Numer. Math.
DOI 10.1007/s00211-007-0112-z

Numerische Mathematik

Bivariate Lagrange interpolation at the Padua points: the ideal theory approach

Len Bos · Stefano De Marchi · Marco Vianello · Yuan Xu

Received: 28 March 2006 / Revised: 13 August 2007

Abstract The Padua points are a family of points on the square $[-1, 1]^2$ given by explicit formulas that admits unique Lagrange interpolation by bivariate polynomials. Interpolation polynomials and cubature formulas based on the Padua points are studied from an ideal theoretic point of view, which leads to the discovery of a compact formula for the interpolation polynomials. The L^p convergence of the interpolation polynomials is also studied.

Mathematics Subject Classification (2000) 41A05 · 41A10

S. De Marchi and M. Vianello were supported by the "ex-60%" funds of the University of Padua and by the INdAM GNCS (Italian National Group for Scientific Computing). Y. Xu was partially supported by NSF Grant DMS-0604056.

L. Bos
Department of Mathematics and Statistics, University of Calgary,
Calgary, AB, Canada T2N1N4
e-mail: lpbos@math.ucalgary.ca

S. De Marchi
Department of Computer Science, University of Verona, 37134 Verona, Italy
e-mail: demarchi@sci.univr.it

M. Vianello
Department of Pure and Applied Mathematics, University of Padua,
35131 Padua, Italy
e-mail: marcov@math.unipd.it

Y. Xu (✉)
Department of Mathematics, University of Oregon,
Eugene, OR 97403-1222, USA
e-mail: yuan@math.uoregon.edu

1 Introduction

For polynomial interpolation in one variable, the Chebyshev points (zeros of Chebyshev polynomial) are optimal in many ways. Let $L_n f$ be the nth Lagrange interpolation polynomial based on the Chebyshev points. The Lebesgue constant $\|L_n\|$ in the uniform norm on $[-1, 1]$ has growth of order of $\mathcal{O}(\log n)$, which is the minimal rate of growth of any projection operator from $C[-1, 1]$ onto the space of polynomials of degree at most n. The Chebyshev points are also the knots of the Gaussian quadrature formula for the weight function $1/\sqrt{1-x^2}$ on $[-1, 1]$.

In the case of two variables, it is often difficult to even give explicit point sets for which the polynomial interpolation problem is well-posed, or *unisolvent*. Let Π_n^2 denote the space of polynomials of degree at most n in two variables. It is well known that $\dim \Pi_n^2 = (n+2)(n+1)/2$. Then consider a set V of points in $\mathbb{R}^2$ of cardinality $|V| = \dim \Pi_n^2$. The set V is said to be unisolvent if for any given function f defined on $\mathbb{R}^2$, there is a unique polynomial P such that $P(x) = f(x)$ on V.

The Padua points are a family of interpolation points that may legitimately be considered as an analogue of the Chebyshev points for the square $[-1, 1]^2$. For each $n \geq 0$, they are defined by

$$\mathrm{Pad}_n = \{\mathbf{x}_{k,j} = (\xi_k, \eta_j), \quad 0 \leq k \leq n, \quad 1 \leq j \leq \lfloor \tfrac{n}{2} \rfloor + 1 + \delta_k\}, \tag{1.1}$$

where $\delta_k := 0$ if n is even or n is odd but k is even, $\delta_k := 1$ if n is odd and k is odd, and

$$\xi_k = \cos\frac{k\pi}{n}, \qquad \eta_j = \begin{cases} \cos\frac{2j-1}{n+1}\pi, & k \text{ even} \\ \cos\frac{2j-2}{n+1}\pi, & k \text{ odd} \end{cases} \tag{1.2}$$

It is easy to verify that the cardinality of Pad_n is equal to the dimension of Π_n^2. These points were introduced heuristically in [4] (only for even degrees) and proved to be unisolvent in [3]. The Lebesgue constant of the Lagrange interpolation based on the Padua points in the uniform norm grows like $\mathcal{O}((\log n)^2)$, which is the minimal order of growth for any set of interpolation points in the square. Moreover, there is a cubature formula of degree $2n-1$ based on these points for the weight function $1/(\sqrt{1-x^2}\sqrt{1-y^2})$ on $[-1, 1]^2$ [3]. A family of points studied in [10] also possesses similar properties [1,2,10], but the cardinality of the set is not equal to $\dim \Pi_n^2$ and the interpolation polynomial belongs to a subspace of Π_n^2.

The study in [3] starts from the fact that the Padua points there lie on a single generating curve,

$$\gamma_n(t) = (\cos(nt), \cos((n+1)t), \qquad 0 \leq t \leq \pi.$$

In fact, the points are exactly the distinct points among $\{\gamma_n(\frac{k\pi}{n(n+1)}), 0 \leq k \leq n(n+1)\}$. In the present paper, we study the interpolation on the Padua points using an ideal theoretic approach, which considers the points as the variety of a polynomial ideal and it treats the cubature formula based on the interpolation points simultaneously. The advantage of this approach is that it casts the result on Padua points into a general theoretic framework. The study in [10] used such an approach. In particular, it shows how the compact formula of the interpolation polynomial arises naturally.

The order of the Lebesgue constant of the interpolation is found in [3], which solves the problem about the convergence of the interpolation polynomials on the Padua points in the uniform norm. In the present paper, we show the convergence in a weighted L^p norm.

The paper is organized as follows. In the following section, we review the necessary background and show that the Padua points are unisolvent. The explicit formula of the Lagrange interpolation polynomials is derived in Sect. 3, and the L^p convergence of the interpolation polynomials is proved in Sect. 4.

2 Polynomial ideal and Padua points

2.1 Polynomial ideals, interpolation, and cubature formulas

We first recall some notation and results about polynomial ideals and their relation to polynomial interpolation and cubature formulas. To keep the notation simple we shall restrict to the case of two variables, even though all results in this subsection hold for more than two variables.

