Peano's Kernel Theorem for Vector-Valued Functions: A Weak Version in Normed Spaces*

Stefano De Marchi[1] and **Marco Vianello**[2]

Abstract

Improving a recent result by the authors, a vector-valued version of Peano's Kernel Theorem is proposed, which gives an integral estimate for a class of linear operators $L : C^{n+1}([a,b];X) \to X$, with X general normed space, vanishing on all abstract polynomials of degree $\leq n$. Continuity and derivatives are intended in the weak sense. When the space is complete, the usual integral representation is retrieved. We show that all usual remainder estimates for polynomial, piecewise polynomial, and spline polynomial interpolation, numerical differentiation and numerical quadrature, can be readily transferred in the vector-valued setting.

Keywords and phrases: Peano's Kernel Theorem, vector-valued functions, normed spaces, Banach spaces, weak topology, polynomial interpolation, numerical quadrature, numerical differentiation.

1991 AMS subject classifications: 41A65, 41A80, 65J05.

---

*Work supported, in part, by the MURST Numerical Analysis "40% funds", the MURST "60% funds", and the GNIM-CNR.

[1]Dipartimento di Matematica e Informatica, Università di Udine, via delle Scienze 206, 33100 Udine, Italy; *e-mail*: demarchi@dimi.uniud.it

[2]Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy; *e-mail*: marcov@euler.math.unipd.it

1 Introduction

Vector-valued versions of classical theorems in analysis and numerical analysis have received a renewed attention in recent years, cf., e.g., [1, 3, 5, 6, 8, 14, 16] and references therein. In [3], the authors proved an abstract version of Peano's Kernel Theorem [9, 10], which gave a Bochner integral representation for a class of continuous linear operators

$$L : C^{n+1}([a, b]; X) \to X \ , \tag{1}$$

with X Banach space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, vanishing on all abstract polynomials of degree $\leq n$. Continuity and derivatives were intended in the strong sense. The structure of L was sufficiently general to embody the most usual instances of numerical analysis and approximation theory, cf. [2, Ch.3]. At the same time, a direct proof became possible, avoiding to appeal to Sard's "Quotient Theorem" [12, 13], whose application should require in turn an abstract version of Riesz representation theorem. We also recall here that completeness plays an essential role in the Quotient Theorem. Classical estimates for polynomial interpolation and numerical quadrature were then recovered in the abstract setting, and an application to the convergence analysis of the trapezoidal time-discretization scheme for parabolic evolution equations was given.

In the present paper we extend the quoted result, improving it in three directions:

(i) we consider a more general class of linear operators, i.e. L as in (1), for which there exists a *linear functional* $\ell \in (C^n[a, b])^*$ such that

$$\varphi(L(f)) = \ell(\varphi \circ f) \ , \quad \forall\, \varphi \in X^* \ , \quad \forall\, f \in C^{n+1}([a, b]; X) \ , \tag{2}$$

X^* denoting the dual of X: we may refer to this property as the "*exchange property*";

(ii) we derive an *integral estimate* for $L(f)$, valid in a *general normed space* (completeness plays indeed no role), with *continuity* and *derivatives* intended in the *weak sense* (cf. (4) below);

(iii) when the space is *complete*, we retrieve the classical *integral representation*, again with *weak* replacing *strong* notions (cf. (6) below).

In both (ii) and (iii), the Kernel of the linear functional ℓ is involved. As a consequence, the most usual applications of Peano's Kernel Theorem in approximation theory and numerical analysis, e.g. remainders estimates for *interpolation*, *numerical quadrature* and *numerical differentiation*, can be readily transferred in the (weak) abstract framework.

2 A Weak Version of the Abstract Peano's Kernel Theorem

We begin by introducing some notations about spaces of functions valued in a normed space X over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

- $C_w^k([a,b];X) = \{f : f^{(j)}$ maps $[a,b]$ into X, $f^{(j)}$ is *weakly-continuous*, $j = 0, 1, ..., k\}$, $f^{(j)}$ denoting the j-th *weak derivative*, endowed with the natural norm $\|f\|_{C^k} = \max_{0\le j\le k} \sup_{x\in[a,b]} \|f^{(j)}(x)\|$ (recall that $f^{(j)}$ being weakly continuous is *strongly bounded*);
- $\mathbb{P}_n(X) = \{p : p$ maps $\mathbb{R}$ into X, $p(x) = \sum_{i=0}^{m} c_i x^i$, $c_i \in X$, $i = 0, 1, ..., m \le n\}$, i.e. the space of *abstract polynomials* of degree $\le n$.

