PEANO'S KERNEL THEOREM FOR VECTOR-VALUED FUNCTIONS AND SOME APPLICATIONS

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Abstract

We extend the well-known Peano Kernel Theorem to a class of linear operators $L : \mathcal{C}^{n+1}([a,b];X) \to X$, X being a Banach space, which vanish on abstract polynomials of degree $\leq n$. We then recover, in the abstract setting, classical estimates of remainders in polynomial interpolation and quadrature formulas. Finally, we present an application to the error analysis of the trapezoidal time discretization scheme for parabolic evolution equations.

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1 Introduction

Peano's Kernel Theorem is one of the most important sources for error estimates in approximation theory and numerical analysis. Originally introduced by G. Peano [11, 12] to estimate remainders of quadrature formulas, this theorem has then been extended and applied to a variety of approximation procedures (cf., e.g. [2, 13, 17, 19]). Nevertheless, all applications seem to be restricted to spaces of functions with finite-dimensional domain and range.

In the present paper, we generalize Peano's Kernel Theorem to get an integral representation of certain linear operators $L : \mathcal{C}^{n+1}([a,b];X) \to X$, X being a Banach space, which vanish on all *abstract polynomials* of degree $\leq n$. Our main motivation for such extension came from the need of estimating local truncation errors, as well as the piecewise linear interpolation error, in the trapezoidal time discretization of abstract parabolic evolution equations (cf. [3]).

It is worth recalling that A. Sard in 1948 [16, Th.1] proved a quite general result on the representation of a linear continuous operator between Banach spaces, which vanishes on the kernel of another operator. This theorem is also known as the "Quotient Theorem" (cf. [17, Th.20, p.311]). In the scalar case, such a result allows us to recover Peano's Kernel Theorem, essentially via the well-known *Riesz representation theorem* for functionals on $\mathcal{C}([a,b];\mathbb{R})$, cf. [16, Th.2].

In this work, we follow a different and simpler approach, related to the specific structure of the operator L (see [2]), to avoid the necessity of a Riesz-like representation in the case of functions with infinite-dimensional range. Anyway, the form considered for L, that is Equation (2) below, allows us to include in the abstract setting many of the usual processes of numerical analysis, like *interpolation, quadrature formulas, numerical differentation.* We stress that the kernel is exactly the same as in the scalar case, so that in the estimates we can resort to known formulas, except for those making use of mean value theorems. We also point out that continuity of L seems to be unnecessary in the proof.

Here is the plan of the paper. In Section 2, we prove our main result, which we term the "*Abstract Peano Kernel Theorem*". The proof is based on a lemma concerning the interchange of certain linear operators with the (Bochner) integral symbol. In Section 3, we provide two simple examples, namely, error estimates for abstract polynomial interpolation and abstract quadrature formulas. These examples do not represent a mere generalization of the corresponding scalar instances, but they give a sound basis for the subsequent (§3.1) *error analysis* of the *trapezoidal time discretization scheme* for abstract *parabolic evolution equations.*

2 The Abstract Peano Kernel Theorem

For convenience, we introduce some notations about spaces of functions valued in X, X being a general Banach space:

- $\mathcal{C}([a,b];X) = \left\{ f : f \text{ maps } [a,b] \ \underline{\textit{continuously}} \text{ into } X \right\}$
- $\mathcal{C}^k([a,b];X) = \left\{ f : f^{(j)} \ \textit{maps} \ [a,b] \ \underline{\textit{continuously}} \text{ into } X,\ j = 1, ..., k \right\}$, $f^{(j)}$ standing for the *j-th* strong (Frechet) derivative
- $L^1(\Omega;X) = \left\{ f : f \text{ maps } \Omega \text{ into } X,\ f \text{ is } \underline{\textit{Bochner integrable}} \text{ on } \Omega \right\}$, Ω being a measurable subset of $\mathbb{R}^m$
- $W^{n,1}((a,b);X) = \{ f : D^j f \in L^1((a,b);X),\ j = 0, ..., n \}$, $D^j f$ denoting the *j-th* <u>*weak derivative*</u>
- $PC([a,b];X) = \left\{ f : f \text{ maps } [a,b] \text{ into } X,\ f \text{ is } \underline{\textit{piecewise continuous}} \right\}$

Concerning Bochner integration theory, we refer the reader to [4, 5, 7]. As for abstract Sobolev spaces $W^{n,p}((a,b);X)$, $p \geq 1$, we refer the reader to [6, PartII, Ch. 5]. Moreover, by "abstract polynomial" (of degree $\leq m$) we mean, as usual, any function of the form $p(x) = \sum_{j=0}^{m} c_j x^j$ where $x \in \mathbb{R}$, $c_j \in X$, $j = 0, 1, ..., m$.