Let I be a polynomial ideal in $\mathbb{R}[\mathbf{x}]$ with $\mathbf{x} = (x_1, x_2)$, the ring of all polynomials in two variables. If there are polynomials $f_1, \ldots, f_r$ in $\mathbb{R}[\mathbf{x}]$ such that every $f \in I$ can be written as $f = a_1 f_1 + \cdots + a_r f_r$, $a_j \in \mathbb{R}[\mathbf{x}]$, then we say that I is generated by the basis $\{f_1, \ldots, f_r\}$ and we write $I = \langle f_1, \ldots, f_r \rangle$. Fix a monomial order, say the lexicographical order, and let $\mathrm{LT}(f)$ denote the leading term of the polynomial f in the monomial order. Let $\langle \mathrm{LT}(I) \rangle$ denote the ideal generated by the leading terms of $\mathrm{LT}(f)$ for all $f \in I \setminus \{0\}$. Then it is well known that there is an isomorphism between $\mathbb{R}[\mathbf{x}]/I$ and the space $\mathcal{S}_I := \mathrm{span}\{x_1^k x_2^j : x_1^k x_2^j \notin \langle \mathrm{LT}(I) \rangle$ [5, Chapt. 5]. The codimension of the ideal is defined by $\mathrm{codim}(I) := \dim(\mathbb{R}[\mathbf{x}]/I)$.

For an ideal I of $\mathbb{R}[\mathbf{x}]$, we denote by $V = V(I)$ its real affine variety, i.e., $V(I) = \{x \in \mathbb{R}^2 : p(x) = 0, \forall p \in I\}$. We consider the case of a zero dimensional variety, that is, when V is a finite set of distinct points in $\mathbb{R}^2$. In this case, it is well-known that $|V| \le \mathrm{codim}\, I$. The following result is proved in [11].

Proposition 2.1 *Let I be a polynomial ideal in $\mathbb{R}[\mathbf{x}]$ with finite codimension and let V be its affine variety. If $|V| = \mathrm{codim}(I)$ then there is a unique interpolation polynomial in $\mathcal{S}_I$ based on the points in the variety, i.e., for every "ordinate function" $z : V \to \mathbb{R}$ there exists a unique $p \in \mathcal{S}_I$ such that $p(x) = z(x)$, $\forall x \in V$.*

In this case I coincides with the polynomial ideal $I(V)$, which contains all polynomials in $\mathbb{R}[\mathbf{x}]$ that vanish on V. An especially interesting case is when the ideal is generated by a sequence of quasi-orthogonal polynomials. Let $d\mu$ be a nonnegative measure with finite moments on a subset of $\mathbb{R}^2$. A polynomial $P \in \mathbb{R}[\mathbf{x}]$ is said to be an orthogonal polynomial with respect to $d\mu$ if

$$\int_{\mathbb{R}^2} P(\mathbf{x}) Q(\mathbf{x}) d\mu(\mathbf{x}) = 0, \qquad \forall Q \in \mathbb{R}[\mathbf{x}], \quad \deg Q < \deg P;$$

it is called a $(2n-1)$-orthogonal polynomial if the above integral is zero for all $Q \in \mathbb{R}[\mathbf{x}]$ such that $\deg P + \deg Q \le 2n-1$. In particular, if P is of degree $n+1$ and $(2n-1)$ orthogonal, then it is orthogonal with respect to all polynomials of degree $n-2$ or lower. We need the following result (cf. [11]).

Proposition 2.2 *Let I and V be as in the previous proposition. Assume that I is generated by $(2n-1)$-orthogonal polynomials. If* $\operatorname{codim} I = |V|$ *then there is a cubature formula*

$$\int_{\mathbb{R}^2} f(\mathbf{x}) d\mu(\mathbf{x}) \approx \sum_{\mathbf{x}\in V} \lambda_{\mathbf{x}} f(\mathbf{x})$$

of degree $2n-1$; that is, the above formula is an identity for all $f \in \Pi^2_{2n-1}$.

The cubature formula in this proposition can be obtained by integrating the interpolation polynomial in Proposition 2.1. We will apply this result for $d\mu = W(\mathbf{x})d\mathbf{x}$, where W is the Chebyshev weight function on $[-1,1]^2$,

$$W(\mathbf{x}) = \frac{1}{\pi^2} \frac{1}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}}, \qquad -1 < x_1, x_2 < 1. \tag{2.1}$$

For such a weight function, the cubature formula of degree $2n-1$ exists only if $|V|$ satisfies Möller's lower bound $|V| \ge \dim \Pi^2_{n-1} + [n/2]$. Formulas that attain this lower bound exist [7,8], whose knots are the interpolation points studied in [10].

Recall that the Chebyshev polynomials T_n and U_n of the first and the second kind are given by $T_n(x) = \cos n\theta$ and $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$, respectively. The polynomials $T_n(x)$ are orthogonal with respect to $1/\sqrt{1-x^2}$. Consequently, it is easy to verify that the polynomials

$$T_k(x_1)T_{n-k}(x_2), \qquad 0 \le k \le n, \tag{2.2}$$

are mutually orthogonal polynomials of degree n with respect to the weight function $W(x_1, x_2)$ on $[-1,1]^2$.

