On blossoming in integer Müntz spaces

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Abstract

The aim of the present note is to show that the theory of blossoming in integer Müntz spaces can be revised by means of Schur functions.

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1 Preliminaries on Schur functions

In this section we introduce the basics definitions and properties of Schur functions useful for understanding what follows.

Let $r \geq 0$ be an integer, and e_r the r-th elementary symmetric function, that is $e_0 = 1$ and $e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r}$ for $r \geq 1$.

Definition 1.1. *A partition is a finite or infinite sequence* $\lambda = (\lambda_1, \ldots, \lambda_r, \ldots)$ *of non negative integers in ascending order, that is* $\lambda_1 \geq \ldots \geq \lambda_r \geq \ldots$, *which contains a finite numbers of non vanishing terms.*

If the partition is finite, that is $\lambda = (\lambda_1, \ldots, \lambda_k)$, *then by* $l(\lambda) = k$ *we denote its* length *and by* $|\lambda| = \lambda_1 + \ldots + \lambda_k$ *its* width.

Definition 1.2. *The* diagram *of the partition* λ *is the set* $(i, j) \in \mathbb{Z}^2$ *such that* $1 \leq j \leq \lambda_i$.

For example the diagram of the partition λ=(64331) can be represented in matrix form as follows:

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & & & & & \end{array} \tag{1}$$

Definition 1.3. *The* conjugate *partition* λ' *of the partition* λ *is the one whose diagram is the transposed of the diagram of* λ, *that is*

$$\lambda'_i = Card\{j : \lambda_j \geq i\}. \tag{2}$$

For instance, transposing the diagram of λ=(64331) we have $\lambda' = (544211)$.

Let us consider a finite partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and form the determinantal ratio:

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)}{\det(x_i^{n-j})} \,. \tag{3}$$

Definition 1.4. s_λ *in (3) is called* Schur function *corresponding to the partition* λ *of length* n *and it is a homogeneous polynomial of degree* $|\lambda|$.

Let $\alpha_j = \lambda_j + n - j$ and $\delta_j = n - j$ for all j, we may define two sequences α and δ so that $\lambda = \alpha - \delta$. Moreover, letting $a_\alpha = \det\left(x_i^{\alpha_j}\right)$ and $a_\delta = \det\left(x_i^{\delta_j}\right) := V(x_1, ..., x_n)$, the classical *Vandermonde* determinant, then from (3) we have

$$a_\alpha = s_\lambda \cdot a_\delta = s_\lambda \cdot V(x_1, \ldots, x_n) \,. \tag{4}$$

We note that a_α is nothing that a *generalized Vandermonde determinant.*

Finally, any Schur function can be express by the *elementary symmetric functions* e_r associated to the points $x_1, ..., x_n$ (cf. e.g. [2, p. 34]) . That is

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \le i,j \le n} \tag{5}$$

where $n = l(\lambda')$. The formula (5) is known as *Jacobi-Trudi identity* (cf. [2, p. 34]).

More results on generalized Vandermonde determinants can be found in [1] and references therein.

2 Blossoming in Müntz spaces

In this Section we recall some facts related to the blossoming in Müntz spaces and raccomanding to interested readers to refer to the paper [6] for more details. The polynomial blossom of a function was introduced in the fundamental paper by L. Ramshaw in 1989 (cf. [8]) whiel more developments were done by many authors (cf. e.g. [7, 10, 4, 5, 9]).

Given $s+1$ real numbers $r_0, \dots, r_s$, and given $x_0, \dots, x_s > 0$ we introduce the following

$$\Delta \begin{pmatrix} x_0 & x_1 & \dots & x_s \\ r_0 & r_1 & \dots & r_s \end{pmatrix} := \begin{vmatrix} x_0^{r_0} & x_0^{r_0+r_1} & \dots & x_0^{r_0+\dots+r_s} \\ x_1^{r_0} & x_1^{r_0+r_1} & \dots & x_1^{r_0+\dots+r_s} \\ \vdots & \vdots & \ddots & \vdots \\ x_s^{r_0} & x_s^{r_0+r_1} & \dots & x_s^{r_0+\dots+r_s} \end{vmatrix}. \tag{6}$$