When X is *complete*, we consider also the space

- $L^1((a,b);X) = \{f : f$ maps (a,b) into X, f is *Bochner integrable*$\}$.

In the present context, the notions of *weak continuity* and *differentiability* are those related to the *weak topology* of X (see, e.g., [7, Ch.3]). In the sequel, the integrals of X-valued functions are always intended in the sense of Bochner: As for *Bochner integrability* we refer to [4, 7].

We state now the main result of the paper, that is a *weak version* of the Abstract Peano's Kernel Theorem [3].

Theorem 2.1 *Let us consider a linear operator* $L : C_w^{n+1}([a,b];X) \to X$, *and the usual truncated power function*

$$(x-t)_+^n = \begin{cases} (x-t)^n & x \ge t \\ 0 & x < t \end{cases} \tag{3}$$

with $x, t \in [a,b]$. *Assume that the exchange property* (2) *holds.*

The following statements are equivalent:

(*i*) $\mathbb{P}_n(X) \subset Ker(L)$;

(*ii*) $\mathbb{P}_n(\mathbb{K}) \subset Ker(\ell)$;

(*iii*) $L(f)$ *is estimated as*

$$\|L(f)\| \leq \int_a^b \|f^{(n+1)}(t)\| \, |K_n(t)| dt \, , \tag{4}$$

where

$$K_n(t) = \frac{1}{n!} \ell\{(\cdot - t)_+^n\} \tag{5}$$

is the scalar *Peano's Kernel of the linear functional* ℓ (*as known,* $K_n \in BV[a,b]$, *cf.* [13, p.25]).

If, in addition, X *is complete,* (*ii*) (*or* (*i*), *or* (*iii*)) *implies the* (*Bochner*) *integral representation*

$$L(f) = \int_a^b f^{(n+1)}(t) K_n(t) dt \, . \tag{6}$$

Remark 2.2 Since here ℓ is a general continuous linear functional on $C^n[a,b]$, one might worry about the fact that $(\cdot - t)_+^n$ is not in $C^n[a,b]$ (indeed, $((\cdot - t)_+^n)^{(n)}$ is not in $C[a,b]$); for a discussion on this point, going back to [13, p.135], see [17, §2].

Proof. First, we prove that $(i) \iff (ii)$.

Assuming (ii), we have $\varphi(L(p)) = \ell(\varphi \circ p) = 0$ for every $p \in \mathbb{P}_n(X)$ and $\varphi \in X^*$, as $\varphi \circ p \in \mathbb{P}_n(\mathbb{K})$. Hence $L(p) = 0$ (cf. [7, §2.7]).

Conversely, assume that $L(p) = 0$ for every $p \in \mathbb{P}_n(X)$. Consider a fixed polynomial $q \in \mathbb{P}_n(\mathbb{K})$, $q(x) = \sum_{i=0}^n c_i x^i$, and $n+1$ linearly independent elements of X, say $\xi_0, ..., \xi_n$. Define the linear form

$$\phi : (Span\{\xi_0, ..., \xi_n\}, \| \cdot \|_X) \to \mathbb{K} \, , \quad \phi\left(\sum_{i=0}^n \eta_i \xi_i\right) = \sum_{i=0}^n \eta_i c_i \, ; \tag{7}$$

ϕ is obviously continuous since the space has finite dimension. By the Hahn-Banach theorem [7, §2.7], there exists an extension of ϕ, say $\varphi_\phi \in X^*$. Defining $p \in \mathbb{P}_n(X)$, $p(x) = \sum_{i=0}^n \xi_i x^i$, we have that $p(x) \in Span\{\xi_0, ..., \xi_n\}$ for every fixed x, and $q = \phi \circ p = \varphi_\phi \circ p$. Finally, by (2) we obtain

$$\ell(q) = \ell(\varphi_\phi \circ p) = \varphi_\phi(L(p)) = 0 \; . \tag{8}$$

The proof that the integral estimate (4) implies (i) is immediate, since the $(n+1)$-th derivative of $p \in \mathbb{P}_n(X)$ is zero. Now, let us prove that (ii) implies (iii). In view of (2) and the fact that $\varphi \circ f \in C^{n+1}[a,b]$, by resorting to the scalar version of Peano's Kernel Theorem [13, p.25], we get

$$\varphi(L(f)) = \int_a^b \varphi(f^{(n+1)}(t))K_n(t)dt \; , \tag{9}$$

and thus the estimate

$$\left|J_{L(f)}(\varphi)\right| = |\varphi(L(f))| \leq \|\varphi\|_{X^*} \int_a^b \|f^{(n+1)}(t)\| \, |K_n(t)|dt \; , \tag{10}$$

where $J : X \to X^{**}$, $J_x(\varphi) = \varphi(x)$, is the standard embedding of X into its second dual X^{**}. Observe that the integral in (10) is finite since $f^{(n+1)}$ is weakly continuous and hence *strongly measurable* and *bounded* [7, §3.5]. Now, as J is an *isometry* (cf. [7, §2.8]), we get

$$\|L(f)\| = \left\|J_{L(f)}\right\|_{X^{**}} \leq \int_a^b \|f^{(n+1)}(t)\| \, |K_n(t)|dt \; , \tag{11}$$

that is (4).

