Polynomials arising in factoring generalized Vandermonde determinants II: a condition for monicity

Stefano De Marchi
Dipartimento di Matematica e Informatica,
University of Udine, Via delle Scienze, 206
33100 Udine (Italy), e-mail: demarchi@dimi.uniud.it

Abstract

In our previous paper [4] we observed that generalized Vandermonde determinants of the form

$$V_{n;\mu}(x_1,...,x_s) = |x_i^{\mu_k}|\,, \quad 1 \le i,k \le n\ ,$$

where the x_i are distinct points belonging to an interval $[a,b]$ of the real line, the index n stands for the order, the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_n$, can be factored as a product of the classical Vandermonde determinant and a *Schur function*. On the other hand, we showed that when $x = x_n$ the resulting polynomial in x is a the Schur function which can be factored as a two-factors polynomial: the first is the constant $\prod_{i=1}^{n-1} x_i^{\mu_1}$ times the monic polynomial $\prod_{i=1}^{n-1}(x-x_i)$ while the second is a polynomial $P_M(x)$ of degree $M = m_{n-1} - n + 1$. In this note we first present a typical application in which these factorizations arise and then we discuss a condition under which the polynomial $P_M(x)$ is *monic*.

Keywords: generalized Vandermonde matrices, Schur functions, interpolation.

AMS subject classification: 15A21, 05E05, 65D05

1 Introduction

Let $X_n = \{x_1,...,x_n\}$ be a set of $n \ge 2$ *distinct points* belonging to an interval $[a,b]$ of the real axis. Let us denote by $VDM(x_1,...,x_n)$ the *classical* Vandermonde determinant of order n, that is:

$$VDM(x_1,...,x_n) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix} .$$

It is well-known that (cf. [2, p. 25, (2.1.7)]):

$$VDM(x_1,...,x_n) = VDM(x_1,...,x_{n-1}) \left(\prod_{1 \le i \le n-1} (x_n - x_i) \right) . \tag{1}$$

Now, let $V_{n-1}(x) = VDM(x_1,...,x_{n-1},x)$ be the polynomial of degree $n-1$ obtained when x is used instead of x_n. Then, formula (1) says that $V_{n-1}(x)$ is the product of the classical Vandermonde of order $n-1$ times a monic polynomial of degree $n-1$, that is $\prod_{i=1}^{n-1}(x-x_i)$.

This idea can be used to compute *generalized* Vandermonde determinants of the form:

$$V_{n;\mu}(x_1, ..., x_n) = |x_i^{\mu_k}|, \quad 1 \le i, k \le n , \tag{2}$$

where the x_i are n distinct points belonging to an interval $[a, b]$ of the real line; n stands for the order and the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_n$. Notice that the sequence μ recalls the monomial basis on which (2) is based. In fact, while the classical Vandermonde determinant of order n is based on the monomials $\{1, x, ..., x^n\}$, the generalized ones are based on the basis $\{x^{\mu_1}, ..., x^{\mu_n}\}$ for some positive integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_n$.

Generalized Vandermonde determinants are strictly connected to Schur functions. In fact they can be factored as a product of the classical Vandermonde determinant and a *Schur function* (cf. [1, 4]):

$$V_{n;\mu}(x_1, ..., x_n) = VDM(x_1, ..., x_{n-1}) \cdot S_{m_{n-1}}(x_1, ..., x_n) ,$$

where $m_{n-1} = \mu_n - \mu_1$. Moreover, for $x = x_n$ we proved that

$$S_{m_{n-1}}(x) = \left(\prod_{i=1}^{n-1} x_i^{\mu_1} \right) \left(\prod_{i=1}^{n-1} (x - x_i) \cdot P_M(x) \right) , \tag{3}$$

with $M = m_{n-1} - n + 1$, for some polynomial $P_M(x)$ of degree M. A method for the computation of the coefficients of the polynomial $P_M(x)$ was suggested in [4].

Inspired by a typical example where these kind of factorization arise, that is the restriction of bivariate polynomials of degree n to curves of the type $y = x^m, \ \ m \le n$, in Section 2 we present our main result and discuss some consequences. We also provide an improved version of Algorithm 2 in [4].