Theorem 2.1. **(cf. [6])** *Given* $s+1$ *real numbers* $r_0, \dots, r_s$ *such that* $r_i + \dots + r_j \neq 0$ *for all* i, j*, with* $1 \le i \le j \le s$*, then for all* $x_0, \dots, x_s \ge 0$*,*

$$\Delta \begin{pmatrix} x_0 & x_1 & \dots & x_s \\ r_0 & r_1 & \dots & r_s \end{pmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{s} (x_j - x_i) \tilde{\Delta} \begin{pmatrix} x_0 & x_1 & \dots & x_s \\ r_0 & r_1 & \dots & r_s \end{pmatrix}, \tag{7}$$

where the function $\tilde{\Delta} \begin{pmatrix} \cdot & \cdot & \dots & \cdot \\ r_0 & r_1 & \dots & r_s \end{pmatrix} : (\mathbb{R}_+^*)^{s+1} \longrightarrow \mathbb{R}$*, with* $\mathbb{R}_+^* =]0, +\infty[$*, is continuous, symmetric and homogeneous of degree* $d := \sum_{i=0}^{s} (s+1-i) r_i - \frac{s(s+1)}{2}$*, never vanishes on* $(\mathbb{R}_+^*)^{s+1}$ *and for which holds the following representation:*

$$\tilde{\Delta} \begin{pmatrix} x & x & \dots & x \\ r_0 & r_1 & \dots & r_s \end{pmatrix} = \frac{1}{s!(s-1)! \dots 2!} \partial_{x_s^s x_{s-1}^{s-1} \dots x_1} \Delta \begin{pmatrix} x & x & \dots & x \\ r_0 & r_1 & \dots & r_s \end{pmatrix}. \tag{8}$$

We recall that the property (8) is known as *diagonal property.*

Definition 2.1. *Given $s+1$ reals $r_0, r_1, \dots, r_s$ such that $r_i > 0,\ i \geq 1$ and $r_0 + \ldots + r_i \neq 0,\ i = 0, \dots, s$, the Chebyshev function of order $s+1$ on $\mathbb{R}^*_+$ is the function*

$$\Phi(x) := (x^{r_0}, x^{r_0+r_1}, \dots, x^{r_0+\ldots+r_s})^T. \tag{9}$$

Given distinct $a_0, \dots, a_s \in \mathbb{R}^*_+$, the corresponding *blossom* $\varphi = (\varphi_0, \dots, \varphi_s)^T$ of Φ is expressible as

$$\{\varphi(a_0, \dots, a_s)\} = \bigcap_{i=0}^{s} Osc_s\Phi(a_i). \tag{10}$$

where $Osc_s\Phi(a_i) := \left\{\Phi(a_i) + \sum_{l=1}^{s} \gamma_l \Phi^{(l)}(a_i) \mid \gamma_1, \dots, \gamma_s \in \mathbb{R}\right\}$.

Hence, for instance a point $X = (x_0, \dots, x_s)^T \in \mathbb{R}^{s+1}$ belongs to $Osc_s\Phi(a_0)$ *if and only if* $\det(X - \Phi(a_0), \Phi'(a_0), \dots, \Phi^{(s)}(a)) = 0$. Solving this determinant for a_0 we get

$$\alpha_0 x_0 a_0^{r_1+\ldots+r_s} + \alpha_1 x_1 a_0^{r_2+\ldots+r_s} + \ldots + \alpha_{s-1} x_{s-1} a_0^{r_s} + \alpha_s x_s = \beta a_0^{r_0+\ldots+r_s}, \tag{11}$$

with the coefficients $\alpha_0, \dots, \alpha_s, \beta$ not vanishing. By repeating the process for $a_1, \dots, a_s$ we get the linear system

$$\left\{\begin{array}{lcl} \alpha_0 x_0 a_0^{r_1+\ldots+r_s} + \alpha_1 x_1 a_0^{r_2+\ldots+r_s} + \ldots + \alpha_{s-1} x_{s-1} a_0^{r_s} + \alpha_s x_s & = & \beta a_0^{r_0+\ldots+r_s} \\ & \vdots & \\ \alpha_0 x_0 a_s^{r_1+\ldots+r_s} + \alpha_1 x_1 a_s^{r_2+\ldots+r_s} + \ldots + \alpha_{s-1} x_{s-1} a_s^{r_s} + \alpha_s x_s & = & \beta a_s^{r_0+\ldots+r_s} \end{array}\right.$$

which has solution $\{\varphi_0(a_0, \dots, a_s), \dots, \varphi_s(a_0, \dots, a_s)\}$.

For the blossom $\varphi = (\varphi_0, \dots, \varphi_s)^T$ of Φ the following *general* Theorem holds (cf. [6]).