In the case X be *complete*, $f^{(n+1)} K_n \in L^1((a,b);X)$ since $f^{(n+1)}$ is strongly measurable and bounded [7, §3.5,3.7], and therefore assuming (ii) (or one of the equivalent statements (i), (iii)), resorting to a basic property of the Bochner integral [7, (3.7.5), §3.7] from (9) we obtain

$$\varphi(L(f)) = \varphi\left(\int_a^b f^{(n+1)}(t)K_n(t)dt\right) \; , \; \forall\, \varphi \in X^* \; . \tag{12}$$

The representation (6) follows at once, as $L(f) - \int_a^b f^{(n+1)}(t)K_n(t)dt \in Ker(\varphi)$ for every $\varphi \in X^*$ (cf. [7, §2.7]). $\square$

Remark 2.3 It is immediately seen that a linear operator for which an estimate like (4) holds (with an arbitrary $u \in L^1[a,b]$ replacing K_n), is necessarily continuous on $C_w^{n+1}([a,b];X)$. On the other hand, the exchange property (2) implies directly the continuity of L on $C_w^{n+1}([a,b];X)$. In fact, for every $f \in C_w^{n+1}([a,b];X)$ we have

$$|\varphi(L(f))| = \left|J_{L(f)}(\varphi)\right| = |\ell(\varphi \circ f)| \leq \|\ell\| \, \|\varphi \circ f\|_{C^n} \leq$$

$$\|\ell\| \, \|\varphi\| \, \|f\|_{C^n} \leq \|\ell\| \, \|\varphi\| \, \|f\|_{C^{n+1}} \, , \tag{13}$$

from which follows

$$\left\|J_{L(f)}\right\|_{X^{**}} = \|L(f)\| \leq \|\ell\| \, \|f\|_{C^{n+1}} \, . \tag{14}$$

On the other hand, if X is complete we can easily characterize the continuous linear operators which satisfy the exchange property. Indeed, it is well-known that $\ell \in (C^n[a,b])^*$ if and only if it has the form

$$\ell(g) = \sum_{j=0}^{n} \int_a^b g^{(j)}(t) \, dw_j(t) \, , \tag{15}$$

for suitable $w_j \in BV[a,b]$ (cf. [13, p.144]). It follows that if L satisfies the exchange property, then

$$0 = \varphi(L(f)) - \sum_{j=0}^{n} \int_a^b \varphi(f^{(j)}(t)) \, dw_j(t)$$

$$= \varphi(L(f) - \sum_{j=0}^{n} \int_a^b f^{(j)}(t) \, dw_j(t)) \, , \ \ \forall \, \varphi \in X^* \, , \ \ \forall \, f \in C_w^{n+1}([a,b];X) \, . \tag{16}$$

Thus, we get the representation

$$L(f) = \sum_{j=0}^{n} \int_a^b f^{(j)}(t) \, dw_j(t) \, , \ \ f \in C_w^{n+1}([a,b];X) \, , \tag{17}$$

i.e. L is the restriction to $C_w^{n+1}([a,b];X)$ of the continuous linear operator on $C_w^n([a,b];X)$ defined by the r.h.s. of (17). Conversely, if (17) holds, then the exchange property applies as soon as ℓ is chosen of the form (15).

Remark 2.4 It can be shown that if L admits an integral representation like

$$L(f) = \int_a^b f^{(n+1)}(t)w(t)dt \ , \quad w \in BV[a,b] \ , \tag{18}$$

cf. (6), then necessarily (2) and (i) hold. In fact, (i) is immediate, while (2) is obtained by choosing $\ell \in (C^n[a,b])^*$ as

$$\ell(g) = w(b)\, g^{(n)}(b) - w(a)\, g^{(n)}(a) - \int_a^b g^{(n)}(t)\, dw(t) \ . \tag{19}$$

