Fractal Interpolation Functions for a Class of Finite Elements

S. De Marchi and M. Morandi Cecchi

Abstract. Classical ways to describe *shape functions* for finite element methods make use of interpolating or approximating schemes like for example, Taylor and Lagrange [8,9] or Bézier [5,8]. In this paper we outline the possibility of using iterative schemes first applied in the computation of fractal curves and surfaces. Our attempt will be restricted to shape functions defined over 2-simplices. Because of the fractal nature of the functions, we get only continuous or uniformly continuous functions. We see that they can be found as an *attractor* of a suitable *Iterated Function System (IFS)* [2,7].

§1. Introduction

A finite element $\mathcal{F}_E$ on $\mathbb{R}^n$ is a triplet (E, P, Γ) (see [3]) where E is a closed polyhedron in $\mathbb{R}^n$; $P \subset \mathcal{C}^s(E)$, $s \in \mathbb{N}$, is a finite dimensional space of real value functions (we shall assume that its dimension is M); Γ is a set of linear functions γ_i, $1 \le i \le M$, linearly independent and defined over the set P. The set Γ is P-unisolvent (see [3,8]). In particular, there exist M functions $p_i \in P$, $1 \le i \le M$, such that $\gamma_j(p_i) = \delta_{ij}$, $1 \le i, j \le M$.

In the following, we use the symbol E to identify the finite element $\mathcal{F}_E$ and its defining polyhedron, and E is a n-simplex defined as usual, by

$$E = \{x \in \mathbb{R}^n : \; x_i = \sum_{j=1}^{n+1} s_{ij}\alpha_j; \; 0 \le \alpha_j \le 1, \; 1 \le i \le n, \; 1 \le j \le n+1\}, \quad (1)$$

where $\{\alpha_j\}$ are the barycentric coordinates of the point x. The functions γ_i and p_i are known as *degrees of freedom* and *shape functions* of E, respectively.

As is well-known, to compute the interpolating polynomial on E, we may use the Neville-Aitken's interpolation scheme. This scheme was generalized in [9] to a family $\mathcal{H} = \{h_i\}_{i=1,\ldots,n}$ of continuous functions forming a Chebyshev system over a set of points $\mathcal{S}$, of cardinality n at least.

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Again, the interpolating polynomial based on such a system can be expressed by recurrence formulae. The coefficients $(\eta_l(x))_{l=0,\dots,m}$ in those formulae are a ratio of determinants whose elements $(\beta_{i,l}(x))_{i,l=0,\dots,m}$ are the interpolation *errors* obtained substituting the function which has to be interpolated with one of the functions of the family. Two interesting relations were obtained

$$\begin{cases} \sum_{l=0}^{m} \eta_l(x) = 1 \\ \sum_{l=0}^{m} \beta_{i,l}(x)\eta_l(x) = 0, \quad i = 1, \dots, m. \end{cases} \tag{2}$$

The coefficients $(\beta_{j,i}(x))_{i,j=0,\dots,m}$ are related to the given Chebychev set $\mathcal{H}$ by the following (see [5])

$$\beta_{j,i}(x) = -h_{0,j}^{(i)}(x) = h_j(x) - h_j(x_i)\frac{h_0(x)}{h_0(x_i)}, \qquad i, j = 1, ..., m, \tag{3}$$

assuming $\beta_{0,j}(x) = 1 \;\; j=0,..,m.$

Some properties of this interpolating scheme are proved in [6]. This framework also includes the hierarchical finite elements [3]. The construction of the *attractor* at which the sequence of hierarchical elements converges is a key tool for this class of elements. This paper presents a way of computing the attractor of an *IFS* that interpolates at the element.

Section 2 briefly reviews a way to compute an attractor of an IFS. Section 3 gives an example of this technique for a function defined over a triangle. An algorithm to compute a fractal interpolating surface over a triangle is provided in Section 4.

§2. Attractors and IFS

Let $X = (X, d_X)$ be a compact metric space or a closed subset of $\mathbb{R}^s$, $s > 0$ with metric d_X and let $\mathcal{H}(X)$ be the set of all nonempty closed subsets of X with Hausdorff metric function h.

On $\mathcal{H}(X)$ we consider the finite dimensional set of continuous functions

$$G = \{g_i : X \longrightarrow X, \;\; i = 1, ..., n\}. \tag{4}$$

The pair $G_X = (G, X)$ is known as an *Iterated Function System* (briefly IFS) on X (see [1,2]). If the functions g_i are contractive, then G_X is said to be a *hyperbolic* IFS.