2.2 Polynomial ideals and Padua points

The Padua points (1.1) were introduced in [4] (apart from a misprint that $n-1$ should be replaced by $n+1$) based on a heuristic argument that we now describe. Morrow and Patterson [7] introduced, for $s \ge 1$, an explicit set of knots in $[-1,1]^2$ for a cubature formula of degree $2s$ with respect to the weight function $\sqrt{1-x_1^2}\sqrt{1-x_2^2}$ on $[-1,1]^2$. These Morrow–Patterson points are the common zeros of

$$R^s_j(x_1, x_2) := U_j(x_1)U_{s-j}(x_2) + U_{s-j-1}(x_1)U_j(x_2), \qquad 0 \le j \le s. \tag{2.3}$$

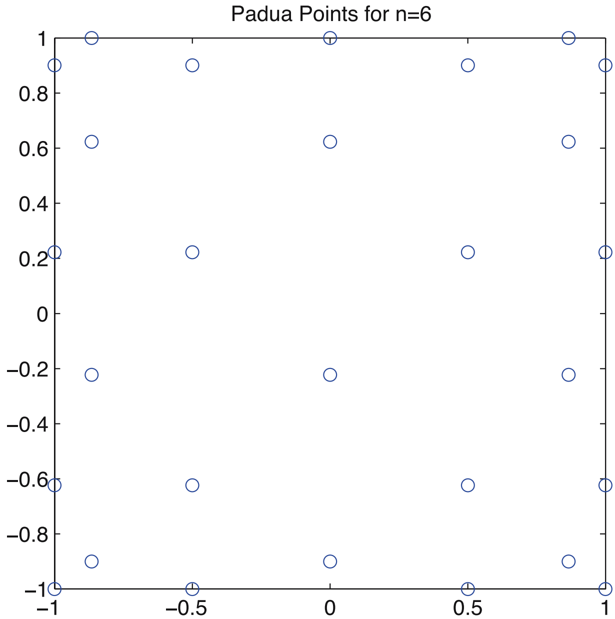

Fig. 1 Padua points for $n = 6$

There turn out to be exactly dim Π_s^2 many such points, and they are all in the *interior* $(-1, 1)^2$. Moreover, they may be seen to lie on a certain collection of vertical and horizontal lines, forming a subset of a certain rectangular grid.

The Padua points of degree n are precisely the Morrow–Patterson points of degree $n-2$, together with the "natural" boundary points of this grid. Figure 1 shows the 28 Padua points of degree 6, consisting of the 15 interior Morrow–Patterson points of degree $6-2=4$, together with 13 additional boundary points.

It should be noted that the Morrow–Patterson points turn out to be sub-optimal in the sense that the associated Lebesgue constants grow faster than best possible. In contrast, the Padua points have Lebesgue constants of optimal growth. See [1] for a discussion of this fact.

As shown in [3] the Padua points can be characterized as self-intersection points (interior points) and the boundary contact points of the algebraic curve $T_n(x_1) + T_{n+1}(x_2) = 0$, $(x_1, x_2) \in [-1, 1]^2$ (the *generating curve*), where T_n denotes the nth Chebyshev polynomial of the first kind. (Actually the generating curve used in [3] is $T_{n+1}(x_1) - T_n(x_2) = 0$.) Depending on the orientation of the x_1 and x_2 directions, there are four families of Padua points, which are the two mentioned above and two others whose generating curves are $T_n(x_1) - T_{n+1}(x_2) = 0$ and $T_{n+1}(x_1) + T_n(x_2) = 0$, respectively. We restrict to the family (1.1) in this paper.

Theorem 2.3 *Let* $Q_k^{n+1} \in \Pi_{n+1}^2$ *be defined by*

$$Q_0^{n+1}(x_1, x_2) = T_{n+1}(x_1) - T_{n-1}(x_1), \tag{2.4}$$

and for $1 \le k \le n+1$*,*

$$Q_k^{n+1}(x_1, x_2) = T_{n-k+1}(x_1)T_k(x_2) + T_{n-k+1}(x_2)T_{k-1}(x_1). \tag{2.5}$$

Then the set Pad_n *in* (1.1) *is the variety of the ideal* $I = \langle Q_0^{n+1}, Q_1^{n+1}, \ldots, Q_{n+1}^{n+1}\rangle$*. Furthermore,* $\mathrm{codim}(I) = |\,\mathrm{Pad}_n\,| = \dim \Pi_n^2$*.*

Proof That the polynomials Q_k^{n+1} vanish on the Padua points (1.1) can be easily verified upon using the following representation

$$Q_0^{n+1}(x_1, x_2) = -2(1 - x_1^2)U_{n-1}(x_1) = -2\sin\theta \sin n\theta,$$

and, for $1 \le k \le n+1$,

$$\begin{aligned} Q_k^{n+1}(x_1, x_2) = {} & \cos(k-1)\theta(\cos(n+1)\phi\cos k\phi + \sin(n+1)\phi\sin k\phi) \\ & + \cos k\phi(\cos n\theta\cos(k-1)\theta + \sin n\theta\sin(k-1)\theta), \end{aligned}$$

where $x_1 = \cos\theta$ and $x_2 = \cos\phi$. Furthermore, the definition of Q_k^{n+1} shows easily that the set $\{\mathrm{LT}(Q_0^{n+1}), \ldots, \mathrm{LT}(Q_{n+1}^{n+1})\}$ is exactly the set of monomials of degree $n+1$ in $\mathbb{R}[\mathbf{x}]$. Hence, it follows that $\dim \mathbb{R}[\mathbf{x}]/I \le \dim \Pi_n^d$. Recall that $\mathrm{codim}(I) = \dim \mathbb{R}[\mathbf{x}]/I \ge |V|$ and $|V| \ge |\,\mathrm{Pad}_n\,| = \dim \Pi_n^d$, we conclude that $\mathrm{codim}(I) = |\,\mathrm{Pad}_n\,|$. □

The basis (2.4) and (2.5) of the ideal was originally identified by using the fact that the interior points of Pad_n are the common zeros of (2.3). This implies that the ideal contains the polynomials $(1-x_1)^2(1-x_2)^2 R_j^{n-1}$, $0 \le j \le n-1$, as well as two specific univariate polynomials, $(1-x_1^2)U_{n-1}(x_1)$, which gives rise to (2.4), and a certain other polynomial in x_2 (the details of which are not important). From these polynomials we were able to derive a simple basis for the ideal, the Q_k^{n+1}. Once the basis is identified, it is of course, after the fact, easier to simply verify it directly, as we did in the proof.