Theorem 2.2. *Given $a_0, \dots, a_s \in \mathbb{R}^*_+$, letting*

$$\tilde{\Delta}_{r_s,\dots,r_1}(a_0, \dots, a_s) = \begin{vmatrix} 1 & a_0^{r_s} & a_0^{r_s+r_{s-1}} & \dots & a_0^{r_s+\dots+r_1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_s^{r_s} & a_s^{r_s+r_{s-1}} & \dots & a_s^{r_s+\dots+r_1} \end{vmatrix}$$

then the blossom φ of the Chebyshev function Φ defined in (9) is

$$\varphi_i(a_0, \dots, a_s) = C_i \frac{\tilde{\Delta}\begin{pmatrix} a_0 & a_1 & \dots & \dots & a_{s-i} & \dots & \dots & a_s \\ 0 & r_s & \dots & r_{i+2} & r_{i+1}+r_i & r_{i-1} & \dots & r_0 \end{pmatrix}}{\tilde{\Delta}_{r_s,\dots,r_1}(a_0, \dots, a_s)}$$

$$\varphi_s(a_0, \dots, a_s) = C_s (a_0 \cdots a_s)^{r_s} \frac{\tilde{\Delta}\begin{pmatrix} a_0 & a_1 & \dots & a_s \\ 0 & r_{s-1} & \dots & r_0 \end{pmatrix}}{\tilde{\Delta}_{r_s,\dots,r_1}(a_0, \dots, a_s)}. \tag{12}$$

where the coefficients C_i are given by

$$C_i = \frac{\prod_{l=1,l\neq i}^{s}\left(\sum_{j=1}^{l} r_{i+1-j}\right)}{\prod_{l=0,l\neq i}^{s}\left(\sum_{j=0}^{l} r_j\right)} \quad i=0,\dots,s\,. \tag{13}$$

It is now clear that the result of Theorem 2.1 can be seen is a special instance of the factorization (4).

Proposition 2.1. *Under the hyphoteses of Theorem 2.1 with the r_j non negative integers,*

$$\tilde{\Delta}\begin{pmatrix} x_0 & x_1 & \dots & x_s \\ r_0 & r_1 & \dots & r_s \end{pmatrix}$$

is a Schur function associated to the points $x_0, ..., x_s$ based on the sequences $\alpha = (\sum_{i=0}^{s} r_i, \sum_{i=0}^{s-1} r_i, \dots, r_0)$, $\delta = (s-1, s-2, \dots, 0)$ and $\lambda = \alpha - \delta$.

Furthermore, as a consequence of the Theorem 2.2, we get an expression for the blossom of an *integer Chebyshev function*, that is in (9) all exponents are integers.

Theorem 2.3. *Given $a_0, \dots, a_s \in \mathbb{R}_+^*$ and integers r_i associated to the Chebyshev function (9). Define sequences $\alpha_i^{s-i} = (\sum_{j=0}^{s} r_j, \sum_{j=1}^{s} r_j, \dots, r_s, 0)$, for $i = 0, ..., s-1$ where the superscript $s-i$ says that in the $(s-i)$-th component the sum is done up to $i-1$, r_i is substituted by $r_i + r_{i+1}$ and untill $j = s$. Moreover, $\alpha_s^0 = (\sum_{j=0}^{s-1} r_j, \sum_{j=1}^{s-1} r_j, \dots, r_{s-1}, 0)$, $\alpha_{s+1} = (\sum_{j=0}^{s-2} r_j, \sum_{j=1}^{s-2} r_j, \dots, r_{s-2}, 0)$, $\delta = (s-1, s-2, ..., 1, 0)$ so that $\lambda_p^* = \alpha_p^* - \delta$. Then the blossom φ is*

$$\varphi_i(a_0, \dots, a_s) = C_i \frac{s_{\lambda_i^{s-i}}(a_0, ..., a_{s-i}, \dots, a_s)}{s_{\lambda_{s+1}}(a_0, \dots, a_s)}\,, \quad i = 0, ..., s-1$$

$$\varphi_s(a_0, \dots, a_s) = C_s \left(\prod_{k=0}^{s} a_k\right)^{r_s} \frac{s_{\lambda_s^0}(a_0, ..., a_s)}{s_{\lambda_{s+1}}(a_0, \dots, a_s)}. \tag{14}$$

where the coefficients C_i, $i = 0, \dots, s$ are given in (13).