Definition 1. *We say the compact set $K \subset \mathbb{R}^s$ is invariant if there exists a hyperbolic IFS, G_X, such that $K = \cup_{i=1}^{n} g_i(K)$.*

An invariant set is mainly determined by an iterative converging sequence of sets which can be constructed from G_X starting from a polygonal region in $\mathbb{R}^s$. The maps g_i have to satisfy some properties that guarantee the convergence of the approximating sequence of sets.

In the case of a hyperbolic IFS, the invariant set is known as the *attractor* of an IFS. If the functions g_i are contraction maps, we can prove the existence and uniqueness of a compact set K invariant with respect to G (see [4]).

Let $\mathcal{S} = \{(x_i, y_i),\ i = 0, .., n;\ n \geq 1\} \subset \mathbb{R}^2$ be a set of points and let $f : \mathcal{D} = [x_0, x_n] \longrightarrow \mathbb{R}$ be the function interpolating on $\mathcal{S}$. Let $\mathcal{G}_f$ be its graph. The graph $\mathcal{G}_f$ can be seen as the *attractor* of an IFS, $W_X = \{w_1, .., w_n\}$, defined on the set $X \subset \mathbb{R}^2$ [1]. The functions w_i are taken to be *contractive affine maps*, with contractivity factor $0 \leq \lambda_i < 1$ of the form

$$w_i(x, y) = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ g_i \end{pmatrix} \tag{5.1}$$

and

$$w_i(x_0, y_0) = (x_{i-1}, y_{i-1}) \tag{5.2}$$

$$w_i(x_n, y_n) = (x_i, y_i). \tag{5.3}$$

These conditions allow us to determine the parameters a_i, c_i, e_i and g_i whereas the d_i remain *free*. If we choose $d_i = 0$, we recover *piecewise linear interpolation*.

Proposition 1. *$\mathcal{G}_f$ is an invariant set in $\mathbb{R}^2$, i.e. $\mathcal{G}_f = \bigcup_{i=1}^{n} w_i(\mathcal{G}_f)$.*

Proof: For each i, W_X has an attractor which is the (fractal) curve between (x_{i-1}, y_{i-1}) and (x_i, y_i). By union of these subsets we get the conclusion. ■

A function $f(x)$ whose graph $\mathcal{G}_f$ is the attractor of an IFS is said to be a *fractal interpolation function* corresponding to the data set $\mathcal{S}$. Given the IFS W_X on $\mathcal{H}(X)$, we define the function $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by $W(S) = \bigcup_i w_i(S)$ for any $S \in \mathcal{H}(X)$. Thus, the attractor of W_X is any set $A \in \mathcal{H}(X)$ that is a fixed point of W, i.e. $W(A) = A$. We have the following question:

"How can we find an IFS in $\mathbb{R}^2$ such that its attractor is a fractal interpolation function $f : [x_0, x_n] \to \mathbb{R}$, and $f(x_i) = y_i$?"

A well-known technique to compute the attractor of a given IFS in $\mathbb{R}^2$ is the *Random Iteration Algorithm* (RIA) [2]. The RIA allows us to compute the coordinates of the points on the curve by choosing at random one of the n maps w_i. The kernel of the algorithm is to find out the attractor, making use of the RIA. It can be summarized as follows:

Algorithm 1

1) Define the affine transformations. That is, we have to determine the numbers a_i, c_i, e_i and g_i corresponding to the i-th tranformation w_i to be a contraction on the interval of definition, i.e. $Range(w_i)|_{[x_0, x_n]} = [y_{i-1}, y_i]$.

2) Apply the RIA to the transformations $w_i(x, y)$, MAX_ITER times.

§3. Fractal Interpolation Functions on General Sets

To extend the theory to $\mathbb{R}^s$, $s > 2$, we have to consider *general fractal interpolation functions*. To this end, let $(\boldsymbol{Y}, d_{\boldsymbol{Y}})$ denote a complete metric space and let $f : K \subset \mathbb{R} \to \boldsymbol{Y}$ be a function.

