are based on the basis $\{x^{\mu_1}, ..., x^{\mu_s}\}$ for some positive integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$. We also point out that this different basis *in some cases* is obtainable from the basis of bivariate polynomials of degree s restricted to some curve $y = x^p$, $p \le s$. (cf. [7]).

Generalized Vandermonde determinants are strictly connected to Schur functions. In fact they can be factored as a product of the classical Vandermonde determinant and a *Schur function* (see Theorem 3.2 or for a different proof [5]):

$$V_{s;\mu}(x_1, ..., x_s) = VDM(x_1, ..., x_{s-1}) \cdot S_{m_{s-1}}(x_1, ..., x_s) ,$$

where $m_{s-1} = \mu_s - \mu_1$. Moreover, when $x = x_s$ we shall prove that

$$S_{m_{s-1}}(x) = \left(\prod_{i=1}^{s-1} x_i^{\mu_1} \right) \left(\prod_{i=1}^{s-1} (x - x_i) \cdot P_M(x) \right) , \tag{3}$$

with $M = m_{s-1} - s + 1$, for some polynomial $P_M(x)$ of degree M.

Our main purpose is the computation of the coefficients of the polynomial $P_M(x)$. Indeed, the polynomial $V_{s,\mu}(x_1, ..., x_s, x)$ is completely determined once we know the coefficients of $P_M(x)$, since $VDM(x_1, ..., x_{s-1})$ is a constant term and the coefficients of $\prod_{i=1}^{s-1}(x - x_i)$ are simply the elementary symmetric functions of the points $x_1, ..., x_{s-1}$.

In Section 2 we present some results concerning the computation of the coefficients of the factors for a simple class of generalized Vandermonde determinants, cf. (4). We start from the polynomial division algorithm which is intrinsically connected with the inversion of a lower triangular Toeplitz matrix, we prove that the coefficients of the polynomial $P_{n-k}(x)$ are *complete elementary symmetric* functions (see Theorem 2.2). Then, we also propose an algorithm to compute these coefficients.

In Section 3 we introduce Schur functions and by using some of their properties we compute the coefficients of the polynomial $P_M(x)$ (cf. (3), for the most general Vandermonde determinants of the type (2). An algorithm to accomplish this task is also presented.

In every Section we provide some examples in order to render the presentation as clear as possible.

2 The polynomial division algorithm and the inverse of a lower triangular Toeplitz matrix

Let us start by considering the polynomial associated to a simple $(k+1)\times(k+1)$ generalized Vandermonde determinant:

$$V_{n;k}(x)=\begin{vmatrix} 1 & x_1 & \dots & x_1^{k-1} & x_1^n \\ 1 & x_2 & \dots & x_2^{k-1} & x_2^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_k & \dots & x_k^{k-1} & x_k^n \\ 1 & x & \dots & x^{k-1} & x^n \end{vmatrix}, \tag{4}$$

where $2<k<n-1$, so that the $n-k$ powers $k,...,n-1$ are missing. This polynomial vanishes on $x_1,x_2,\dots,x_k$, then

$$V_{n;k}(x)=(x-x_1)(x-x_2)\cdots(x-x_k)\;P^*_{n-k}(x)\;, \tag{5}$$

for some *non-monic* polynomial $P^*_{n-k}(x)$ of degree $\le n-k$. Therefore, in order to completely determine $V_{n;k}(x)$ we should know the coefficients of the polynomial $P^*_{n-k}(x)$.
It is not difficult to see (cf. [9])

$$V_{n;k}(x)=VDM(x_1,...,x_k)(x-x_1)(x-x_2)\cdots(x-x_k)\;P_{n-k}(x)\;, \tag{6}$$

for some *monic* $P_{n-k}(x)$.
Let $P_{n-k}(x)=x^{n-k}+\sum_{i=1}^{n-k}b_{i+1}x^{n-k-i}$ and $Q_k(x)=\prod_{i=1}^{k}(x-x_i)$. Since $V_{n;k}(x)$ is a polynomial of degree n and $b_1=1$, then dividing both sides by $VDM(x_1,...,x_k)$ (which is not zero by hypothesis on the nodes), we have:

$$P_{n-k}(x)\cdot Q_k(x)=\left(x^{n-k}+\sum_{i=1}^{n-k}b_{i+1}x^{n-k-i}\right)\left(\prod_{i=1}^{k}(x-x_i)\right)=x^n-R_{k-1}(x)\;, \tag{7}$$

where $R_{k-1}(x)$ is a polynomial of degree $\le k-1$ and $b_1=1$. Moving $R_{k-1}(x)$ to the left side, the equation also says that *the polynomial $P_{n-k}(x)$ is the quotient of the division of x^n by $Q_k(x)$.*
Thus, by means of the *division algorithm for polynomials* in which the dividend is the monomial x^n and the divisor is $Q_k(x)$, we may determine the coefficients of $P_{n-k}(x)$. But this is not a good way to proceed: we can do better.
In fact, as proved in [9], the coefficients of $P_{n-k}(x)$ can be found as the solution of a linear system whose matrix is a *lower triangular Toeplitz* matrix generated by the *elementary symmetric functions* of the points involved. Since lower triangular Toeplitz matrices are structured matrices (generated by a vector, see below), then our problem reduces to finding the first column of the inverse of the lower triangular Toeplitz matrix of order $n-k+1$ generated by the vector whose first $k+1$ elements are the coefficients of the polynomial $Q_k(x)$ and the remaining $n-2k$ are set to zero (see also cf. [10, Ch. 2, §5]), and it is this point that we want to investigate a little bit more deeply.

DEFINITION 2.1 **(cf. [10])** *$T=(t_{i,j})_{1\le i,j\le n}$ is a lower triangular Toeplitz matrix if it can be written as*

$$T=\sum_{i=1}^{n}a_i\mathcal{S}^{i-1}\;,$$

where $\mathbf{a} = (a_1, ..., a_n)$ *is a real or complex vector and the matrix* $\mathcal{S} = \mathcal{S}^1$ *is the* **down-shift** *matrix, that is the matrix*

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Thus a lower triangular Toeplitz matrix is *completely determined by its first column or its last row.* Moreover, this set of matrices is an *algebra* generated by the down-shift matrix $\mathcal{S}$. If a lower triangular Toeplitz is invertible, then the inverse is still a lower triangular Toeplitz matrix. Hence, the inverse of an invertible lower triangular Toeplitz matrix is completely defined by the knowledge for example of its first column.

Given the lower triangular $n \times n$ Toeplitz matrix generated by the vector

$$\mathbf{a} = (1, a_1, a_2, ..., a_{n-1})^T \ ,$$

that is the matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \ , \tag{8}$$

the *first* column of the matrix $C = A^{-1}$, the vector $\mathbf{c} = (c_{i,1}, ..., c_{n,1})$, can be computed simply by solving the system:

$$A\mathbf{c} = \mathbf{e}_1 \ ,$$

where $\mathbf{e}_1 = (1, 0, ..., 0)^T$.

The solution provides the following useful recurrence:

$$\begin{aligned} c_{1,1} &= 1 \ ; \\ c_{i,1} &= -\sum_{j=1}^{i-1} c_{j,1} a_{i-j}, \quad i = 2, ..., n \ . \end{aligned} \tag{9}$$

Hence the recurrence (9) applied to the polynomial division $x^n : (x - x_1) \cdots (x - x_k)$ gives the coefficients b_i in (7) of the polynomial $P_{n-k}(x)$, in terms of the coefficients $a_p, p = 0, 1, .., k$ of the polynomial $Q_k(x)$:

$$\begin{aligned} b_1 &= 1 \ ; \\ b_i &= -\sum_{j=1}^{i-1} b_j a_{i-j}, \quad i = 2, ..., n-k+1 \ . \end{aligned} \tag{10}$$

Furthermore, $a_s = 0, \ k+1 \leq s \leq n-k$.

To conclude this part we give a simple example.

Example 2.1 Let us take $n = 5$ and $k = 2$, so that we consider the division of x^5 by $Q_2(x) = x^2 + a_1 x + a_2$. The quotient, is $P_3(x) = x^3 - a_1 x^2 + (a_1^2 - a_2)x + (-a_1^3 + 2a_1 a_2)$.