Proof. Apply the Proposition 2.1 to each $\tilde{\Delta}$ appearing in Theorem 2.2. Computed the C_i, the blossom $\varphi = (\varphi_0, \dots, \varphi_s)$ associated to a Chebyshev function Φ is obtained by applying the Jacobi-Trudi identity to each $s_{\lambda_p^*}$. □

2.1 An example: blossoming in integer Müntz spaces

In this section we present a complete example that computes the blossom in a polynomial Müntz spaces of dimension 5 (i.e. with integer Chebyshev function) firstly by using the *Mazure approach* (cf. Theorem 2.2) and then by using our observation based on Schur functions (cf. Theorem 2.3).

Example 2.1. Given the integers r_1, r_2, r_3 such that $r_1 r_2 r_3 (r_1+r_2)(r_2+r_3)(r_1+r_2+r_3) \neq 0$, by using formula (6) and for all x, y, $z \in \mathbb{R}_+^*$ get

(15)
$$\begin{aligned} \Delta \begin{pmatrix} x & y & z & 0 \\ 0 & r_1 & r_2 & r_3 \end{pmatrix} &= -(xyz)^{r_1} \Delta \begin{pmatrix} x & y & z \\ 0 & r_2 & r_3 \end{pmatrix} \\ &= (xyz)^{r_1} \Delta_{r_2,r_3}(x,y,z) \\ &= (xyz)^{r_1}(z-x)(z-y)(y-x)\tilde{\Delta}_{r_2,r_3}(x,y,z). \end{aligned}$$

where we use the shorthand notations $\Delta_{r_2,r_3}(x,y,z)$ and $\tilde{\Delta}_{r_2,r_3}(x,y,z)$ instead of the corresponding determinant Δ. In particular, from formula (15) we have a conseqence

(16)
$$\Delta_{r_2,r_3}(x,y,z) = V(x,y,z) \cdot \tilde{\Delta}_{r_2,r_3}(x,y,z) = (z-y)(z-x)(y-x) \cdot \tilde{\Delta}_{r_2,r_3}(x,y,z) \; ;$$

which, as we shall see later, can be generalized to longer sequences of indeces r.

Thus, by means of (7)

$$\Delta \begin{pmatrix} x & y & z & 0 \\ 0 & r_1 & r_2 & r_3 \end{pmatrix} = -xyz(y-x)(z-x)(z-y)\tilde{\Delta} \begin{pmatrix} x & y & z & 0 \\ 0 & r_1 & r_2 & r_3 \end{pmatrix}, \tag{17}$$

and proceeding as before we obtain

$$\tilde{\Delta} \begin{pmatrix} x & y & z & 0 \\ 0 & r_1 & r_2 & r_3 \end{pmatrix} = (xyz)^{r_1-1} \tilde{\Delta}_{r_2,r_3}(x,y,z) \, . \tag{18}$$

Let $\Phi(x) := (x^2, x^3, x^4, x^5)$ be a Chebyshev function for which we want to compute its blossoms $\varphi := (\varphi_0, \varphi_1, \varphi_2, \varphi_3)$. From Theorem 2.2 we get

$$\begin{aligned} \varphi_0(x,y,z,t) &= C_0 \frac{\tilde{\Delta} \begin{pmatrix} x & y & z & t \\ 0 & r_3 & r_2 & r_0+r_1 \end{pmatrix}}{\tilde{\Delta}_{r_3,r_2,r_1}(x,y,z,t)} \\ \varphi_1(x,y,z,t) &= C_1 \frac{\tilde{\Delta} \begin{pmatrix} x & y & z & t \\ 0 & r_3 & r_1+r_2 & r_0 \end{pmatrix}}{\tilde{\Delta}_{r_3,r_2,r_1}(x,y,z,t)} \\ \varphi_2(x,y,z,t) &= C_2 \frac{\tilde{\Delta} \begin{pmatrix} x & y & z & t \\ 0 & r_2+r_3 & r_1 & r_0 \end{pmatrix}}{\tilde{\Delta}_{r_3,r_2,r_1}(x,y,z,t)} \\ \varphi_3(x,y,z,t) &= C_3 (xyzt)^{r_3} \frac{\tilde{\Delta} \begin{pmatrix} x & y & z & t \\ 0 & r_2 & r_1 & r_0 \end{pmatrix}}{\tilde{\Delta}_{r_3,r_2,r_1}(x,y,z,t)}, \end{aligned} \tag{19}$$

Since $r_0 = 2$ e $r_1 = r_2 = r_3 = 1$, then the denominator in (19) is $\tilde{\Delta}_{1,1,1}(x, y, z, t) = 1$ and using formula (13) the C_i are

$$C_0 = \frac{1}{10},\ C_1 = \frac{1}{20},\ C_2 = \frac{1}{15},\ C_3 = \frac{1}{4}. \tag{20}$$