Definition 2. *A set of generalized data is the set*

$$GD = \{(x_i, Y_i) \in \boldsymbol{X} = \mathbb{R} \times \boldsymbol{Y}, \quad x_i \in K \text{ in ascending order for } i \in \mathcal{I}\}.$$

where $\mathcal{I} = \{0, 1, ..., n\}$, $n \geq 2$. The continuous function $f : [x_0, x_n] \longrightarrow \boldsymbol{Y}$ such that $f(x_i) = Y_i$, $\forall\, i$ is an *interpolation function* corresponding to this set.

On the set GD, we define n linear functions $L_i : \mathbb{R} \to \mathbb{R}$ where $L_i(x) = a_i x + e_i$ for $i \in \mathcal{I}_0 = \mathcal{I} \backslash \{0\}$. Let c and σ be real numbers such that $c > 0$ and $0 \leq \sigma < 1$. For each $i \in \mathcal{I}_0$ let $M_i : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be a function which is Lipschitz in the first variable

$$d_{\boldsymbol{Y}}(M_i(a, y), M_i(b, y)) \leq c|a - b|, \qquad \forall a, b \in \mathbb{R}, \quad y \in \boldsymbol{Y}, \tag{6}$$

and contractive in the second one

$$d_{\boldsymbol{Y}}(M_i(x, a), M_i(x, b)) \leq \sigma d_{\boldsymbol{Y}}(a, b) \quad \forall x \in \mathbb{R},\ a, b \in \boldsymbol{Y}. \tag{7}$$

From L_i and M_i we define the transformation

$$\begin{aligned} &w_i : \boldsymbol{X} \longrightarrow \boldsymbol{X} \\ &(x, Y) \mapsto w_i(x, Y) = (L_i(x), M_i(x, Y)) \end{aligned} \tag{8}$$

One can prove that the IFS, $W_{\boldsymbol{X}} = \{\boldsymbol{X}, w_i,\ i \in \mathcal{I}_0\}$ is hyperbolic with respect to the metric $d_{\boldsymbol{X}}(X_1, X_2) = |x_1 - x_2| + \lambda|Y_1 - Y_2|$, where $X_1 = (x_1, Y_1)$, $X_2 = (x_1, Y_1)$ and $\lambda > 0$ (see [4]).

3.1. Fractal interpolation surfaces

Here we wish to present a technique to construct a fractal interpolation surface over a 2-simplex. We start from the *canonical* 2-simplex Σ that has vertices $\sigma_0 = (0,0)$, $\sigma_1 = (1,0)$ and $\sigma_2 = (0,1)$. Let $K = [0,1] \times [0,1]$. On the set $A = K \times \mathbb{R}$ we define the functions $w_i : A \longrightarrow A$ as

$$w_i(x, y, z) = (L_i(x, y), M_i(x, y, z)) \quad \text{for } i \in \mathcal{I}_0, \tag{9}$$

where $L_i : K \to K$ and $M_i : A \to \mathbb{R}$. We require that $w_i(\Sigma) = \Sigma_i$, that is they map the simplex onto its $i - th$ subsimplex Σ_i, $L_i(\Sigma) = \Sigma_i$ is a homeomorphism and M_i is Lipschitz in x, y and contractive in z. We also assume $L_1(\sigma_0) = \sigma_0$, $M_1(0,0,0) = 0$ and there exist indices $i_0, i_1 \in \mathcal{I}_0$ such that $L_{i_0}(\sigma_1) = \sigma_1$, $L_{i_1}(\sigma_2) = \sigma_2$ and $M_{i_0}(1,0,0) = M_{i_1}(0,1,0) = 0$. These conditions are analogous to (5.2) and (5.3).

Let consider the IFS $W_A = \{A, w_i, i \in \mathcal{I}_0\}$. It is straightforward to prove that W_A is hyperbolic on the space A with distance function $d_{A,\alpha}$ given by

$$d_{\Sigma,\alpha}((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_1 - x_2| + |y_1 - y_2| + \frac{1-\tilde{c}}{\alpha L}|z_1 - z_2|.$$

Here $\tilde{c} = \max\{c_1, c_2\}$ with c_1 is the contraction constant of $L_i(x, \cdot)$ and c_2 the corresponding for $L_i(\cdot, y)$; while $L = \max\{L_1, L_2\}$, where L_1 is the Lipschitz constant of $M_i(x, \cdot, \cdot)$, and L_2 the Lipschitz constant for $M_i(\cdot, y, \cdot)$. $\alpha > 1$ is a fixed real number. Since W_A is hyperbolic, we are interested in its attractor. The following proposition characterizes this set.