Consider the linear operators

$L_1 = \delta_{\bar{x}} : X^{[a,b]} \to X$ (evaluation at $\bar{x} \in [a,b]$);

$L_2 = I_\alpha : L^1((a,b); X) \to X$,

$$I_\alpha(f) = \int_a^b \alpha(x) f(x) dx, \quad \text{where} \quad \alpha \in PC([a,b]; \mathbb{R}) \ ;$$

$L_3 = D : W^{1,1}((a,b); X) \to L^1((a,b); X)$ (the weak derivative operator).

The following Lemma will be needed to extend *Peano's Kernel Theorem* in spaces of X-valued functions.

Lemma 2.1 *Let ϕ be any function in $L^1((a,b)^2; X)$ that has these two properties:*

(i) $\phi(\cdot, t) \in W^{1,1}((a,b); X)$ for almost any $t \in (a,b)$;

(ii) $\|D_x \phi(x, \cdot)\| \leq h(\cdot)$, $h(\cdot) \in L^1((a,b); \mathbb{R})$, for each $x \in (a,b)$.

Then the operators L_1, L_2, and L_3 defined above have the following commutativity property:

$$L_i \int_a^b \phi(\cdot, t) dt = \int_a^b L_i \phi(\cdot, t) dt \quad i = 1, 2, 3 \ . \tag{1}$$

Proof. As for the operator L_1 the result is obvious. Concerning the operator L_2, we observe that we can apply the generalized version of Fubini's theorem for the Bochner integral (cf. [7, Th. 3.7.13]), to obtain

$$\int_a^b \alpha(x) \left(\int_a^b \phi(x,t) dt \right) dx = \int_a^b \int_a^b \alpha(x) \phi(x,t) dx dt = \int_a^b \left(\int_a^b \alpha(x) \phi(x,t) dx \right) dt \ ,$$

since $\alpha(x)\phi(x,t) \in L^1((a,b)^2; X)$, $\alpha \in PC([a,b]; \mathbb{R})$.

Finally, the operator L_3 commutes with the integral by an easy generalization (to Bochner integrals) of the classical theorem of *differentiation under the integral symbol* (cf. [7, Ch. 3]). $\square$

In the following, we will be concerned with the linear operator

$$L : W^{n,1}((a,b); X) \to X \ ,$$

defined by

$$L(f) = \int_a^b \sum_{j=0}^{n} \alpha_j(x) D^j f(x) dx + \sum_{j=0}^{n} \sum_{i_j=0}^{n_j} \beta_{i_j} D^j f(x_{i_j}), \quad x_{i_j} \in [a,b] \ \forall i, j, \quad \beta_{i_j} \in \mathbb{R} \ , \tag{2}$$

where $\alpha_j \in PC([a,b];X)$. Such an operator is sufficiently general to include many of the usual processes of numerical analysis, such as interpolation, numerical quadrature, and numerical differentiation.

Now, we state and prove the main result of this paper, which we term the "*Abstract Peano Kernel Theorem*".

Theorem 2.1 *Suppose that the operator L defined in (2) vanishes on all abstract polynomials of degree $\leq n$. Moreover, consider the usual truncated power function*

$$(x-t)_+^n = \begin{cases} (x-t)^n & x \geq t \\ 0 & x < t \end{cases} \tag{3}$$

where $x, t \in [a,b]$, and the functional

$$\hat{L} : W^{n,1}((a,b);\mathbb{R}) \to \mathbb{R},$$

associated in the natural way to L in (2).