Since the product Chebyshev polynomials (2.2) are orthogonal with respect to W in (2.1), it follows readily that $Q_k^{n+1}(x_1, x_2)$ are orthogonal to the polynomials in Π_{n-2}^2 with respect to W on $[-1, 1]^2$. In particular, they are $(2n-1)$-orthogonal polynomials. Hence, as a consequence of the theorem and Propositions 2.1 and 2.2, we have the following:

Corollary 2.4 *The set of Padua points* Pad_n *is unisolvent for* Π_n^d*. Furthermore, there is a cubature formula of degree* $2n-1$ *based on the Padua points in* Pad_n*.*

We denote the unique interpolation polynomial based on Pad_n by $\mathcal{L}_n f$, which can be written as

$$\mathcal{L}_n f(\mathbf{x}) = \sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} f(\mathbf{x}_{k,j})\ell_{k,j}(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2), \tag{2.6}$$

where $\ell_{k,j}(\mathbf{x})$ are the fundamental interpolation polynomials uniquely determined by

$$\ell_{k,j}(\mathbf{x}_{k',j'}) = \delta_{k,k'}\delta_{j,j'}, \quad \forall \mathbf{x}_{k,j} \in \mathrm{Pad}_n, \quad \text{and} \quad \ell_{k,j} \in \Pi_n^d. \tag{2.7}$$

In the following section we derive an explicit formula for the fundamental polynomials.

3 Construction of the Lagrange interpolation polynomials

To derive explicits formulas for the fundamental Lagrange polynomials of (2.6), we will use orthogonal polynomials and follow the strategy in [8,10]. We note that the method in [8] works for the case that $|V| = \dim \Pi_{n-1}^2 + \sigma$ with $\sigma \le n$. In our case $\sigma = n + 1$, so that the general theory there does not apply. We remind the reader that [3] gives an alternate derivation.

Let $\mathcal{V}_n^d$ denote the space of orthogonal polynomials of degree n with respect to (2.1) on $[-1, 1]^2$. An orthonormal basis for $\mathcal{V}_n^d$ is given by

$$P_n^k(x_1, x_2) := \tilde{T}_{n-k}(x_1)\tilde{T}_k(x_2), \qquad 0 \le k \le n,$$

where $\widetilde{T}_0(x) = 1$ and $\tilde{T}_k(x) = \sqrt{2}T_k(x)$ for $k \ge 1$. We introduce the notation

$$\mathbb{P}_n = \left[P_0^n, P_1^n, \ldots, P_n^n\right]$$

and treat it both as a set and as a column vector. The reproducing kernel of the space Π_n^d in $L^2(W, [-1, 1]^2)$ is defined by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} [\mathbb{P}_k(\mathbf{x})]^t \mathbb{P}_k(\mathbf{y}) = \sum_{k=0}^{n}\sum_{j=0}^{k} P_j^k(\mathbf{x}) P_j^k(\mathbf{y}). \tag{3.1}$$

There is a Christoffel-Darboux formula (cf. [6,8]) which states that

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \frac{\left[A_{n,i}\mathbb{P}_{n+1}(\mathbf{x})\right]^t \mathbb{P}_n(\mathbf{y}) - \left[A_{n,i}\mathbb{P}_{n+1}(\mathbf{y})\right]^t \mathbb{P}_n(\mathbf{x})}{x_i - y_i}, \quad i = 1, 2\,, \tag{3.2}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $A_{n,i} \in \mathbb{R}^{(n+1)\times(n+2)}$ are matrices defined by

$$A_{n,1} = \frac{1}{2}\begin{pmatrix} 1 & & \bigcirc & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ \bigcirc & & 1 & 0 & 0 \\ 0 & \ldots & 0 & \sqrt{2} & 0 \end{pmatrix}, \quad A_{n,2} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 & \ldots & 0 \\ 0 & 0 & 1 & & \bigcirc \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \bigcirc & & 1 \end{pmatrix}.$$

The fundamental interpolation polynomials are given in terms of $\mathbf{K}_n(\mathbf{x}, \mathbf{y})$. To this end, we will try to express $\mathbf{K}_n(\mathbf{x}, \mathbf{y})$ in terms of the polynomials in the ideal I of Theorem 2.3.

Recall that the Padua points are the common zeros of polynomials Q_k^{n+1} defined in (2.4)and (2.5). We also denote

$$\mathbb{Q}_{n+1} := \left[\sqrt{2}Q_0^{n+1}, 2Q_1^{n+1}, \ldots, 2Q_n^{n+1}, \sqrt{2}Q_{n+1}^{n+1}\right]. \tag{3.3}$$

The definition of Q_k^{n+1} shows that we have the relation

$$\mathbb{Q}_{n+1} = \mathbb{P}_{n+1} + \Gamma_1 \mathbb{P}_n + \Gamma_2 \mathbb{P}_{n-1}, \tag{3.4}$$

where $\Gamma_1 \in \mathbb{R}^{(n+2)\times(n+1)}$ and $\Gamma_2 \in \mathbb{R}^{(n+2)\times n}$ are matrices defined by

$$\Gamma_1 = \begin{pmatrix} 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \sqrt{2} \\ \bigcirc & & 1 & 0 \\ & \ddots & & \vdots \\ 1 & & \bigcirc & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_2 = \begin{pmatrix} -1 & \bigcirc \\ \bigcirc & \bigcirc \end{pmatrix}.$$