Now let us consider the lower triangular Toeplitz matrix A of dimension 4:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \end{pmatrix} . \tag{11}$$

The first column of the inverse C is determined by using the recurrence (9) applied to the vector $\mathbf{a} = (1,\ a_1,\ a_2, 0)^T$, obtaining $c_{1,1} = 1;\ c_{2,1} = -a_1;\ c_{3,1} = a_1^2 - a_2;\ c_{4,1} = -a_1^3 + 2a_1a_2;$.

The above steps are then equivalent to find the polynomial of degree 5

$$V_{5;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^5 \\ 1 & x_2 & x_2^5 \\ 1 & x & x^5 \end{vmatrix} . \tag{12}$$

Since $Q_2(x) = x^2 - (x_1 + x_2)x + x_1x_2$ then $\mathbf{a} = (1,\ -(x_1 + x_2),\ x_1x_2,\ 0)^T$ and by means of (10), $P_3(x)$ is:

$$P_3(x) = x^3 + (x_1 + x_2)x^2 + (x_1^2 + x_1x_2 + x_2^2)x + (x_2^3 + x_1x_2^2 + x_1^2x_2 + x_1^3) .$$

Finally, $V_{5;2}(x) = VDM(x_1, x_2) \cdot (x - x_1)(x - x_2) \cdot P_3(x)$.

2.1 An algorithm for computing the coefficients and some consequences

We recall that the coefficients of the polynomial

$$Q_k(x) = \prod_{i=1}^{k} (x - x_i)$$

in (7) are up to the sign, the *elementary symmetric functions*, $e_r^{(k)}$, of the points $x_i,\ i = 1, ..., k$, so defined:

$$\begin{aligned} e_0^{(k)} &= 1 \\ e_r^{(k)} &= \sum_{i_1 < ... < i_k} x_{i_1} \cdots x_{i_k}, \quad 1 \le r \le k \\ e_r^{(k)} &= 0, \quad r < 0, \quad \text{or} \quad r > k . \end{aligned}$$

The latter is usually assumed *by definition.*

Thanks to this definition we are ready to present a simple algorithm that computes the coefficients of the quotient polynomial $P_{n-k}(x)$ in (6).

Algorithm 1

`Inputs:` $n,\ k$ and the points $x_i,\ i = 1, ..., k$.

1. Form the $n - k + 1$ dimensional array $\mathbf{a}$, that is

$$\mathbf{a} = (\underbrace{e_0^{(k)}}_{=1}, -e_1^{(k)}, e_2^{(k)}, ..., (-1)^k\ e_k^{(k)}, \underbrace{e_{k+1}^{(k)}, ... , e_{n-k}^{(k)}}_{=0}) ;$$

2. By means of the recurrence (10) compute the array $\mathbf{b}$.

`Output:` the array $\mathbf{b}$ of the coefficients of $P_{n-k}(x)$.

$\diamond\diamond$

Let λ be any partition of length $\leq r$ a multivariate polynomial is usually defined as

$$m_\lambda(x_1, ..., x_r) = \sum_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r} \ ,$$

where α varies on all **distinct** permutations of λ. This definition helps us to introduce the *complete elementary symmetric functions* of the set of q points, that is the homogeneous symmetric polynomials

$$\begin{aligned} h_s^{(q)} &= \sum_{|\lambda|=s} m_\lambda \\ h_s^{(q)} &= 0, \quad s < 0, \quad \text{or} \quad s > q \ . \end{aligned}$$

Thus $h_s^{(q)}$ is the sum of all monomials of total degree s in the variables $x_1, ..., x_q$. In particular $h_0^{(q)} = e_0^{(q)}$, $h_1^{(q)} = e_1^{(q)}$.

Lemma 2.1

$$h_s^{(q)} = det\ (e^{(q)}_{\lambda_i - i + j}), \qquad 1 \leq i, j \leq s \ ,$$

and $\lambda = \underbrace{(1, ..., 1)}_{s}$.

Proof. The proof is based on [11, p. 21, formula (2.6')] by solving for h_n (cf. also [11, p. 28]). □

Finally, we state our first Theorem.