Proposition 2. *The attractor of W_A is the graph of a continuous function $f : \Sigma \longrightarrow \mathbb{R}$ that passes through the vertices of the subsimplices Σ_i, $i \in \mathcal{I}_0$.*

Definition 3. *We call the graph of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ that is a fractal a fractal surface.*

As an example, in generalizing (5.1) to 3D, we consider affine maps of the form

$$w_i(x, y, z) = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ g_i & h_i & l_i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \\ m_i \end{pmatrix}.$$

The functions L_i and M_i required in (9) are

$$L_i(x, y) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}. \qquad (10)$$

$$M_i(x, y, z) = g_i x + h_i y + l_i z + m_i$$

The constants $a_i, b_i, c_i, d_i, g_i, h_i, e, f_i$ and g_i are found by the conditions on the functions w_i, that is $w_i(\Sigma) = \Sigma_i$, while the parameters l_i, $|l_i| < 1$ are arbitrarily chosen (see previous the section).

Example. Let $\mathcal{B}$ be the triangle depicted in Fig. 1. Assume that on the nodes, the values of the functions are $0, 0, \frac{1}{2}, 1$, respectively. Let $\{\mathcal{B}; w_1, w_2\}$ be the IFS defined on $\mathcal{B}$, with code given in the next table, assuming that the related probabilities are the same, i.e. $p_1 = p_2 = 0.5$.

w	a	b	c	d	g	h	e	f	m
1	1.0	0.5	1.0	1.0	0.5	0.0	0.0	0.0	0.0
2	0.5	1.0	0.5	1.0	1.0	0.0	0.0	0.0	0.0

Table 1. IFS code for the triangle of Fig. 1.

It is easy to see that its attractor is the set $\mathcal{G}$ that is the fractal surface interpolating at $\mathcal{B}$.

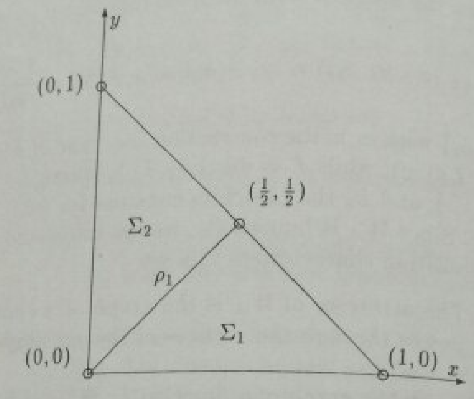

Fig. 1. The *canonical* simplex subdivided in two subtriangles.

§4. An Algorithm for Constructing a Fractal Surface

Let Ω be a closed nondegenerate polygonal region in $\mathbb{R}^2$ containing $n+1$ distinct points $\{(x_i, y_i)\}_{i=0}^n$. We want to find a self-affine fractal interpolating surface of the form $z = \gamma(x, y)$ which interpolates the data $\{(x_i, y_i, z_i)\}_{i=0}^n$. We assume that Ω is triangulable and $\Sigma_\Omega = \{\Sigma_i\}_{i=1}^p$ is such a triangulation with $\mathcal{S} = \{\sigma_1, ..., \sigma_m\}$ as the set of the vertices of Σ_Ω.

Now, let γ be the fractal surface representing the attractor of a given IFS, and let L_i, M_i be the functions defined in the previous section, such that $L_i : \Omega \to \Sigma_i$ and $M_i : \Omega \times \mathbb{R} \to \mathbb{R}$ for $i = 1, ..., p$. By using functions L_i, M_i, we define the mapping $\Psi : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ by

$$(\Psi\gamma)(x) = M_i(L_i^{-1}(x), \gamma(L_i^{-1}(x))) \quad x \in \Sigma_i \quad i = 1, ..., p. \tag{11}$$

Ψ is well-defined and contractive in the sup-norm $\|\cdot\|_\infty$ on a suitable subset of $\mathcal{C}(\Omega)$ [7]. Thus, it has a fixed point $\tilde{\gamma}$ that defines the fractal interpolation surface. We show that the $graph(\tilde{\gamma})$ is self-affine and there exists a *deterministic* algorithm to construct it.