By using the representation (3.4) we can rewrite (3.2) as

$$\begin{aligned} (x_i - y_i)\mathbf{K}_n(\mathbf{x}, \mathbf{y}) &= \left[\mathbb{Q}_{n+1}(\mathbf{x}) - \Gamma_1 \mathbb{P}_n(\mathbf{x}) - \Gamma_2 \mathbb{P}_{n-1}(\mathbf{x})\right]^t A_{n,i}^t \mathbb{P}_n(\mathbf{y}) \\ &\quad - \mathbb{P}_n^t(\mathbf{x}) A_{n,i} \left[\mathbb{Q}_{n+1}(\mathbf{y}) - \Gamma_1 \mathbb{P}_n(\mathbf{y}) - \Gamma_2 \mathbb{P}_{n-1}(\mathbf{y})\right] \\ &= \mathbf{S}_{1,i}(\mathbf{x}, \mathbf{y}) + \mathbf{S}_{2,i}(\mathbf{x}, \mathbf{y}) + \mathbf{S}_{3,i}(\mathbf{x}, \mathbf{y}), \quad i = 1, 2, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \mathbf{S}_{1,i}(\mathbf{x}, \mathbf{y}) &= \mathbb{Q}_{n+1}^t(\mathbf{x}) A_{n,i}^t \mathbb{P}_n(\mathbf{y}) - \mathbb{P}_n^t(\mathbf{x}) A_{n,i} \mathbb{Q}_{n+1}(\mathbf{y}), \\ \mathbf{S}_{2,i}(\mathbf{x}, \mathbf{y}) &= \mathbb{P}_n^t(\mathbf{x}) \left(A_{n,i}\Gamma_1 - \Gamma_1^t A_{n,i}^t\right) \mathbb{P}_n(\mathbf{y}), \\ \mathbf{S}_{3,i}(\mathbf{x}, \mathbf{y}) &= \mathbb{P}_n^t(\mathbf{x}) A_{n,i} \Gamma_2 \mathbb{P}_{n-1}(\mathbf{y}) - \mathbb{P}_{n-1}^t(\mathbf{x}) \Gamma_2^t A_{n,i}^t \mathbb{P}_n(\mathbf{y}) . \end{aligned}$$

If both $\mathbf{x}$ and $\mathbf{y}$ are in Pad_n, then $\mathbf{S}_{1,i}$ will be zero. We now work out the other two terms. First, observe that $A_{n,1}\Gamma_1$ is a symmetrix matrix,

$$A_{n,1}\Gamma_1 = \begin{pmatrix} 0 & \ldots & & 0 \\ \bigcirc & & & \sqrt{2} \\ & & 1 & \\ & \ddots & & \\ & 1 & & \\ \sqrt{2} & & & \bigcirc \\ 0 & \ldots & & 0 \end{pmatrix},$$

and

$$A_{n,1}\Gamma_2 = \frac{1}{2}\begin{pmatrix} -1 & \bigcirc \\ \bigcirc & \bigcirc \end{pmatrix}.$$

It follows that $\mathbf{S}_{2,1}(\mathbf{x},\mathbf{y}) \equiv 0$ and

$$\begin{aligned}\mathbf{S}_{3,1}(\mathbf{x},\mathbf{y}) &= -\frac{1}{2}\,P_0^n(\mathbf{x})P_0^{n-1}(\mathbf{y}) + \frac{1}{2}\,P_0^n(\mathbf{y})P_0^{n-1}(\mathbf{x}) \\ &= -T_n(x_1)T_{n-1}(y_1) + T_n(y_1)T_{n-1}(x_1). \end{aligned} \tag{3.6}$$

We also have $A_{n,2}\Gamma_2 = 0$, which entails that $\mathbf{S}_{3,2}(\mathbf{x},\mathbf{y}) \equiv 0$. Finally, we have

$$A_{n,2}\Gamma_1 - \Gamma_1^t A_{n,2}^t = \frac{1}{2}\begin{pmatrix} \bigcirc & & 1 \\ & \bigcirc & \\ -1 & & \bigcirc \end{pmatrix},$$

from which we get

$$\mathbf{S}_{2,2}(\mathbf{x},\mathbf{y}) = T_n(x_1)T_n(y_2) - T_n(x_2)T_n(y_1). \tag{3.7}$$

The terms $\mathbf{S}_{3,1}$ and $\mathbf{S}_{2,2}$ do not vanish on the Padua points. We try to make them part of the left hand side of (3.5) by seeking a polynomial $h_n(\mathbf{x},\mathbf{y})$ such that

$$\begin{aligned} &\mathbf{S}_{3,1}(\mathbf{x},\mathbf{y}) - (x_1 - y_1)h_n(\mathbf{x},\mathbf{y}) = 0, \qquad \mathbf{x},\mathbf{y} \in \mathrm{Pad}_n\,, \\ &\mathbf{S}_{2,2}(\mathbf{x},\mathbf{y}) - (x_2 - y_2)h_n(\mathbf{x},\mathbf{y}) = 0, \qquad \mathbf{x},\mathbf{y} \in \mathrm{Pad}_n\,. \end{aligned} \tag{3.8}$$

It is easy to check that

$$h_n(\mathbf{x},\mathbf{y}) := T_n(x_1)T_n(y_1)$$

satisfies (3.8). This can be verified directly using the three term relation $2tT_n(t) = T_{n+1}(t) + T_{n-1}(t)$ as follows. By (3.6),

$$\begin{aligned} &\mathbf{S}_{3,1}(\mathbf{x},\mathbf{y}) - (x_1 - y_1)h_n(\mathbf{x},\mathbf{y}) \\ &\quad = \frac{1}{2}T_n(x_1)\left(T_{n+1}(y_1) - T_{n-1}(y_1)\right) - \frac{1}{2}T_n(y_1)\left(T_{n+1}(x_1) - T_{n-1}(x_1)\right) \\ &\quad = \frac{1}{2}T_n(x_1)Q_0^{n+1}(\mathbf{y}) - \frac{1}{2}T_n(y_1)Q_0^{n+1}(\mathbf{x})\,, \end{aligned}$$

using the definition of Q_0^{n+1} at (2.4). Similarly, by (3.7),

$$\begin{aligned}&\mathbf{S}_{2,2}(\mathbf{x},\mathbf{y})-(x_2-y_2)h_n(\mathbf{x},\mathbf{y})\\&\quad=-T_n(y_1)(x_2T_n(x_1)+T_n(x_2))+T_n(x_1)(y_2T_n(y_1)+T_n(y_2))\\&\quad=-T_n(y_1)Q_1^{n+1}(\mathbf{x})+T_n(x_1)Q_1^{n+1}(\mathbf{y}),\end{aligned}$$

using the definition of Q_1^{n+1} at (2.5). Since $Q_k^{n+1}(\mathbf{x})$ vanishes on the Padua points, both equations in (3.8) are satisfied. Consequently, we have proved the following proposition:

Proposition 3.1 *For* $\mathbf{x}=(x_1,x_2)$ *and* $\mathbf{y}=(y_1,y_2)$, *define*

$$\mathbf{K}_n^*(\mathbf{x},\mathbf{y}):=\mathbf{K}_n(\mathbf{x},\mathbf{y})-T_n(x_1)T_n(y_1). \tag{3.9}$$

Then

$$\begin{aligned}(x_1-y_1)\mathbf{K}_n^*(\mathbf{x},\mathbf{y})&=\mathbb{Q}_{n+1}^t(\mathbf{x})A_{n,1}^t\mathbb{P}_n(\mathbf{y})-\mathbb{P}_n^t(\mathbf{x})A_{n,1}\mathbb{Q}_{n+1}(\mathbf{y})\\&\quad+\frac{1}{2}T_n(x_1)Q_0^{n+1}(\mathbf{y})-\tfrac{1}{2}T_n(y_1)Q_0^{n+1}(\mathbf{x}),\\(x_2-y_2)\mathbf{K}_n^*(\mathbf{x},\mathbf{y})&=\mathbb{Q}_{n+1}^t(\mathbf{x})A_{n,2}^t\mathbb{P}_n(\mathbf{y})-\mathbb{P}_n^t(\mathbf{x})A_{n,2}\mathbb{Q}_{n+1}(\mathbf{y})\\&\quad-T_n(y_1)Q_1^{n+1}(\mathbf{x})+T_n(x_1)Q_1^{n+1}(\mathbf{y}).\end{aligned}$$

In particular, $\mathbf{K}^*(\mathbf{x},\mathbf{y})=0$ *if* $\mathbf{x},\mathbf{y}\in\mathrm{Pad}_n$ *and* $\mathbf{x}\neq\mathbf{y}$.

Since $P_n^n(x_1,x_2)=T_n(x_1)$, we evidently have $K_n^*(\mathbf{x},\mathbf{x})\geq 1>0$. Consequently, a compact formula for the fundamental interpolation polynomials follows immediately from the above proposition.

Theorem 3.2 *The fundamental interpolation polynomials* $\ell_{k,j}$ *in* (2.6) *associated with the Padua points* $\mathbf{x}_{k,j}=(\xi_k,\eta_j)$ *are given by*

$$\ell_{k,j}(\mathbf{x})=\frac{\mathbf{K}_n^*(\mathbf{x},\mathbf{x}_{k,j})}{\mathbf{K}_n^*(\mathbf{x}_{k,j},\mathbf{x}_{k,j})}=\frac{\mathbf{K}_n(\mathbf{x},\mathbf{x}_{k,j})-T_n(x_1)T_n(\xi_k)}{\mathbf{K}_n(\mathbf{x}_{k,j},\mathbf{x}_{k,j})-[T_n(\xi_k)]^2}. \tag{3.10}$$

In fact, using the representation of $\mathbf{K}_n^*$ in Proposition 3.1, the conditions (2.7) can be verified readily, which proves the theorem.

We can say more about the formula (3.10). In fact, a compact formula for the reproducing kernel $\mathbf{K}_n$ appeared in [9], which states that

$$\begin{aligned}\mathbf{K}_n(\mathbf{x},\mathbf{y})&=D_n(\theta_1+\phi_1,\theta_2+\phi_2)+D_n(\theta_1+\phi_1,\theta_2-\phi_2)\\&\quad+D_n(\theta_1-\phi_1,\theta_2+\phi_2)+D_n(\theta_1-\phi_1,\theta_2-\phi_2),\end{aligned} \tag{3.11}$$

where $\mathbf{x}=(\cos\theta_1,\cos\theta_2)$, $\mathbf{y}=(\cos\phi_1,\cos\phi_2)$, and

$$D_n(\alpha,\beta)=\frac{1}{2}\frac{\cos\left(\left(n+\frac{1}{2}\right)\alpha\right)\cos\frac{\alpha}{2}-\cos\left(\left(n+\frac{1}{2}\right)\beta\right)\cos\frac{\beta}{2}}{\cos\alpha-\cos\beta}.$$

Hence, the formula (3.10) provides a compact formula for the fundamental interpolation polynomials $\ell_{k,j}$. Furthermore, the explicit formula (3.11) allows us to derive an explicit formula for the denominator $\mathbf{K}_n^*(\mathbf{x}_{k,j}, \mathbf{x}_{k,j})$. The Padua points (1.1) naturally divide into three groups,

$$\mathrm{Pad}_n = A_v \cup A_b \cup A_{in},$$

where A_v consists of the two vertex points $\left((-1)^k, (-1)^{n-1}\right)$ for $k = 0, n$, A_b consists of the other points on the boundary of $[-1, 1]^2$, and A_{in} consists of interior points inside $(-1, 1)^2$ (see Fig. 1).