Indeed,

$$\ell(g) = \int_a^b w(t)\, dg^{(n)}(t) = \int_a^b g^{(n+1)}(t)w(t)dt \ , \tag{20}$$

where the second equality holds as $g^{(n)} \in AC[a,b]$, while the first can be proved integrating by parts in the sense of Riemann-Stieltjes [13, p.501], and then making use of the well-known relation between Riemann-Stieltjes and Lebesgue-Stieltjes integration (cf. [13, §34,44]). Following the lines of the previous Remark, it is also easy to show that (18) is equivalent to

$$L(f) = w(b)\, f^{(n)}(b) - w(a)\, f^{(n)}(a) - \int_a^b f^{(n)}(t)\, dw(t) \ , \quad f \in C_w^{n+1}([a,b];X) \ , \tag{21}$$

i.e. it is the restriction to $C_w^{n+1}([a,b];X)$ of the continuous linear operator on $C_w^n([a,b];X)$ defined by the r.h.s. of (21).

Remark 2.5 It is worth noting that the *exchange property* (2) indeed characterizes a *proper* subclass of continuous linear operators $L : C_w^{n+1}([a,b];X) \to X$, if $dim(X) > 1$. Consider for example, $X = \mathbb{R}^m$, $m > 1$, and $L(f) = \{L_i(f)\}_{1 \le i \le m}$, with $L_i(f) = c_i f_i(x_0)$, $x_0 \in [a,b]$, where $c_i \neq c_j$ for some i, j. Let us take $g \in C^{n+1}([a,b];\mathbb{R})$ with $g(x_0) \neq 0$, and $f = \{f_j\}_{1 \le j \le m} \in C^{n+1}([a,b];\mathbb{R}^m)$ such that $f_j = \delta_{ij} g$ for an arbitrary but fixed i, $i = 1, ..., m$. By the arbitrarity of φ, the exchange property (2) should lead to the contradiction $\ell(g) = c_i g(x_0)$ for every i. This simple counterexample suggests that, when the space possesses a basis, a componentwise version of the Kernel Theorem should be seeked, in order to estimate or to represent more general linear operators.

As an application of Theorem 2.1, consider the *linear* operator

$$\Lambda_1 : C_w^{n+1}([a,b];X) \to X \ , \ \ \Lambda_1(f) = \sum_{j=0}^{n} \sum_{i_j=0}^{n_j} \beta_{i_j} f^{(j)}(x_{i_j}) \ , \tag{22}$$

where $x_{i_j} \in [a,b] \ \forall\, i,j$, and $\beta_{i_j} \in \mathbb{K}$. Moreover, when X is a *Banach space*, we may also define the linear operator

$$\Lambda_2 : C_w^{n+1}([a,b];X) \to X \ , \ \ \Lambda_2(f) = \Lambda_1(f) + \sum_{j=0}^{n} \int_a^b \alpha_j(x) f^{(j)}(x) dx \ , \tag{23}$$

where $\alpha_j \in L^1((a,b);\mathbb{K})$. We notice that the operators Λ_1 and Λ_2 resemble, in an abstract setting, the most typical instances of *numerical analysis* and *approximation theory*, such as *interpolation*, *numerical quadrature* and *numerical differentiation* (cf., e.g., [2, Ch.III]).

These operators do satisfy the exchange property (2). This is immediate for Λ_1, in view of the definition of weak derivative. As for Λ_2, the fact that every functional φ commutes with the integral symbol is a basic property of the Bochner integral, cf. [7, §3.7]. Therefore we get the estimate (4) for $L = \Lambda_1$ or $L = \Lambda_2$, and the representation (6) for $L = \Lambda_2$ when X is complete, as soon as *the linear functionals ℓ_1 and ℓ_2 corresponding respectively to Λ_1 and Λ_2 for $X = \mathbb{K}$*, vanish on all scalar polynomials of degree $\leq n$. In other words, comparing to the extension of Peano's Kernel Theorem proved in [3], with the present point of view it is *not* necessary to check *directly* that $\Lambda_i(p) = 0$ on all abstract polynomials p. At the same time, derivatives are intended in the weak sense, and $\Lambda_1(f)$ can be estimated in absence of completeness.

As a consequence, all usual scalar formulas for polynomial, piecewise polynomial, and spline polynomial interpolation (cf., e.g., [11]), numerical differentiation and numerical quadrature (cf., e.g., [15]), can be readily transferred in the abstract setting, together with the corresponding remainders estimates. In fact, for such remainders we have

$$|\ell_i(g)| \leq |g^{(n+1)}(\xi)| \int_a^b |K_{n,i}(t)| dt \leq \|g^{(n+1)}\|_\infty \int_a^b |K_{n,i}(t)| dt \ , \ \ \xi \in [a,b] \ , \tag{24}$$

$i = 1, 2$, and correspondingly

$$\|L_i(f)\| \leq \sup_{x \in [a,b]} \|f^{(n+1)}(x)\| \int_a^b |K_{n,i}(t)| dt \ , \tag{25}$$

where the quantity $\|K_{n,i}\|_1$ in (24), (25) is usually termed *Peano's error constant* (cf. [15, p.168]).