The blossom φ is then:

$$\begin{aligned}
\varphi_0(x, y, z, t) &= \frac{1}{10}\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 3 \end{pmatrix}, \\
\varphi_1(x, y, z, t) &= \frac{1}{20}\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 2 & 2 \end{pmatrix}, \\
\varphi_2(x, y, z, t) &= \frac{1}{15}\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 2 & 1 & 2 \end{pmatrix}, \\
\varphi_3(x, y, z, t) &= \frac{1}{4}(xyzt)\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 2 \end{pmatrix}.
\end{aligned} \tag{21}$$

It suffices to explicit $\tilde{\Delta}$ in (21) by computing $\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 3 \end{pmatrix}$. To this aim we recall the equality (16) that can be explicitly written as (cf. [6, formula (4.4)])

$$\tilde{\Delta}_{r_1,r_2}(x, y, z) = \sum_{i=0}^{r_1-1}\sum_{j=0}^{r_2-1} y^{i+j} \sum_{l=0}^{r_2+i-j-1} x^{r_1+l-1-i} z^{r_1+r_2-2-j-l}. \tag{22}$$

Then,

$$\tilde{\Delta}\begin{pmatrix} x & y & z & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix} \overset{(18)}{=} \tilde{\Delta}_{1,3}(x, y, z) \overset{(22)}{=} x^2 + y^2 + z^2 + xz + xy + yz. \tag{23}$$

Since $\tilde{\Delta}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & 3 \end{pmatrix}$ is homogeneous and symmetric of total degree 2 and degree 2 in every variable, necessarily we have

$$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 3 \end{pmatrix} = x^2 + y^2 + z^2 + t^2 + xz + xy + yz + xt + yt + zt. \tag{24}$$

In a similar manner we compute the remaining terms:

$$\begin{aligned}
\tilde{\Delta}\begin{pmatrix} x & y & z & 0 \\ 0 & 1 & 2 & 2 \end{pmatrix} &\overset{(18)}{=} \tilde{\Delta}_{2,2}(x, y, z) \\
&\overset{(22)}{=} x^2(y+z) + y^2(x+z) + z^2(x+y) + 2xyz, \\
\tilde{\Delta}\begin{pmatrix} x & y & z & 0 \\ 0 & 2 & 1 & 2 \end{pmatrix} &\overset{(18)}{=} (xyz)\tilde{\Delta}_{1,2}(x, y, z) \overset{(22)}{=} xyz(x+y+z), \\
\tilde{\Delta}\begin{pmatrix} x & y & z & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} &\overset{(18)}{=} \tilde{\Delta}_{1,2}(x, y, z) \overset{(22)}{=} x + y + z.
\end{aligned} \tag{25}$$

For the symmetry and homogeneity we conclude:

$$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 2 & 2 \end{pmatrix} = x^2(y+z+t)+y^2(x+z+t)+z^2(x+y+t) \\ +t^2(x+y+z)+2xyz+2xyt+2xzt+2yzt, \tag{26}$$

$$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 2 & 1 & 2 \end{pmatrix} = x^2(yz+zt+yt)+y^2(xz+zt+xt)+z^2(xy+yt+xt) \\ +t^2(xy+yz+zx)+Axyzt, \tag{27}$$

$$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 2 \end{pmatrix} = x+y+z+t. \tag{28}$$

It remains to compute A that appears in formula (27). By resorting to the formula that expresses the diagonal property of $\tilde{\Delta}$ (cf. [6, Th. 5.2, formula (5.4)]), that is

$$\tilde{\Delta}\begin{pmatrix} x & x & \dots & x \\ r_0 & r_1 & \dots & r_s \end{pmatrix} = \frac{x^d}{s!(s-1)!\dots 2!}\prod_{j=0}^{s-1}\prod_{k=1}^{s-j}(r_k+\dots+r_{k+j}). \tag{29}$$

we get

$$\tilde{\Delta}\begin{pmatrix} x & x & x & x \\ 0 & 2 & 1 & 2 \end{pmatrix} = 12x^2+Ax^2 \overset{(29)}{=} 15x^2, \tag{30}$$

this gives $A=3$.