Theorem 2.2 *Given k distinct points, then the coefficients of the quotient polynomial $P_{n-k}(x)$ in the factorization (6), i.e. the elements of the array* **b**, *are the complete elementary symmetric functions:*

$$\mathbf{b}_i = h_{i-1}^{(k)} \ , \quad i = 1, ..., n - k + 1.$$

Proof. Let us consider the array **a** as above, made up of the elementary symmetric functions of the k points $x_1, ..., x_k$. From the recurrence (10) we recognize that

$$\mathbf{b}_i = \det \begin{pmatrix} e_1^{(k)} & e_2^{(k)} & \dots & \dots & e_{i-1}^{(k)} \\ e_0^{(k)} & e_1^{(k)} & \dots & \dots & e_{i-2}^{(k)} \\ e_{-1}^{(k)} & e_0^{(k)} & e_1^{(k)} & \dots & e_{i-3}^{(k)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ e_{-i-2}^{(k)} & \dots & \dots & e_0^{(k)} & e_1^{(k)} \end{pmatrix} \ ,$$

which by the Lemma 2.1, is exactly $h_{i-1}^{(k)}$. □

Corollary 2.3 *The quotient polynomial $P_{n-k}(x)$ can be written in determinantal form, that is*

$$P_{n-k}(x) = \begin{vmatrix} e_0^{(k)} & -e_1^{(k)} & \dots & \dots & (-1)^{n-k} e_{n-k}^{(k)} \\ 0 & e_0^{(k)} & -e_1^{(k)} & \dots & (-1)^{n-k-1} e_{n-k-1}^{(k)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & e_0^{(k)} & -e_1^{(k)} \\ 1 & x & \dots & x^{n-k-1} & x^{n-k} \end{vmatrix} . \tag{13}$$

Example 2.2 Given the polynomial

$$V_{7;4}(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^7 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^7 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^7 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^7 \\ 1 & x & x^2 & x^3 & x^7 \end{vmatrix} . \tag{14}$$

Thanks to our results we can say:

(i) $V_{7;4}(x) = VDM(x_1, x_2, x_3, x_4)\ (x - x_1)(x - x_2)(x - x_3)(x - x_4)\ P_3(x)$,

(ii) and $P_3(x) = x^3 + h_1^{(4)} x^2 + h_2^{(4)} x + h_3^{(4)}$.

3 Schur functions and generalized Vandermonde determinants

A *Schur function* is defined as follows (cf. [11]).

DEFINITION 3.1 *Given a partition $\lambda = (\lambda_1, ..., \lambda_n)$, $|\lambda| \leq n$, the Schur function s_λ, defined on $\mathbb{R}^n$, is*

$$s_\lambda(x_1, ..., x_n) = \frac{det\ (x_i^{\lambda_j + n - j})}{det\ (x_i^{n-j})}, \quad 1 \leq i, j \leq n. \tag{15}$$

Notice that the denominator is nothing else than the classical Vandermonde determinant.
Schur functions provides a very useful tool to study and compute generalized Vandermonde determinants (cf. [9, 8, 5]). To fit our setting in this framework, we simply set $x_n = x$.

Example 3.1 Let us consider the determinant

$$V_{4;2} = \begin{vmatrix} 1 & x_1 & x_1^4 \\ 1 & x_2 & x_2^4 \\ 1 & x_3 & x_3^4 \end{vmatrix} . \tag{16}$$

From our results and after a few calculations,

$$\begin{aligned} V_{4;2} &= (x_1-x_3)(x_2-x_3)[(x_2^3-x_1^3)+(x_2^2-x_1^2)x_3+(x_2-x_1)x_3^2] \\ &= (x_1-x_3)(x_2-x_3)(x_2-x_1)[x_2^2+x_1x_2+x_1^2+x_2x_3+x_1x_3+x_3^2]\ . \end{aligned}$$

which shows once more that we can factor the classical Vandermonde $VDM(x_1,x_2,x_3)$ out of the determinant.

Hence, let $\alpha=(4,1,0)$ and $\delta=(2,1,0)$ be the partitions consisting of the degrees of the monomials involved considered in *reverse order*, then letting $\lambda=\alpha-\delta=(2,0,0)$, the above determinant divided by $VDM(x_1,x_2,x_3)$ is the following Schur function:

$$\frac{\begin{vmatrix} 1 & x_1 & x_1^4 \\ 1 & x_2 & x_2^4 \\ 1 & x_3 & x_3^4 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}} = x_2^2+x_1x_2+x_1^2+x_2x_3+x_1x_3+x_3^2 = s_\lambda(x_1,x_2,x_3)\ . \tag{17}$$

Remark. We recognize that the above Schur function is $h_2^{(3)}$. This again is a general setting for these determinants as we shall see in the next Example.