Proposition 3.3 *For* $\mathbf{x}_{k,j} \in \mathrm{Pad}_n$,

$$\mathbf{K}_n^*(\mathbf{x}_{k,j}, \mathbf{x}_{k,j}) = n(n+1) \begin{cases} \frac{1}{2}, & \textit{if} \quad \mathbf{x}_{k,j} \in A_v \\ 1, & \textit{if} \quad \mathbf{x}_{k,j} \in A_b \\ 2, & \textit{if} \quad \mathbf{x}_{k,j} \in A_{in}. \end{cases}$$

The proof can be derived from the explicit formula of the Padua points (1.1) and the compact formula (3.11) by a tedious verification. We refer to [3] for a proof using the generating curve.

Since integration of the Lagrange interpolation polynomial gives the quadrature formula, it follows from the above proposition and Corollary 2.4 that:

Proposition 3.4 *A cubature formula of degree* $2n - 1$ *based on the Padua points is given by*

$$\frac{1}{\pi^2} \int_{[-1,1]^2} f(x_1, x_2) \frac{dx_1 dx_2}{\sqrt{1 - x_1^2}\sqrt{1 - x_2^2}} \approx \sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} f(\mathbf{x}_{k,j})$$

where $w_{k,j} = 1/\mathbf{K}_n^*(\mathbf{x}_{k,j}, \mathbf{x}_{k,j})$.

We note that this cubature formula is not a minimal cubature formula, since there are cubature formulas of the same degree with a fewer number of nodes.

4 Convergence of the Lagrange interpolation polynomials

In [3] it is shown that the Lagrange interpolation projection $\mathcal{L}_n f$ in (2.6) has Lebesgue constant $\mathcal{O}((\log n)^2)$; that is, as an operator from $C([-1, 1]^2)$ to itself, its operator norm in the uniform norm is $\mathcal{O}((\log n)^2)$. This settles the problem of uniform convergence of $\mathcal{L}_n f$. In this section we prove that $\mathcal{L}_n f$ converges in L^p norm. For the Chebyshev weight function W defined in (2.1), we define $L^p(W)$ as the space of Lebesgue measurable functions for which the norm

$$\|f\|_{W,p} := \left(\int_{[-1,1]^2} |f(x_1, x_2)|^p W(x_1, x_2) dx_1 dx_2 \right)^{1/p}$$

is finite. We keep this notation also for $0 < p < 1$, even though it is no longer a norm for p in that range. The main result in this section is

Theorem 4.1 *Let $\mathcal{L}_n f$ be the Lagrange interpolation polynomial (2.6) based on the Padua points. Let $0 < p < \infty$. Then*

$$\lim_{n\to\infty} \|\mathcal{L}_n f - f\|_{W,p} = 0, \qquad \forall f \in C([-1,1]^2).$$

In fact, let $E_n(f)_\infty$ be the error of the best approximation of f by polynomials from Π_n^2 in the uniform norm; then

$$\|\mathcal{L}_n f - f\|_{W,p} \le c_p E_n(f)_\infty \qquad \forall f \in C([-1,1]^2).$$

The proof follows the approach in [10], where the mean convergence of another family of interpolation polynomials is proved. We shall be brief whenever the same proof carries over. First we need a lemma on the Fourier partial sum, $S_n f$, defined by

$$S_n f(\mathbf{x}) = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j^k(f) P_j^k(\mathbf{x}) = \frac{1}{\pi^2} \int_{[-1,1]^2} \mathbf{K}_n f(\mathbf{x},\mathbf{y}) W(\mathbf{y}) d\mathbf{y}$$

where $a_j^k(f)$ is the Fourier coefficient of f with respect to the orthonormal basis $\{P_j^k\}$ in $L^2(W)$. In the following we let c_p denote a generic constant that depends on p only, its value may be different from line to line.

Lemma 4.2 *Let $1 < p < \infty$. Then*

$$\|S_n f\|_{W,p} \le c_p \|f\|_{W,p}, \qquad \forall f \in L^p(W). \tag{4.1}$$

This is [10, Lemma 3.4]. We will need another lemma whose proof follows almost verbatim from that of [10, Lemma 3.5]. In the following we write

$$N = |\,\mathrm{Pad}_n\,| = (n+1)(n+2)/2.$$

Lemma 4.3 *Let $1 \le p < \infty$. Let $\mathbf{x}_{k,j}$ be the Padua points. Then*

$$\frac{1}{N} \sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} |P(\mathbf{x}_{k,j})|^p \le c_p \int_{[-1,1]^p} |P(\mathbf{x})|^p W(\mathbf{x}) d\mathbf{x}, \qquad \forall P \in \Pi_n^2. \tag{4.2}$$

The main tool in the proof of Theorem 4.1 is a converse inequality of (4.2), which we state below and give a complete proof, even though the proof is similar to that of [10, Theorem 3.3].

Proposition 4.4 *Let $1 < p < \infty$. Let $\mathbf{x}_{k,j}$ be the Padua points. Then*

$$\int_{[-1,1]^2} |P(\mathbf{x})|^p W(\mathbf{x}) d\mathbf{x} \le c_p \frac{1}{N} \sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} |P(\mathbf{x}_{k,j}|^p, \qquad \forall P \in \Pi_n^d.$$

Proof Let $P \in \Pi_n^d$. For $p > 1$, we have

$$\begin{aligned} \|P\|_{W,p} &= \sup_{\|g\|_{W,q}=1} \int_{[-1,1]^2} P(\mathbf{x}) g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} \\ &= \sup_{\|g\|_{W,q}=1} \int_{[-1,1]^2} P(\mathbf{x}) S_n g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \end{aligned} \tag{4.3}$$

where the second equality follows from the orthogonality. We write

$$S_n g = S_{n-1} g + \mathbf{a}_n^t(g) \mathbb{P}_n, \quad \text{where} \quad \mathbf{a}_n = \int_{[-1,1]^2} g(\mathbf{x}) \mathbb{P}_n(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}.$$