2 The main result

In order to understand more about the topic, we recommend the reader to refer to our previous paper [4] also to become confident with the notation used. For more details on symmetric and Schur functions see also the monograph [7].

First, we observe that generalized Vandermonde determinants of the form (2) can be factored as follows:

$$V_{n;\mu} = \left(\prod_{i=1}^{n} x_i^{\mu_1} \right) \det \left(x_i^{\mu_j - \mu_1} \right)_{1 \le i,j \le n} . \tag{4}$$

Furthermore, we may consider the associated polynomial

$$V_{m_{n-1};n-1}(x) = \begin{vmatrix} 1 & x_1^{m_1} & \dots & x_1^{m_{n-1}} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n-1}^{m_1} & \dots & x_{n-1}^{m_{n-1}} \\ 1 & x^{m_1} & \dots & x^{m_{n-1}} \end{vmatrix} , \tag{5}$$

with obvious meaning for the symbols used. The main result proved in [4] was the following Theorem.

Theorem 2.1 [4, Th. 3.2] *Let $\alpha = (m_{n-1}, m_{n-2}, ..., m_1, 0)$ and $\delta = (n-1, n-2, ..., 1, 0)$ be the partitions of length n associated to the determinant (5). Let $\lambda = \alpha - \delta$ and λ' be its conjugate partition. Then,*

$$V_{m_{n-1};n-1}(x) = VDM(x_1, ..., x_{n-1}) \left(\prod_{i=1}^{n-1} (x - x_i) \right) P_M(x) \; ; \tag{6}$$

where $M = m_{n-1} - n + 1$. Moreover, if $\lambda_1 \geq n$, then $M \geq n$ and

$$P_M(x) = \begin{vmatrix} (-1)^{\lambda'_1-1} e_{\lambda'_1-1} & (-1)^{\lambda'_1} e_{\lambda'_1} & \cdots & (-1)^{\lambda'_1+M-1} e_{\lambda'_1+M-1} \\ \vdots & \vdots & \vdots \;\; \vdots & \\ \vdots & \vdots & \vdots \;\; \vdots & \\ (-1)^{\lambda'_n-M} e_{\lambda'_n-M} & (-1)^{\lambda'_n-M+1} e_{\lambda'_n-M+1} & \cdots & (-1)^{\lambda'_n} e_{\lambda'_n} \\ 1 & x & \cdots & x^M \end{vmatrix} . \tag{7}$$

otherwise λ' has $n - \lambda_1$ zeros and $M < n$.

Here the e_p are the *elementary symmetric* functions of the points $x_1, ..., x_{n-1}$.

A first remark is that the polynomial $P_M(x)$ is in general *not monic*, as it will appear evident from the following two examples.

Example 2.1 Let us consider the basis of bivariate polynomials of degree n restricted to the curve $y = x^m, \;\; m \leq n$. The resulting space consists of univariate polynomials of degree at most mn. The dimension of this space, say N, is in general *strictly* less then $mn+1$ (when $m > 2$). Moreover this space is generated by a set of monomials of degree at most mn where there are some *missing powers* or *gaps*. One can also determine the number of these gaps which form a sequence of integers: the sequence λ.

More precisely, let

$$\{x^{\alpha_1}, x^{\alpha_2}, \cdots, x^{\alpha_N}\}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ be the basis (cf. §1, the Introduction), and consider for given N and distinct points $x_1 < x_2 < \cdots < x_N$, the associated generalized Vandermonde determinant

$$V_\alpha(x_1, \cdots, x_N) := \det[x_i^{\alpha_j}]_{1 \leq i,j \leq N}. \tag{8}$$

Here $\alpha_i = \mu_{N-i+1}$ w.r.t. the basis in (2).
Since the α_j are positive integers we must have $\alpha_j \geq N - j$ and thus by taking $\lambda_j = \alpha_j - (N - j)$ we have

$$V_\alpha(x_1, \cdots, x_N) = \det(x_i^{\lambda_j+N-j}) = VDM(x_1, ..., x_N) \; s_\lambda(x_1, \cdots, x_N) \; . \tag{9}$$

Of importance is the fact that λ depends only on m i.e., is constant with respect to N. More precisely we have:

Proposition 2.1 [3, Lemma 1] *If $\lambda_j = \alpha_j - (N - j)$ then*

$$\lambda = (\underbrace{\sum_{k=1}^{m-2} k}_{1}, \underbrace{\sum_{k=1}^{m-3} k, \sum_{k=1}^{m-3} k}_{2}, \underbrace{\sum_{k=1}^{m-4} k, \sum_{k=1}^{m-4} k, \sum_{k=1}^{m-4} k}_{3}, \cdots, \underbrace{\sum_{k=1}^{1} k, \cdots, \sum_{k=1}^{1} k}_{m-2}, 0, \cdots, 0).$$

For instance, when $m = n = 4$,

$$\begin{array}{rcl} \alpha & = & (16,13,12,10,9,8,7,6,5,4,3,2,1,0) \\ \delta & = & (13,12,11,10,9,8,7,6,5,4,3,2,1,0) \\ \lambda & = & (3,1,1,0,0,0,0,0,0,0,0,0,0,0) \ . \end{array}$$

Thus, λ represents the number of gaps between the sequences α and δ. For more details on some properties of bivariate polynomials of degree n restricted to curves $y = x^m$, $m \le n$, the reader can refer to [3, §2].

Example 2.2 Let us take the polynomial

$$V_{7;3}(x) = \begin{vmatrix} 1 & x_1^3 & x_1^4 & x_1^7 \\ & & & \\ 1 & x_2^3 & x_2^4 & x_2^7 \\ & & & \\ 1 & x_3^3 & x_3^4 & x_3^7 \\ & & & \\ 1 & x^3 & x^4 & x^7 \end{vmatrix} .$$

already considered in [4, Example 3.3]. By the Theorem 2.1, $V_{7;3}(x) = VDM(x_1, x_2, x_3)\ (x - x_1)(x - x_2)(x - x_3)\ P_4(x)$. Here, $\alpha = (7,4,3,0)$, $\delta = (3,2,1,0)$, $\lambda = (4,2,2,0)$ and $\lambda' = (3,3,1,1)$, then the determinantal form of $P_4(x)$ is:

$$P_4(x) = \begin{vmatrix} e_2 & -e_3 & 0 & 0 & 0 \\ -e_1 & e_2 & -e_3 & 0 & 0 \\ 0 & 0 & 1 & -e_1 & e_2 \\ 0 & 0 & 0 & 1 & -e_1 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix} ;$$

since $e_j = 0, j > 3$ and $j < 0$. Then computing the leading coefficient of $P_4(x)$, one can see that it is not 1.

◇◇

It seems clear that the form of the sequence α plays an important role on the monicity $P_M(x)$. We are now ready to show that if α and consequently λ, have a "special" form, then $P_M(x)$ is monic.

Theorem 2.2 *Given* $\alpha = (m_{n-1}, m_{n-2}, ..., m_1, 0)$ *let us assume* $m_{n-i} = n - i, \quad i = 2, ..., n-2$ *and* $m_{n-1} = m_{n-2} + M + 1$, *for some positive* M. *Then* $\lambda = (M, 0^{(n-1)})$ *and* $P_M(x)$ *is a monic polynomial of exact degree* M.

Proof. By hyphotesis $\alpha = (n - 1 + M, n - 2, ..., 1, 0)$ and $\delta = (n - 1, n - 2, ..., 1, 0)$. Then $\lambda = \alpha - \delta = (M, 0, ..., 0)$. From [4, Prop. 3.1], we know that for this special λ the corresponding Schur function is $h_M^{(M)}$, i.e. the *complete elementary symmetric* function of "degree" M on the points $(x_1, ..., x_M)$. Then, by bordering the corresponding matrix as we did in Theorem 2.1, we obtain

$$P_M(x) = \begin{vmatrix} 1 & | & & & \\ 0 & | & h_M^{(M)} & & \\ \vdots & | & & & \\ 0 & -- & --- & & - \\ 1 & x & \ldots\ \ldots & & x^M \end{vmatrix} . \tag{10}$$

Therefore, the leading coefficient (of x^M) is 1 since

$$h_M^{(M)} = \det(e^{(M)}_{\lambda^*-i+j}), \ \ 1 \leq i,j \leq M, \ \ \lambda^* = \underbrace{(1, ..., 1)}_{M} .$$

This concludes the proof. □
The symbol $0^{(n-1)}$ stands for the sequence $\underbrace{(0, ..., 0)}_{n-1}$.