Example 3.2 The case of a group of k missing powers can easily be managed in the framework of Schur functions.

Proposition 3.1 *Let us consider the generalized Vandermonde determinant,*

$$V_{n;n-k}=\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-k-1} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-k-1} & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-k} & x_{n-k}^2 & \dots & x_{n-k}^{n-k-1} & x_{n-k}^n \\ 1 & x_{n-k+1} & x_{n-k+1}^2 & \dots & x_{n-k+1}^{n-k-1} & x_{n-k+1}^n \end{vmatrix}\ ;$$

when the k powers $n-k,...,n-1$ are missed. Then

$$V_{n;n-k}=s_\lambda\ \cdot\ VDM(x_1,...,x_{n-k})=h_k^{(n-k+1)}\ \cdot\ VDM(x_1,...,x_{n-k})$$

with $\lambda=(k,0^{(n-k)})=(k,\underbrace{0,...,0}_{n-k})$ and $h_k^{(n-k+1)}$ which is the k-th complete symmetric function of the $n-k+1$ variables $x_1,...,x_{n-k+1}$.

Proof. The `Jacobi-Trudi identity` (cf. [11, formula (3.4), p. 41]) says:

$$s_\lambda=\det(h_{\lambda_i-i+j}^{(n-k+1)})\ ,\qquad 1\le i,j\le n-k+1\ . \tag{18}$$

We claim that $s_\lambda=h_k^{(n-k+1)}$. In fact

$$s_\lambda=\begin{vmatrix} h_k^{(n-k+1)} & h_{k+1}^{(n-k+1)} & h_{k+2}^{(n-k+1)} & \dots & h_n^{(n-k+1)} \\ h_{-1}^{(n-k+1)} & h_0^{(n-k+1)} & h_1^{(n-k+1)} & \dots & h_{n-k-1}^{(n-k+1)} \\ h_{-2}^{(n-k+1)} & h_{-1}^{(n-k+1)} & h_0^{(n-k+1)} & \dots & h_{n-k-2}^{(n-k+1)} \\ \vdots & & \ddots & & \vdots \\ h_{-n+k}^{(n-k+1)} & h_{-n+k+1}^{(n-k+1)} & \dots & \dots & h_0^{(n-k+1)} \end{vmatrix}\ ,$$

and since $h_p^{(n-k+1)} = 0,\ p \leq -1$ we get the claimed result. □

Remark. From Lemma 2.1 we also observe that $h_k^{(n-k+1)}$ is a function of the elementary symmetric functions that is,

$$h_k^{(n-k+1)} = \det\left(e_{\lambda_i^*-i+j}^{(n-k+1)}\right), \quad 1 \leq i,j \leq n-k+1\ , \tag{19}$$

where here $\lambda^* = \underbrace{(1,...,1)}_{n-k+1}$. Therefore we we may express h_k as a determinant of a (Hessenberg) matrix whose elements are the elementary symmetric functions $e_0, ..., e_k$.

◇◇

Now we are ready to extend our results to very general Vandermonde determinants and its associated polynomials as we did in the previous Section.
To be precise, given a sequence of integers, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_n$, and a set of n distinct points, $x_1, ..., x_n$, we can consider determinants of the form

$$V_{n;\mu} = \left|x_i^{\mu_j}\right|_{1\leq i,j\leq n}\ . \tag{20}$$

It is obvious that these determinants can be rewritten

$$V_{n;\mu} = \left(\prod_{i=1}^{n} x_i^{\mu_1}\right) \det\left(x_i^{\mu_j-\mu_1}\right)_{1\leq i,j\leq n}\ , \tag{21}$$

and so for our purposes *it is enough* to consider the associated polynomials of the type

$$V_{m_{n-1};n-1}(x) = \begin{vmatrix} 1 & x_1^{m_1} & \ldots & x_1^{m_{n-1}} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n-1}^{m_1} & \ldots & x_{n-1}^{m_{n-1}} \\ 1 & x^{m_1} & \ldots & x^{m_{n-1}} \end{vmatrix}\ , \tag{22}$$

with obvious meaning for the symbols used.

We need another concept related to partitions: that is the *diagram* of a partition (cf. [11, p. 2]).