Making use of the *recurrent IFS formalism* [2], consider the set $\mathcal{H}$ of all nonempty compact subsets of $\mathbb{R}^3$. We construct its p-cartesian product $\mathcal{H}^p = \underbrace{\mathcal{H} \times \mathcal{H} \times \ldots \times \mathcal{H}}_{p}$. Set $\chi(i) = \{j : \Sigma_j \subset \rho_{\kappa(i)}\}$ where κ is a mapping from $\{1, ..., m\}$ into $\{1, ..., r\}$ such that $L_i(\rho_{\kappa(i)}) = \Sigma_i$. This means that given the triangulation Σ_Ω, we construct the triangulation $R_\Omega = \{\rho_k\}_{k=1}^r$ made by r triangular subregions of Ω obtained as the union of some triangles of Σ_Ω. The vertices of the triangulation Σ_Ω are ordered in such a way that $\{\sigma_1, ..., \sigma_l\} \subset \mathcal{S}$ is the set of the vertices of R_Ω.

On $\mathcal{H}^p$ we define the function $F : \mathcal{H}^p \to \mathcal{H}^p$ whose i-th component is given by

$$F_i(A_1, ..., A_p) = w_i \left(\bigcup_{j \in \chi(i)} A_j \right), \qquad i = 1, ..., p, \tag{12}$$

where the functions $\{w_i\}_{i=1}^{p}$ are given as in (9). We need the following definitions.

Definition 4. *Let ϑ be a nonvertical plane in $\mathbb{R}^3$. We denote by $\mathcal{C}_\vartheta(\Omega)$ the set*

$$\mathcal{C}_\vartheta(\Omega) = \{f \mid f : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \text{ with } (x, f(x)) \in \vartheta, \ \forall x \in \partial\Omega\}$$

Definition 5. *On Ω we consider the set of points $\sigma_i = (x_i, y_i)$, and let z_i be the values of a function which interpolates the set $\{(\sigma_i, z_i)\}_{i=1}^{n}$. We denote by $\mathcal{C}_B(\Omega)$ the set*

$$\mathcal{C}_B(\Omega) = \{f \mid f : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \text{ with } f(\sigma_j) = z_j, \ \forall \sigma_j \in \partial\Omega\}.$$

We note that Ψ is contractive on both $\mathcal{C}_\vartheta$ and $\mathcal{C}_B$. Finally, we state the main result (proved in [7]) which suggested the algorithm to us:

Proposition 3. *Let $\gamma : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be given, and suppose the following properties hold:*

(a) *$\gamma \in \mathcal{C}_\vartheta(\Omega)$ or $\gamma \in \mathcal{C}_B(\Omega)$;*

(b) *given the triangulation Σ_Ω we assume that the graph associated with Σ_Ω has chromatic number equal to 3. Then*

$$graph(\Psi\gamma_{|\Sigma_i}) = F_i(graph(\gamma)_{\Sigma_1}, graph(\gamma)_{\Sigma_2}, ..., graph(\gamma)_{\Sigma_p}) \quad i = 1, ..., p \tag{13}$$

Since $F_i\left(graph(\gamma)_{\Sigma_1}, graph(\gamma)_{\Sigma_2}, ..., graph(\gamma)_{\Sigma_p}\right) = w_i \left(\bigcup_{j \in \chi(i)} A_j \right)$, the previous result leads to the following algorithm:

Algorithm 2

1) Set $A_0 = (graph(\gamma)_{\Sigma_1}, graph(\gamma)_{\Sigma_2}, \ldots, graph(\gamma)_{\Sigma_p})$;
2) For $n = 1, 2, \ldots$, set $A_n = F(A_{n-1})$
3) Then, $A_n \to \tilde{\gamma}$, where $\tilde{\gamma}$ is a fractal surface.

To show the convergence in step 3), we have to check whether F is contractive in $\mathcal{H}^p$, or in view of (13), whether $\|\Psi^{\circ k}\gamma - \tilde{\gamma}\|_H < \epsilon$ for each $\epsilon > 0$. But Ψ is contractive on $\mathcal{C}_\vartheta$ and $\mathcal{C}_B$, so we can conclude that

$$A_n \to (graph(\tilde{\gamma})_{\Sigma_1}, graph(\tilde{\gamma})_{\Sigma_2}, ..., graph(\tilde{\gamma})_{\Sigma_p})$$

and $\bigcup_{k=1}^{n} A_k^{(i)} \to graph(\tilde{\gamma})$.

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