Then, for each $f \in \mathcal{C}^{n+1}([a,b];X)$ we have

$$L(f) = \int_a^b f^{(n+1)}(t) K_n(t) dt, \tag{4}$$

where

$$K_n(t) = \frac{1}{n!} \hat{L}_x \left\{ (x-t)_+^n \right\} \tag{5}$$

is the usual scalar *Peano's Kernel, $\hat{L}_x$ denoting that the functional $\hat{L}$ is applied to $(x-t)_+^n$ as a function of x.*

Proof. Using Taylor's formula for X-valued functions (cf. [9]), we can write

$$f(x) = \sum_{j=0}^{n} \frac{(x-a)^j}{j!} f^{(j)}(a) + \frac{1}{n!} \int_a^b (x-t)_+^n f^{(n+1)}(t) dt .$$

Then,

$$L(f) = \frac{1}{n!} L \left(\int_a^b (x-t)_+^n f^{(n+1)}(t) dt \right),$$

since by hypothesis L vanishes on all abstract polynomials of degree $\leq n$. Now observe that, by using the operators introduced in Lemma 2.1, we can write L as

$$L(f) = \sum_{j=0}^{n} I_{\alpha_j} \left(D^j f \right) + \sum_{j=0}^{n} \sum_{i_j=0}^{n_j} \beta_{i_j} \delta_{x_{i_j}} \left(D^j f \right) . \tag{6}$$

Thus, by the fact that $(\cdot - t)_+^n \in W^{n,1}((a,b);X)$ and $f^{(n+1)} \in \mathcal{C}([a,b];X)$, the weak partial derivatives $D_x^j \left(f^{(n+1)}(t)(x-t)_+^n \right)$ satisfy all the hypotheses of Lemma 2.1 for $j = 0, ..., n-1$, whence it is immediately shown that

$$L(f) = \frac{1}{n!} \int_a^b L_x \left\{ f^{(n+1)}(t)(x-t)_+^n \right\} dt .$$

The proof is completed by observing that for any $g \in W^{n,1}((a,b);\mathbb{R})$ and for any $c \in X$, $L(cg) = c\hat{L}(g)$. In fact, for any $\gamma \in X$ and $u : [a,b] \to \mathbb{R}$, $\delta_{\bar{x}}(\gamma u) = \gamma u(\bar{x}) = \gamma \delta_{\bar{x}}(u)$. Moreover, for any $\gamma \in X$ and $u \in L^1((a,b);\mathbb{R})$, $\gamma u \in L^1((a,b);X)$ and $I_\alpha(\gamma u) = \int_a^b \alpha(x)(\gamma u)(x)dx = \gamma \int_a^b \alpha(x)u(x)dx = \gamma I_\alpha(u)$. Finally, for any $\gamma \in X$ and $u \in W^{1,1}((a,b);\mathbb{R})$, $\gamma u \in W^{1,1}((a,b);X)$ and $D(\gamma u) = \gamma Du$. $\square$

Remark 2.2 We stress that the Peano Kernel, $K_n(t)$, appearing in Theorem 2.1 is *exactly the same* as in the classical scalar case [2, Th. 3.7.1], essentially by the fact that $L(cg) = c\hat{L}g$ for any $g \in W^{n,1}((a,b);\mathbb{R})$, $c \in X$. This implies that, as we shall see in the next section, most of the classical applications of Peano's Kernel Theorem can be translated into the abstract framework, by resorting to known formulas *except* for those making use of *mean value theorems.*

3 Examples and applications

We begin this section with two simple examples, which are almost straightforward extensions of the corresponding scalar instances.

Example 3.1 *Abstract polynomial interpolation.*
Let $f \in \mathcal{C}^{n+1}([a,b];X)$, $s_0 < s_1 < ... < s_n$ be fixed in $[a,b]$. Consider the abstract polynomial

$$p_n(f;s) = \sum_{j=0}^{n} l_{j,n}(s) f(s_j) \; , \tag{7}$$

$l_{j,n}(s)$ being the $j-th$ scalar elementary Lagrangean polynomial.