Consider, for example, the abstract polynomial interpolation formula: the remainder estimate

$$\|f(x) - \sum_{j=0}^{n} \lambda_{j,n}(x) f(x_j)\| \leq \sup_{x \in [a,b]} \|f^{(n+1)}(x)\| \frac{1}{(n+1)!} \prod_{j=0}^{n} |x - x_j| \ , \qquad (26)$$

where $x_j \in [a,b]$ are distinct and $\lambda_{j,n}(x)$ represents the j-th elementary Lagrangean polynomial of degree n, is still valid in an arbitrary normed space X, with $f^{(n+1)}(x)$ denoting either the strong or the weak $(n+1)$-th derivative (cf. [3, Ex.3.1]).

Conclusion. We have extended Peano's Kernel Theorem, in the form of an *integral estimate*, to a class of continuous linear operators $L : C_w^{n+1}([a,b];X) \to X$, X being a *general normed space*. When X is a *Banach space*, Theorem 2.1 together with Remarks 2.3 and 2.4 provide the following important characterization: *L admits an integral representation like* (18) (*which is equivalent to* (21)) *if and only if it has the form* (17) *and the corresponding linear functional* (15) *vanishes on* $\mathbb{P}_n(\mathbb{K})$. Moreover, $w(t)$ can be replaced with $K_n(t)$ in (18), (21); correspondingly, the first two summands in (21) vanish, as $K_n(a) = K_n(b) = 0$, cf. (5).

Acknowledgements. We wish to thank Dr. S. Waldron for having attracted our attention on his paper [17], which clarified us some basic facts concerning the scalar version of Peano's Kernel Theorem.

References

[1] W. Arendt, *Vector-Valued Versions of Some Representation Theorems in Real Analysis*, Lecture Notes in Pure and Applied Mathematics **150**, Marcel Dekker, New York, 1994, pp. 33-50.

[2] P.J. Davis, *Interpolation and Approximation*, Dover, New York, 1975.

[3] S. De Marchi and M. Vianello, *Peano's Kernel Theorem for Vector-Valued Functions and Some Applications*, Numer. Funct. Anal. and Optim. **17** (1996), 57-64.

[4] J. Diestel and J.J. Uhl, *Vector Measures*, American Mathematical Society, Providence (RI), 1977.

[5] L. Furi and M. Martelli, *A Multidimensional Version of Rolle's Theorem*, Amer. Math. Monthly **102** (1995), 243-249.

[6] H. König, *Vector-Valued Lagrange Interpolation and Mean Convergence of Hermite Series*, Lecture Notes in Pure and Applied Mathematics **150**, Marcel Dekker, New York, 1994, pp. 227-247.

[7] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, AMS Colloquium Publications, Providence (RI), 1957.

[8] M. Niga and A.D. Choudary-Raza, *Stolz-Cesaro theorem in Banach algebras of continuous functions*, Stud. Cerc. Mat. **44** (1992), 127-133.

[9] G. Peano, *Resto delle formule di quadratura espresso con un integrale definito*, Atti Reale Accademia dei Lincei, Rend. 5, **22** (1913), 562-569.

[10] G. Peano, *Residuo in Formulas de Quadratura*, Mathesis **4** (1914), 5-10.

[11] M.J.D. Powell, *Approximation Theory and Methods*, Cambridge University Press, Cambridge, 1981.

[12] A. Sard, *Integral representation of remainders*, Duke Math. J. **15** (1948), 333-345.

[13] A. Sard, *Linear Approximation*, American Mathematical Society, Providence (RI), 1963.

[14] R. Spigler and M. Vianello, *Extending L'Hôpital's theorem to B-modules*, J. Math. Anal. Appl. **179** (1993), 638-645.

[15] A. Stroud, *Numerical Quadrature and Solution of Ordinary Differential Equations*, Springer-Verlag, New York, 1974.

[16] M. Vianello, *A generalization of L'Hôpital's rule via absolute continuity and Banach modules*, Real Anal. Exchange **18** (1993), 557-567.

[17] S. Waldron, *Refinements of the Peano kernel theorem*, Technical Report, Department of Mathematics of the Israel Institute of Technology, Haifa, May 1996.