DEFINITION 3.2 *Given a partition λ, its diagram is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$.*

DEFINITION 3.3 *The* **conjugate** *of a partition λ, is the partition λ' whose diagram is the transpose of the diagram of λ obtained by reflection in the main diagonal.*

For example, if $\lambda = (3,2,0)$, then $\lambda' = (2,2,1)$. The definition of conjugate of a partition allows to introduce an equivalent relation to the Jacobi-Trudi identity for Schur functions. That is, for any partition λ of n,

$$s_\lambda = \det\left(e_{\lambda'_i-i+j}\right)_{1\leq i,j\leq n}\ , \tag{23}$$

with λ' its conjugate partition.

We are now ready to generalize the Theorem 2.2 and the Corollary 2.3.

Theorem 3.2 *Let $\alpha = (m_{n-1}, m_{n-2}, ..., m_1, 0)$ and $\delta = (n-1, n-2, ..., 1, 0)$ be the partitions of length n associated to the determinant (22). Let $\lambda = \alpha - \delta$ and λ' be its conjugate partition. Then,*

$$V_{m_{n-1};n-1}(x) = VDM(x_1, ..., x_{n-1}) \left(\prod_{i=1}^{n-1} (x - x_i) \right) P_M(x) \; ; \tag{24}$$

where $M = m_{n-1} - n + 1$. Moreover, if $\lambda_1 \geq n$, then $M \geq n$ and

$$P_M(x) = \begin{vmatrix} (-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1} & (-1)^{\lambda'_1} e_{\lambda'_1} & \cdots & (-1)^{\lambda'_1 + M - 1} e_{\lambda'_1 + M - 1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{\lambda'_n - M} e_{\lambda'_n - M} & (-1)^{\lambda'_n - M + 1} e_{\lambda'_n - M + 1} & \cdots & (-1)^{\lambda'_n} e_{\lambda'_n} \\ 1 & x & \cdots & x^M \end{vmatrix} . \tag{25}$$

otherwise λ' has $n - \lambda_1$ zeros and $M < n$.

Proof. The equality (24) is simply proved by resorting to the definition of Schur functions when $x = x_n$.

Since $\lambda_1 \geq n$, then λ' has no zeros. To get the determinant (25) we observe that $P_M(x)$ is the determinant of the matrix obtained by bordering the matrix

$$\left(e_{\lambda'_i - i + j} \right)_{1 \leq i, j \leq M} \; , \tag{26}$$

with the column consisting of the elementary symmetric functions translated by 1 to the left starting from the first column of the matrix (26), i.e

$$\left((-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1}, \; ..., (-1)^{\lambda'_n - M} e_{\lambda'_n - M}, 1 \right)^T$$

and with the last row consisting of the vector of the powers $(1, x, \; ..., x^M)$.

If $\lambda_1 < n$, then λ' has $n - \lambda_1$ zeros. Therefore in the matrix (26) there are not the corresponding rows. □

Example 3.3 Given the polynomial:

$$V_{7;3}(x) = \begin{vmatrix} 1 & x_1^3 & x_1^4 & x_1^7 \\ 1 & x_2^3 & x_2^4 & x_2^7 \\ 1 & x_3^3 & x_3^4 & x_3^7 \\ 1 & x^3 & x^4 & x^7 \end{vmatrix} .$$

By the Theorem 3.2, $V_{7;3}(x) = VDM(x_1, x_2, x_3) \; (x - x_1)(x - x_2)(x - x_3) \; P_4(x)$. Since $\alpha = (7, 4, 3, 0)$, $\delta = (3, 2, 1, 0)$, $\lambda = (4, 2, 2, 0)$ and $\lambda' = (3, 3, 1, 1)$, then the determinantal form for $P_4(x)$ is:

$$P_4(x) = \begin{vmatrix} e_2 & -e_3 & e_4 & -e_5 & e_6 \\ -e_1 & e_2 & -e_3 & e_4 & -e_5 \\ e_{-2} & -e_{-1} & e_0 & -e_1 & e_2 \\ -e_{-3} & e_{-2} & -e_{-1} & e_0 & -e_1 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix} .$$