By Theorem 2.1, we get the usual *Kowalewski remainder representation* [2, Ex.1, p.71]

$$R_n(f;s) = f(s) - p_n(f;s) = \frac{1}{n!}\sum_{j=0}^{n} l_{j,n}(s)\int_a^b (s_j - t)^n f^{(n+1)}(t)dt \; , \quad s \in [a,b] \; . \tag{8}$$

In order to derive estimates for the interpolation error, we go back to the integral representation of the remainder given in (4), obtaining

$$\|R_n(f;s)\| \leq \max_{t\in[a,b]} \|f^{(n+1)}(t)\| \int_a^b |K_n(t)|\, dt \; . \tag{9}$$

Now, since the kernel $K_n(t)$ is exactly the same as in the scalar case, it does not change sign in $[a,b]$, in view of a well known result (cf. [19], p. 271). Therefore, by Theorem 4 in [19, §2.7], we have

$$\int_a^b |K_n(t)|\, dt = \frac{1}{(n+1)!}\prod_{i=0}^{n} |s - s_i| \; , \tag{10}$$

and thus the usual error estimate for polynomial interpolation.

Example 3.2 *Abstract quadrature formulas.*
Let $f \in \mathcal{C}^{n+1}([a,b];X)$, and consider a quadrature formula of degree at least n

$$\int_a^b w(t)f(t)dt = \sum_{i=0}^{n} w_{in} f(t_{in}) + E_n(f) \; , \tag{11}$$

where $w \in L^1((a,b);\mathbb{R})$, $w_{in} \in \mathbb{R}$, $a \leq t_{0n} < \ldots < t_{nn} \leq b$. Then, the error term $E_n(f)$ can be estimated as

$$\|E_n(f)\| \leq \frac{1}{n!} \max_{t \in [a,b]} \|f^{(n+1)}(t)\| \int_a^b |w(t)K_n(t)| \, dt \; , \tag{12}$$

which corresponds to the usual scalar bound when the Kernel $K_n(t)$ does not exhibit constant sign in $[a,b]$.

In particular, as for the "*abstract trapezoidal formula*", we obtain $|K_1(t)| = \dfrac{(t-a)(b-t)}{2}$ and hence

$$\| \int_a^b f(t)dt - \frac{b-a}{2}(f(b)+f(a)) \| \leq \max_{t \in [a,b]} \|f''(t)\| \frac{(b-a)^3}{12} \tag{13}$$

for any $f \in \mathcal{C}^2([a,b];X)$.

3.1 Application to the error analysis of the trapezoidal time discretization method for parabolic PDEs

Semidiscretization in time (also known as Rothe's method, cf. [14, 15]), is a classical and powerful approach in the numerical solution of evolutionary PDEs. Compared to semidiscretization in space, it presents the feature of retaining a *great flexibility* in the choice of the *space discretization* method at each step, allowing at the same time local control of the space discretization error. In other words, the *space* and the *time errors* can be *controlled separately*, and the space discretization evolves in time.

The stability and convergence analysis of *time discretization schemes* is usually placed in suitable functional-analytic frameworks, and related to abstract formulations of the evolution problems; cf., e.g., [8, 14, 18, 20] and references therein.

Here, we analyze a generalized version of the *trapezoidal method* (also known as *Crank-Nicholson method*), for approximating *classical* solutions of *linear parabolic evolution equations* in *Hilbert space*. Consider the abstract IVP

$$u' = Au + f(t) \; , \; t \in (t_0, T] \; ; \;\; u(t_0) = u_0 \; , \tag{14}$$

where $A : (D(A) \subseteq \mathcal{H}) \to \mathcal{H}$, with $\mathcal{H}$ real or complex Hilbert space, is a *linear*, *self-adjoint* and *dissipative* operator, with *compact resolvent* (cf. [1, 10]). We assume that $f : [t_0, T] \to \mathcal{H}$ and $u_0 \in D(A)$ satisfy suitable regularity and compatibility conditions, ensuring existence of a classical solution $u \in \mathcal{C}^3([t_0,T];\mathcal{H})$.