Since the cubature formula is of degree $2n-1$ and $PS_{n-1}g$ is of degree $2n-1$, we have

$$\begin{aligned} \left| \int_{[-1,1]^2} P(\mathbf{x}) S_{n-1}(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} \right| &= \left| \sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} P(\mathbf{x}_{k,j}) S_{n-1}(\mathbf{x}_{k,j}) \right| \\ &\le \left(\sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} |P(\mathbf{x}_{k,j})|^p \right)^{1/p} \left(\sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} |S_{n-1}(\mathbf{x}_{k,j})|^q \right)^{1/q}. \end{aligned} \tag{4.4}$$

By Lemma 4.2 and Lemma 4.3,

$$\sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} |S_{n-1}(\mathbf{x}_{k,j})|^q \le c \|S_{n-1} g\|_{W,q} \le c \|g\|_{W,q} = c.$$

Hence, using the fact that $w_{k,j} \sim N^{-1}$, it follows readily that

$$\left| \int_{[-1,1]^2} P(\mathbf{x}) S_{n-1}(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} \right| \le c \left(\sum_{\mathbf{x}_{k,j} \in \mathrm{Pad}_n} w_{k,j} |P(\mathbf{x}_{k,j})|^p \right)^{1/p}. \tag{4.5}$$

We need to establish the similar inequality for the $\mathbf{a}_n^t \mathbb{P}_n$ term. Since $\mathcal{L}_n P = P$ as P is of degree n, it follows from (3.10) and the orthogonality that

$$\int_{[-1,1]^2} P(\mathbf{x})\mathbf{a}_n^t(g)\mathbb{P}_n(\mathbf{x})W(\mathbf{x})d\mathbf{x} = \int_{[-1,1]^2} \mathcal{L}_n P(\mathbf{x})\mathbf{a}_n^t(g)\mathbb{P}_n(\mathbf{x})W(\mathbf{x})d\mathbf{x}$$

$$= \sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} P(\mathbf{x}_{k,j}) \int_{[-1,1]^2} \mathbf{K}_n^*(\mathbf{x}_{k,j},\mathbf{x})\mathbf{a}_n^t(g)\mathbb{P}_n(\mathbf{x})W(\mathbf{x})d\mathbf{x}$$

$$= \sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} P(\mathbf{x}_{k,j})\mathbf{a}_n^t(g)\mathbb{P}_n(\mathbf{x}_{k,j}) - \frac{1}{2}a_n^n(g) \sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} P(\mathbf{x}_{k,j})\widetilde{T}_n(\xi_{k,j}).$$

For the first term we can write $\mathbf{a}^t(g)\mathbb{P}_n = S_n g - S_{n-1} g$ and apply Hölder's inequality as in (4.4), so that the proof of (4.5) can be carried out again. For the second term we use the fact that $|\widetilde{T}_n(x)| \le \sqrt{2}$ to conclude that

$$|a_n^n(g)| \le \int_{[-1,1]^2} |\widetilde{T}_n(\mathbf{x})g(\mathbf{x})|W(\mathbf{x})d\mathbf{x} \le \sqrt{2}\|g\|_{W,q} \le \sqrt{2}.$$

so that the sum is bounded by

$$\left| a_n^n(g) \sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} P(\mathbf{x}_{k,j})\widetilde{T}_n(\xi_{k,j}) \right| \le \sqrt{2} \left(\sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} |P(\mathbf{x}_{k,j})|^p \right)^{1/p}.$$

In this way we have established the inequality

$$\left| \int_{[-1,1]^2} P(\mathbf{x})\mathbf{a}_n^t(g)\mathbb{P}_n(\mathbf{x})W(\mathbf{x})d\mathbf{x} \right| \le c \left(\sum_{\mathbf{x}_{k,j}\in \mathrm{Pad}_n} w_{k,j} |P(\mathbf{x}_{k,j})|^p \right)^{1/p}.$$

Together with (4.5), this completes the proof of the proposition. □

As shown in the proof of Theorem 3.1 of [10], the proof of Theorem 4.1, including the cases $0 < p < 1$, follows as an easy consequence of Proposition 4.4.

References

1. Bos, L., Caliari, M., De Marchi, S., Vianello, M.: A numerical study of the Xu polynomial interpolation formula in two variables. Computing **76**, 311–324 (2005)
2. Bos, L., De Marchi, S., Vianello, M.: On the Lebesgue constant for the Xu interpolation formula. J. Approx. Theory **141**, 134–141 (2006)
3. Bos, L., Caliari, M., De Marchi, S., Vianello, M., Xu, Y.: Bivariate Lagrange interpolation at the Padua points: the generating curve approach. J. Approx. Theory **143**, 15–25 (2006)
4. Caliari, M., De Marchi, S., Vianello, M.: Bivariate polynomial interpolation on the square at new nodal sets. Appl. Math. Comput. **165**, 261–274 (2005)

5. Cox, D., Little, J., O'Shea, D.: Ideals, Varieties, and Algorithms, 2nd edn. Springer, Berlin (1997)
6. Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables. Encyclopedia of Mathematics and its Applications, 81. Cambridge University Press, Cambridge (2001)
7. Morrow, C.R., Patterson, T.N.L.: Construction of algebraic cubature rules using polynomial ideal theory. SIAM J. Numer. Anal. **15**, 953–976 (1978)
8. Xu, Y.: Common zeros of polynomials in several variables and higher dimensional quadrature. Pitman Research Notes in Mathematics Series. Longman, Essex (1994)
9. Xu, Y.: Christoffel functions and Fourier series for multivariate orthogonal polynomials. J. Approx. Theory **82**, 205–239 (1995)
10. Xu, Y.: Lagrange interpolation on Chebyshev points of two variables. J. Approx. Theory **87**, 220–238 (1996)
11. Xu, Y.: Polynomial interpolation in several variables, cubature formulae, and ideals. Adv. Comput. Math. **12**, 363–376 (2000)