Finally, since $e_j = 0, j > 3$ and $j < 0$ then

$$P_4(x) = \begin{vmatrix} e_2 & -e_3 & 0 & 0 & 0 \\ -e_1 & e_2 & -e_3 & 0 & 0 \\ 0 & 0 & 1 & -e_1 & e_2 \\ 0 & 0 & 0 & 1 & -e_1 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix} .$$

Example 3.4 Let us consider the polynomial:

$$V_{4;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^4 \\ 1 & x_2 & x_2^4 \\ 1 & x & x^4 \end{vmatrix} .$$

By the Theorem 3.2, $V_{4;2}(x) = VDM(x_1, x_2)\ (x-x_1)(x-x_2)\ P_2(x)$. Since $\alpha = (4, 1, 0)$, $\delta = (2, 1, 0)$, $\lambda = (2, 0, 0)$ and $\lambda' = (1, 1, 0)$, then the determinantal form for $P_2(x)$ is:

$$P_2(x) = \begin{vmatrix} e_0 & -e_1 & e_2 \\ -e_{-1} & e_0 & -e_1 \\ 1 & x & x^2 \end{vmatrix} .$$

Remarks. In Example 3.3, $\lambda_1 = 4 \geq n = 4$ then λ' has no zeros and $M = n = 4$. In the Example 3.4, $\lambda_1 = 2 < n = 3$ then λ' has some zeros and consequently $M = 2 < n$.

Finally we can write down an algorithm for computing the coefficients of $P_M(x)$ based on the Theorem 3.2.

Algorithm 2

`Inputs:` the partitions α, δ of M and the points $x_i,\ i = 1, ..., M$.

1. Let $\lambda = \alpha - \delta$ and λ' its conjugate.

Build the matrix

$E = \left(e_{\lambda'_i - i + j}\right)_{1 \leq i,\, j \leq M}$

where e_p is the p-th elementary symmetric function of M points.

2. Build the matrix E_1 by bordering the matrix E:

add as first column the array

$$\left((-1)^{\lambda'_1-1}e_{\lambda'_1-1},\ ...,(-1)^{\lambda'_n-M}e_{\lambda'_n-M},1\right)^T$$

and as last row $(1, x,\ ..., x^M)$.

`Output:` the coefficients of $P_M(x)$, which are the minors of order $M-1$

of the matrix E_1.

Acknowledgments. This work was done during my stay at the University of Dortmund (Germany) with the support of the Italian Research Council (CNR) within the program "*short-term mobility*", pos. 140.2, year 1999. Furthermore I would like to express my special thanks to Dr. Ulrike Maier and Prof. M. Reimer who invited me to spend a pleasant and productive time in the *Fachbereich Mathematik der Universität Dortmund.*

References

[1] E. R. Heineman. Generalized Vandermonde determinants. *Proc. Royal Soc. Edinburgh*, 464–476 (1929).

[2] R. P. Flowe and G. A. Harris. A note on generalized Vandermonde determinants. *SIAM J. Matrix Anal. Appl.*, 14(4):1146–1151 (1993).

[3] L. Kuipers and B. Meulenbeld. Symmetric polynomials with non negative coefficients. *Proc. AMS*, 6:88–93 (1956).

[4] P. C. Rosenbloom. Some properties of absolutely monotonic functions. *Bull. AMS*, 52:458–462 (1946).

[5] R. C. King. Generalized Vandermonde determinants and Schur functions. *Proc. AMS*, 48(1): 53–56 (1975).

[6] P. J. Davis. *Interpolation & Approximation.* Dover Publications, 1963.

[7] L. Bos and S. De Marchi. Fekete points for bivariate polynomials restricted to $y = x^m$. *East J. Approx.*, 6(2): 189–200 (2000).

[8] S. De Marchi. On computing the factors of generalized Vandermonde determinants, *Recent Advances in Applied and Theoretical Mathematics*, World Scientific Engineering Society, N. Mastorakis Ed., 140–144 (2000).

[9] S. De Marchi. Generalized Vandermonde determinants, Toeplitz matrices and the Polynomial Division Algorithm, Universität Dortmund, Ergebnisberichte Angewandte Mathematik, nr. 176, June 1999.

[10] D. Bini and V. Pan. *Polynomial and Matrix Computations*, Volume 1, Fundamentals Algorithms. Birkhäuser, Boston, 1994.

[11] I. G. Macdonalds. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs, Second Edition, 1995.