A typical instance of equation (14) is represented by a diffusion process modelled by the linear parabolic PDE

$$\frac{\partial v}{\partial t} = \mathcal{L}v + g(x,t)\ ,\quad x \in \Omega\ ,\ t \in (t_0, T]\ ,$$
$$v(x,t_0) = v_0(x),\quad v(x,t) \equiv 0\ \text{ for } x \in \partial\Omega\ , \tag{15}$$

where $\Omega \subset \mathbb{R}^m$ is open, smooth and bounded, $-\mathcal{L} = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right) + c(x)$ is a linear *strongly elliptic* diffusion operator, with $a_{i,j}(\cdot) = a_{j,i}(\cdot) \in \mathcal{C}^1(\overline{\Omega};\mathbb{R})$, $0 \le c(\cdot) \in \mathcal{C}(\overline{\Omega};\mathbb{R})$, and $g(x,t)$ is a source term. Equation (15) can be put into the abstract form (14) by setting, for example, $\mathcal{H} = L^2(\Omega)$, $A = \mathcal{L}$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $u(t) = v(\cdot,t)$, $f(t) = g(\cdot,t)$. In this case A is indeed self-adjoint, maximal and dissipative (cf. [10]); moreover its resolvent $(\sigma I - A)^{-1}$, $\sigma > 0$, turns out to be compact (cf. [21, Ch.6, §6]).

As for the regularity required for the solution $u(t)$ of (14), we only mention that $u \in \mathcal{C}^3([t_0,T];\mathcal{H})$ whenever $Au_0 \in D(A)$, $f(t_0) \in D(A)$ and $A^2u_0 + Af(t_0) + f'(t_0) \in D(A)$, and moreover $f \in W^{3,1}([t_0,T];\mathcal{H})$. The latter is verified, for instance, when $f'' \in Lip([t_0,T];\mathcal{H})$, in particular when $f \in \mathcal{C}^3([t_0,T];\mathcal{H})$. We shall not dwell upon this aspect here, for the sake of brevity. In any event, the proof can be easily obtained by extending the result in [20], within the linear semigroups theory (cf. also [3]).

The *trapezoidal discretization scheme* for (14) is given by

$$u_{n+1} = u_n + \frac{h}{2}A\left(u_n + u_{n+1}\right) + \frac{h}{2}\left(f(t_n) + f(t_{n+1})\right)\ , \tag{16}$$

with $n = 0,1,...,N-1$, $h = (T-t_0)/N$, $t_n = t_0 + nh$. First of all, we notice that such a scheme is well-defined since each equation in (16) has a unique solution in $D(A)$, in view of the properties of the operator A. Consider also the following "perturbed" version of (16)

$$\tilde{u}_{n+1} = \tilde{u}_n + \frac{h}{2}A\left(\tilde{u}_n + \tilde{u}_{n+1}\right) + \frac{h}{2}\left(f(t_n) + f(t_{n+1})\right) + \delta_{n+1}\ , \tag{17}$$

for $n = 0,1,...,N-1$, with $\delta_j \in \mathcal{H}$, $0 \le j \le N$, $\delta_0 = \tilde{u}_0 - u_0$. Setting $\epsilon_n = \tilde{u}_n - u_n$, from (16) and (17) we can write

$$\epsilon_{n+1} = A_h B_h \epsilon_n + B_h \delta_{n+1}\ ,\ n = 0,1,...,N-1\ ,$$

where $A_h = \left(I - \frac{h}{2}A\right)^{-1}$, $B_h = \left(I + \frac{h}{2}A\right)$. By resorting to well-known properties of the *Yosida regularization* of the monotone operator A, cf. [1, Ch. 7], it can be shown that $A_hB_hv = (2A_h - I)v$ for every $v \in D(A)$, where $A_h : \mathcal{H} \to \mathcal{H}$ is compact and self-adjoint. Then, we have $\|A_h\| = \rho(A_h)$, $\|2A_h - I\| = \rho(2A_h - I)$, $\rho(\cdot)$ denoting the spectral radius. Moreover, the spectrum of such operators consists of real eigenvalues, say μ and ν respectively, where

$$\mu = \frac{1}{1 + \frac{h}{2}|\lambda|}\ ,\qquad \nu = \frac{1 - \frac{h}{2}|\lambda|}{1 + \frac{h}{2}|\lambda|}\ ,$$

λ being an eigenvalue of A (recall that $\lambda < 0$, since A is dissipative). Therefore, we have $|\mu|, |\nu| < 1$, from which follows $\|A_h\| < 1$, $\|2A_h - I\| < 1$.