We have then computed in a unique way the blossoms φ of Φ and one can trivially prove that the following fundamental equality holds:

$$\Phi(x) = \begin{pmatrix} x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = \begin{pmatrix} \varphi_0(x,x,x,x) \\ \varphi_1(x,x,x,x) \\ \varphi_2(x,x,x,x) \\ \varphi_3(x,x,x,x) \end{pmatrix} = \varphi(x,x,x,x). \tag{31}$$

$$\Diamond\ \square\ \Diamond$$

We use the results obtained up to formula (21) and we now compute the functions $\tilde{\Delta}$ by mean of our approach based on Schur functions, being $n=4$, $s=3$. To make the treatment more simple we omit the superscripts and indeces in the λs and we refer to the notations introduced in Theorem 2.3.

Firstly we should factor

$$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & x & x^2 & x^5 \\ 1 & y & y^2 & y^5 \\ 1 & z & z^2 & z^5 \\ 1 & t & t^2 & t^5 \end{pmatrix}. \tag{32}$$

which is a generalized Vandermonde determinant. Thus, $\alpha = (5,2,1,0)$, $\delta = (3,2,1,0)$ and

$$\lambda = \alpha - \delta = (5,2,1,0) - (3,2,1,0) = (2,0,0,0) \text{ and } \lambda' = (1,1)\,. \tag{33}$$

thanks to the Jacobi-Trudi identity (5), the corresponding Schur function $s_\lambda = s_\lambda(x,y,z,t)$ is

$$\begin{aligned} s_\lambda &\overset{(5)}{=} \det(e_{\lambda'_i - i + j})_{1\le i,j\le 2} = \det\begin{pmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} \end{pmatrix} \\ &= \det\begin{pmatrix} e_1 & e_2 \\ e_0 & e_1 \end{pmatrix} = \begin{vmatrix} x+y+z+t & xy+xz+xt+yz+yt+zt \\ 1 & x+y+z+t \end{vmatrix} \\ &= x^2+y^2+z^2+t^2+xy+xz+xt+yz+yt+zt. \end{aligned} \tag{34}$$

In the same way we compute the remaining Schur functions $\tilde{\Delta}$. $\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 2 & 2 \end{pmatrix}$ has $\lambda = (2,1,0,0)$ and $\lambda' = (2,1)$ then

$$\begin{aligned} s_\lambda &\overset{(5)}{=} \det(e_{\lambda'_i - i + j})_{1\le i,j\le 2} = \det\begin{pmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} \end{pmatrix} \\ &= \begin{vmatrix} e_2 & e_3 \\ e_0 & e_1 \end{vmatrix} = \begin{vmatrix} xy+xz+xt+yz+yt+zt & xyz+xyt+yzt+xzt \\ 1 & x+y+z+t \end{vmatrix} \\ &= x^2(y+z+t)+y^2(x+z+t)+z^2(x+y+t)+t^2(x+y+z) \\ &\quad +2xyz+2xyt+2yzt+2xzt. \end{aligned} \tag{35}$$

$\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 2 & 1 & 2 \end{pmatrix}$ has $\lambda = (2,1,1,0)$ and $\lambda' = (3,1)$, then

$$\begin{aligned} s_\lambda &\overset{(5)}{=} \det(e_{\lambda'_i - i + j})_{1\le i,j\le 2} = \det\begin{pmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} \end{pmatrix} \\ &= \det\begin{pmatrix} e_3 & e_4 \\ e_0 & e_1 \end{pmatrix} = \begin{vmatrix} xyz+xyt+yzt+xzt & xyzt \\ 1 & x+y+z+t \end{vmatrix} \\ &= x^2(yz+yt+zt)+y^2(xz+xt+zt)+z^2(xy+xt+yt) \\ &\quad +t^2(xy+xz+yz)+3xyzt. \end{aligned} \tag{36}$$

Finally, $\tilde{\Delta}\begin{pmatrix} x & y & z & t \\ 0 & 1 & 1 & 2 \end{pmatrix}$ has $\lambda = (1,0,0,0)$ and $\lambda' = 1$ so that

$$s_\lambda \stackrel{(5)}{=} \det(e_{\lambda'-1+1}) = e_1 = x + y + z + t. \tag{37}$$

Finally the denominator appearing in Theorem 2.3 has $\lambda = (0,0,0,0)$ and $\lambda' = (0)$. Then $\det(e_{\lambda'_i - i + j})_{1 \le i,j \le 1} = 1$.

3 Conclusion

The main conclusion of the paper is that to determine the blossom ϕ of a Chebyshev function $\Phi(x)$, the Schur functions approch makes the computations more elegant and intuitive.

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