Remarks

- In reading carefully the previous Theorem, it can be seen as a Corollary of Theorem 3.2 in [4].
- The bordering process in (10) is based on the conjugate partition λ'. The monicity of $P_M(x)$ is also related to λ', as we state in the following Corollary.

Corollary 2.1 *Given a sequence λ associated to the polynomial (6). If $\lambda' = (1^{(M)}, 0^{(n-1)})$ then $P_M(x)$ is monic.*

Proof. Since the conjugate partition of λ' is λ, hence the conjugate of

$$\lambda' = (1^{(M)}, 0^{(n-M)}) , \tag{11}$$

is $\lambda = (M, 0^{(n-1)})$. By applying the Theorem 2.2 we conclude. □

Example 2.3 A simple but instructive example comes easily from Example 2.1. Let us take the basis of bivariate polynomials of degree $n \geq 3$ restricted to the curve $y = x^3$. The resulting space consists of univariate polynomials of degree $3n$ where the only *missing power* is $3n-1$, i.e. x^{3n-1} is not in the basis, and its dimension is $3n$.
Then, $\lambda = \lambda' = (1, 0^{(3n-1)})$, so that $M = 1$ and $P_1(x) = x - h_1^{(3n-1)} = x - e_1^{(3n-1)}$.

2.1 Algorithmic aspects

Finally we suggest an algorithm for the computation of the coefficients of the (monic) polynomial $P_M(x)$ which is based on the Theorems 3.2 in [4] and 2.2.

Algorithm

`Inputs:` the partitions α, δ of M and the points x_i, $i = 1, ..., M$.

1.Let $\lambda = \alpha - \delta$ and λ' its conjugate.

2. Check if λ' fulfills the hyphotesis of the Corollary 2.1.

3. Build the matrix

$$E = \left(e_{\lambda'_i - i + j}\right)_{1 \le i,j \le M}$$

where e_p is the p-th elementary symmetric function of M points.

4. Build the matrix E_1 by bordering the matrix E:

if λ' has not the form (11) then add as first column the array

$$\left((-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1},\ \ldots, (-1)^{\lambda'_n - M} e_{\lambda'_n - M}, 1\right)^T$$

and as last row $(1, x,\ \ldots, x^M)$.

otherwise add as first column the array

$$(1, 0,\ \ldots, 1)^T$$

and as last row $(1, x,\ \ldots, x^M)$.

`Output`: the coefficients of $P_M(x)$, which are the minors of order $M - 1$

of the matrix E_1.

Remark. The construction of the sequences λ and λ' can be easily done with the help of matrices with elements 0 and 1. For instance a sequence $\lambda = (4, 3, 2)$ can be represented as the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Hence λ' is simply the transpose matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In the special case considered in this note, the matrix corresponding to $\lambda = (M, 0^{(n-1)})$ is the vector $\underbrace{(1, \ldots, 1)}_{M}$ and so λ' is still a vector. Therefore, in the monic case the construction of λ and λ' has a much smaller complexity.

References

[1] R. C. King. Generalized Vandermonde determinants and Schur functions. *Proc. AMS*, 48(1): 53–56 (1975).

[2] P. J. Davis. *Interpolation & Approximation.* Dover Publications, 1963.

[3] L. Bos and S. De Marchi. Fekete points for bivariate polynomials restricted to $y = x^m$. *East J. Approx.*, 6(2): 189–200 (2000).

[4] S. De Marchi. Polynomials arising in factoring generalized Vandermonde determinants: an algorithm for computing their coefficients. To appear in *Appl. Math. Model.*, 2001.

[5] S. De Marchi. On computing the factors of generalized Vandermonde determinants, *Recent Advances in Applied and Theoretical Mathematics*, World Scientific Engineering Society, N. Mastorakis Ed., 140–144 (2000).

[6] S. De Marchi. Generalized Vandermonde determinants, Toeplitz matrices and the Polynomial Division Algorithm, Universität Dortmund, Ergebnisberichte Angewandte Mathematik, nr. 176, June 1999.

[7] I. G. Macdonalds. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs, Second Edition, 1995.