We are now ready to give the estimate

$$\begin{aligned} \|\epsilon_{n+1}\| &\leq \|A_h B_h \epsilon_n\| + \|A_h\| \|\, \delta_{n+1}\| \\ &\leq \|2A_h - I\|\|\epsilon_n\| + \|\delta_{n+1}\| \leq \|\epsilon_n\| + \|\delta_{n+1}\| \ , \ \ 0 \leq n \leq N-1 \ . \end{aligned} \tag{18}$$

From (18), we obtain by induction on n the *stability estimate*

$$\|\epsilon_n\| = \|\tilde{u}_n - u_n\| \leq \sum_{j=0}^{N} \|\delta_j\| \ , \ \ 0 \leq n \leq N \ , \tag{19}$$

which shows that, as in the case of linear dissipative systems of ODEs, the abstract trapezoidal scheme is *unconditionally stable*.

In order to prove *convergence* of the trapezoidal method, we observe that the solution $u(t)$ of (14) satisfies itself a perturbed iterative scheme like (17), i.e.

$$u(t_{n+1}) = u(t_n) + \frac{h}{2} A\left(u(t_n) + u(t_{n+1})\right) + \frac{h}{2}\left(f(t_n) + f(t_{n+1})\right) + \sigma_{n+1}(h) \ , \tag{20}$$

for $n = 0, 1, ..., N-1$, so that convergence is obtained if $\sum_{j=1}^{N} \|\sigma_j(h)\| \to 0$ as $h \to 0$. We stress that only additive perturbations like those in (17) have been considered in the stability analysis, since here we are mainly interested in the propagation of the *local truncation errors*, $\sigma_j(h)$. A full stability analysis should take into account also the local errors introduced in solving approximately each equation of the scheme (cf. also [18]).

Iteration (20) corresponds to applying, at each step n, the *abstract trapezoidal quadrature formula* to the function u' (cf. Example 3.2). In fact,

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} u'(t)dt = \frac{h}{2}\left(u'(t_n) + u'(t_{n+1})\right) + \sigma_{n+1}(h) \ ,$$

i.e. (20) since $u' = Au + f(t)$. Using estimate (13) with $a = t_n$, $b = t_{n+1}$, and recalling that $u \in \mathcal{C}^3\left([t_0, T]; \mathcal{H}\right)$, we get

$$\|\sigma_{n+1}(h)\| \leq \max_{t \in [t_n, t_{n+1}]} \|u'''(t)\| \frac{h^3}{12} \leq \max_{t \in [t_0, T]} \|u'''(t)\| \frac{h^3}{12} \ . \tag{21}$$

Thus, the *local consistency order* is $O(h^3)$, the constant of the $O(\cdot)$ symbol being independent of n. Finally, by means of (21) and (19), the *convergence estimate*

$$\max_{1 \leq n \leq N} \|u(t_n) - u_n\| \leq \sum_{j=1}^{N} \|\sigma_j(h)\| \leq (T - t_0) \max_{t \in [t_0, T]} \|u'''(t)\| \frac{h^2}{12} \tag{22}$$

is obtained. This shows that, as in the finite dimensional case, the trapezoidal method is (globally) of the *second order*. We emphasize the *key role* played by the Abstract Peano Kernel Theorem in the error analysis above. Here, for the sake of simplicity, only linear parabolic equations have been considered. The stability and convergence analysis of the trapezoidal discretization method on abstract *semilinear* parabolic equations will appear in [3].

In the present framework, Theorem 2.1 has another important application. Suppose that we want to approximate the solution $u(t)$ on the whole interval $[t_0, T]$. Then, it is natural to use an interpolation scheme having (globally) the same order of the time discretization. Indeed, if we consider the *piecewise linear interpolant* on the values u_n, say $S_{1,h}(t)$, then (9) and (10) in Example 3.1 with $n = 1$, together with (22), give

$$\max_{t\in[t_0,T]} \|u(t) - S_{1,h}(t)\| \leq \max_{t\in[t_0,T]} \|u''(t)\| \frac{h^2}{8} + (T - t_0) \max_{t\in[t_0,T]} \|u'''(t)\| \frac{h^2}{12}\,, \tag{23}$$

that is a uniform approximation scheme of order $O(h^2)$.

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